MATH 1036 - Calculus I

Sample problems for Midterm Examinations and their SOLUTIONS

1. Find the domain of the function $f(x) = \dfrac{5x+4}{\sqrt{2+x-x^2}}$.

Solution f(x) is defined if and only if denominator is $\neq 0$ AND the expression under the square root is ≥ 0. Summarizing this conditions, we have: $2+x-x^2 > 0 \Leftrightarrow x^2-x-2 < 0$. Roots of the equation $x^2-x-2=0$ are -1 and 2. Therefore the solution of inequality $x^2-x-2<0$ is the interval $(-1,2)$.

Answer: $(-1, 2)$

2. Find the limit $\lim\limits_{x\to 2} \dfrac{x^4-16}{x-2}$.

Solution

$\lim\limits_{x\to 2} \dfrac{x^4-16}{x-2} = \lim\limits_{x\to 2} \dfrac{(x^2-4)(x^2+4)}{x-2} = \lim\limits_{x\to 2} \dfrac{(x-2)(x+2)(x^2+4)}{x-2} = \lim\limits_{x\to 2}(x+2)(x^2+4) =$ (now we can use direct substitution) $= (2+2)(2^2+4) = 32$.

Answer: 32

3. Find the limit $\lim\limits_{x\to 0} x^2 \left[1+\cos\left(1+\dfrac{1}{x}\right)\right]$.

Solution

Note that $-1 \leq \cos\left(1+\frac{1}{x}\right) \leq 1$, since for ANY t we have $-1 \leq \cos t \leq 1$. Therefore $0 \leq 1+\cos\left(1+\frac{1}{x}\right) \leq 2$. This implies that $0 \cdot x^2 \leq x^2\left[1+\cos\left(1+\frac{1}{x}\right)\right] \leq 2x^2$ and therefore

$$0 \leq x^2 \left[1+\cos\left(1+\frac{1}{x}\right)\right] \leq 2x^2$$

Note that $\lim\limits_{x\to 0} 0 = \lim\limits_{x\to 0} 2x^2 = 0$. Hence the Squeeze Theorem implies that $\lim\limits_{x\to 0} x^2 \left[1+\cos\left(1+\dfrac{1}{x}\right)\right] = 0$.

Answer: 0

4. Differentiate the function $f(x) = \sqrt{2}\,x + \pi\,x\sqrt{x} + 1.5$

Solution

$(\sqrt{2}\,x + \pi\,x\sqrt{x} + 1.5)' =$ (the sum rule) $= (\sqrt{2}\,x)' + (\pi\,x\sqrt{x})' + (1.5)' =$ (the constant multiple rule for the first and second terms; derivative of constant is zero for the last term) $= \sqrt{2}\,(x)' + \pi\,(x^{3/2})' + 0 =$ (the power rule) $= \sqrt{2} + \pi \cdot \dfrac{3}{2}x^{\frac{3}{2}-1} = \sqrt{2} + \dfrac{3\pi}{2}x^{1/2} = \sqrt{2} + \dfrac{3\pi}{2}\sqrt{x}$.

Answer: $\sqrt{2} + \frac{3\pi}{2}\sqrt{x}$

5. Differentiate the function $g(\theta) = (\theta + \tan\theta)\cos\theta$.

<u>1st Solution</u>

$g(\theta) = (\theta + \tan\theta)\cos\theta =$
$\theta\cos\theta + \tan\theta \cdot \cos\theta = \theta\cos\theta + \frac{\sin\theta}{\cos\theta} \cdot \cos\theta = \theta\cos\theta + \sin\theta$. Therefore $g'(\theta) = (\theta\cos\theta + \sin\theta)' =$ (the sum rule)$= (\theta\cos\theta)' + (\sin\theta)' =$ (the product rule for the first term) $= (\theta)'\cos\theta + \theta(\cos\theta)' + \cos\theta = \cos\theta + \theta\cdot(-\sin\theta) + \cos\theta = 2\cos\theta - \theta\sin\theta$.

<u>2nd Solution</u>

$g'(\theta) = [(\theta+\tan\theta)\cos\theta]' =$ (the product rule) $= (\theta+\tan\theta)'\cos\theta + (\theta+\tan\theta)\cdot(\cos\theta)' = ((\theta)' + (\tan\theta)')\cos\theta + (\theta + \tan\theta)\cdot(-\sin\theta) = (1+\sec^2\theta)\cos\theta - (\theta+\tan\theta)\sin\theta = \cos\theta + \sec^2\theta\cos\theta - \theta\sin\theta - \tan\theta\cdot\sin\theta = \cos\theta + \frac{1}{\cos^2\theta}\cdot\cos\theta - \theta\sin\theta - \frac{\sin\theta}{\cos\theta}\cdot\sin\theta = \cos\theta + \frac{1}{\cos\theta} - \theta\sin\theta - \frac{\sin^2\theta}{\cos\theta} = \cos\theta + \frac{1-\sin^2\theta}{\cos\theta} - \theta\sin\theta = \cos\theta + \frac{\cos^2\theta}{\cos\theta} - \theta\sin\theta = \cos\theta + \cos\theta - \theta\sin\theta = 2\cos\theta - \theta\sin\theta$

Answer: $2\cos\theta - \theta\sin\theta$

6. Differentiate the function $y = \sin(\cos^2 x)$.

<u>Solution</u>

$y = \sin(\cos^2 x) = f(g(x))$, where $f(x) = \sin x$ and $g(x) = \cos^2 x$. Applying the Chain Rule we obtain: $y' = f'(g(x))\cdot g'(x)$. Note that $f'(x) = \cos x$ and therefore $f'(g(x)) = \cos(\cos^2 x))$. To find $g'(x) = (\cos^2 x)'$ we apply the Chain Rule again. Indeed, $\cos^2 x = (\cos x)^2 = h(k(x))$, where $h(x) = x^2$ and $k(x) = \cos x$. Therefore $g'(x) = 2\cos x \cdot (\cos x)' = -2\cos x\cdot\sin x$. Thus $y' = (\sin(\cos^2 x))' = \cos(\cos^2 x)\cdot(-2\cos x\sin x) = -2\cos x\cdot\sin x\cdot\cos(\cos^2 x)$.

Answer: $-2\cos x\,\sin x\,\cos(\cos^2 x)$

7. Find the constant a that makes the following function continuous on $(-\infty, \infty)$

$$f(x) = \begin{cases} x^2 + ax & \text{if } x \le 4 \\ \dfrac{x-4}{\sqrt{x}-2} & \text{if } x > 4 \end{cases}$$

<u>Solution</u>

Note that $x^2 + ax$ is a polynomial and therefore is continuous in $(-\infty, \infty)$. Thus $f(x)$ is continuous for all $x < -4$ for ANY choice of a. Similarly, $\dfrac{x-4}{\sqrt{x}-2}$ is a ratio of two functions, where $x - 4$ is continuous for all x and $\sqrt{x} - 4$ is continuous for all $x \ge 0$. Therefore their ratio is continuous for all $x \ge 0$ which do not make denominator equal to zero, i.e. for all $x \ge 0$ and $x \ne 4$. This implies that $f(x)$ is continuous for all $x > 4$ (again, for any choice of a). Therefore we need to find a that makes $f(x)$ continuous at 4. In order to do this, we equate two one-sided limits:

$$(*) \qquad \lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$$

If $x < 4$ then $f(x) = x^2 + ax$ and therefore

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (x^2 + ax) = 4^2 + 4a = 16 + 4a$$

(we can use direct substitution since the function is a polynomial).

If $x > 4$ then $f(x) = \dfrac{x-4}{\sqrt{x}-2}$ and therefore

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4^+} \frac{(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} =$$

$$= \lim_{x\to 4^+} \frac{(x-4)(\sqrt{x}+2)}{x-4} = \lim_{x\to 4^+} (\sqrt{x}+2) = \sqrt{4}+2 = 4$$

Substituting these results in $(*)$ we obtain equation for a:

$$16 + 4a = 4$$

Solving for a, we find $a = -3$.

Answer: -3

8. Prove that there is at least one solution of the equation
$\sin(x+1) = x$.

Solution Note that $\sin(x+1) = x$ is equivalent to $\sin(x+1) - x = 0$. Consider $f(x) = \sin(x+1) - x$. Note that $f(x)$ is continuous on $(-\infty, \infty)$. Further, $f(0) = \sin(0+1) - 0 = \sin 1 > 0$ (since $0 < 1 < \pi$, and $\sin t$ is positive in the interval $(0, \pi)$) and $f(2) = \sin(2+1) - 2 = \sin 3 - 2 < 0$ (since $\sin t \le 1$ for any t). Thus $f(0) > 0$ and $f(2) < 0$. The Intermediate Value Theorem implies that $f(x)$ has a root between 0 and 2, as required.

9. Find the x-coordinates of points $x > -1$ at which the tangent line to the graph of function
$y = (x^2-1)^5\sqrt{x+1}$ is horizontal.

Solution Note that the tangent line is horizontal if and only if its slope is 0. Further, the slope of tangent line equals to the derivative. Thus, we have to find all points where $y' = 0$. Compute y':

$$y' = [(x^2-1)^5\sqrt{x+1}]' = [(x^2-1)^5(x+1)^{1/2}]' = \text{(the Product Rule)}$$

$$= [(x^2-1)^5]' \cdot (x+1)^{1/2} + (x^2-1)^5 \cdot [(x+1)^{1/2}]' =$$

(combain the Power Rule and the Chain Rule)

$$= 5(x^2-1)^4 \cdot [(x^2-1)]' \cdot (x+1)^{1/2} + (x^2-1)^5 \cdot \frac{1}{2}(x+1)^{1/2-1} \cdot [x+1]' =$$

$$= 5(x^2-1)^4 \cdot 2x \cdot (x+1)^{1/2} + (x^2-1)^5 \cdot \frac{1}{2}(x+1)^{1/2-1} \cdot 1 = 10x(x^2-1)^4 \cdot (x+1)^{1/2} + (x^2-1)^5 \cdot \frac{1}{2}(x+1)^{-1/2}$$

$$= (x^2-1)^4 \cdot (x+1)^{-1/2} \cdot [10x \cdot (x+1)^{1/2+1/2} + \frac{1}{2}(x^2-1)] = (x^2-1)^4 \cdot (x+1)^{-1/2} \cdot [10x \cdot (x+1) + \frac{1}{2}(x^2-1)]$$

$$= (x^2-1)^4 \cdot (x+1)^{-1/2} \cdot [10x^2 + 10x + \frac{x^2}{2} - \frac{1}{2}] = \frac{1}{2}(x^2-1)^4 \cdot (x+1)^{-1/2} \cdot [20x^2 + 20x + x^2 - 1] =$$

$$= \frac{1}{2}(x^2-1)^4 \cdot (x+1)^{-1/2} \cdot [21x^2+20x-1] = \frac{(x^2-1)^4}{2\sqrt{x+1}} \cdot [21x^2+20x-1]$$

Thus

$$y' = \frac{(x^2-1)^4(21x^2+20x-1)}{2\sqrt{x+1}}$$

and therefore $y' = 0$ is equivalent to

$$\frac{(x^2-1)^4(21x^2+20x-1)}{2\sqrt{x+1}} = 0$$

The last equation is equivalent to $(x^2-1)^4(21x^2+20x-1) = 0$ AND $2\sqrt{x+1} \neq 0$ AND $2\sqrt{x+1}$ is defined. Further, $(x^2-1)^4(21x^2+20x-1) = 0$ implies $x^2-1 = 0$ or $21x^2+20x-1 = 0$. Solving these equations we obtain $x = 1$, $x = -1$ (from first equation) and $x = -1$, $x = 1/21$ (from the second equation). Thus $y' = 0$ and therefore the tangent line is horizontal if $x = 1$ or $x = 1/21$.

Note: as for $x = -1$, although it makes the denominator $\sqrt{x+1}$ equal to 0, it is still possible to compute the right derivative of the function using the definition of derivative, which is equal to 0. Thus $y' = 0$ at $x = -1$ as well.

10. Find equations of the tangent lines to the graph of the function $y = \dfrac{x-1}{x+1}$ that are perpendicular to the line $x + 2y = 1$.

Solution Note that $x + 2y = 1$ is equivalent to $y = -\dfrac{1}{2}x + \dfrac{1}{2}$. Therefore the slope of the line is $-1/2$ and hence any line, which is perpendicular to this given line, has slope 2 (recall that two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 \cdot m_2 = -1$). Thus we need to find equations of tangent lines which have slope 2. Since the slope of tangent line equals to the derivative, we need to find points at which $y' = 2$. To find y' we use the Quotient Rule:

$$y' = \left[\frac{x-1}{x+1}\right]' = \frac{(x-1)' \cdot (x+1) - (x-1) \cdot (x+1)'}{(x+1)^2} = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

Thus $y' = 2$ is equivalent to $\dfrac{2}{(x+1)^2} = 2$. This implies that $(x+1)^2 = 1$ and therefore $x+1 = 1$ or $x+1 = -1$. Hence $x = 0$ or $x = -2$. Note that $y = -1$ when $x = 0$, and $y = 3$ when $x = -2$. Therefore we obtain two lines: $y = -1 + 2x$ and $y = 3 + 2(x-(-2)) = 3 + 2(x+2) = 2x + 7$.

Answer: $y = 2x - 1$ and $y = 2x + 7$

11. Sketch the curve $y = x^4 - x^2$.

Solution

Let $f(x) = x^4 - x^2$.

- Domain: $(-\infty, \infty)$ (the function is a polynomial).
- Intercepts: $y = x^4 - x^2 = x^2(x^2-1) = x^2(x-1)(x+1)$ and therefore $x = 0, 1, -1$.

- Symmetry: $f(-x) = (-x)^4 - (-x)^2 = x^4 - x^2$ and hence the function is even. Therefore the graph is symmetric with respect to the y-axis.

- Limits at infinity: $\lim_{x\to\infty}(x^4 - x^2) = \lim_{x\to\infty} x^2(x^2-1) = \infty$ since both x^2 and $x^2 - 1$ approach ∞ as x approaches ∞. By symmetry, $\lim_{x\to-\infty}(x^4 - x^2) = \infty$.

- Asymptotes: no vertical asymptotes (function is continuous everywhere), no horizontal asymptotes (see the previous item).

- Intervals of increase and decrease, local maximums and minimums:

  $y' = 4x^3 - 2x = 4x(x^2 - 1/2) = 4x(x - 1/\sqrt{2})(x + 1/\sqrt{2})$ and critical numbers found by letting $y' = 0$ are $x = 0$ and $x = \pm 1/\sqrt{2}$. We use the following chart to determine the sing of y'.

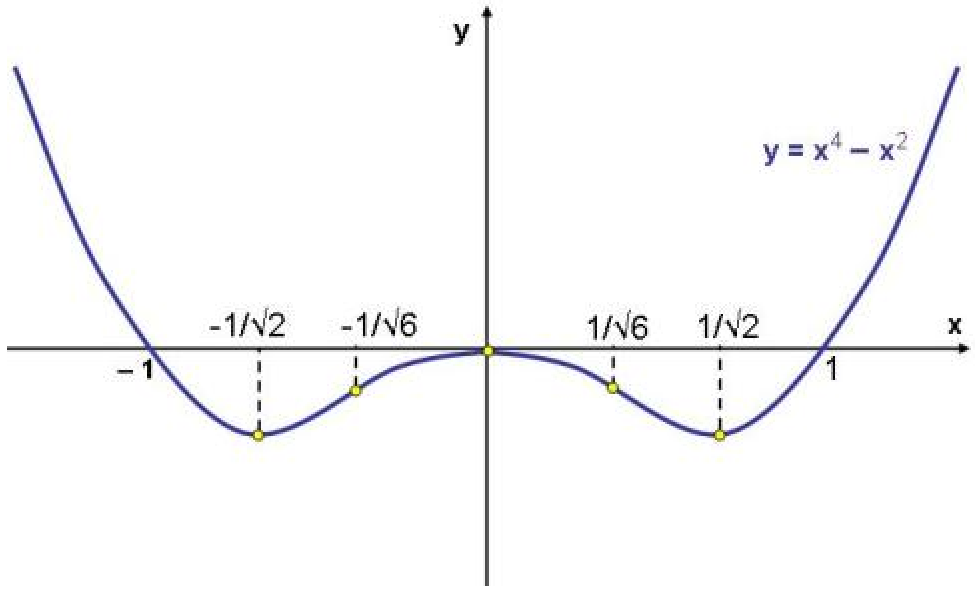
intervals	$(-\infty, -1/\sqrt{2})$	$(-1/\sqrt{2}, 0)$	$(0, 1/\sqrt{2})$	$(1/\sqrt{2}, \infty)$
test points	-1	$-1/2$	$1/2$	1
$y' = 4x^3 - 2x$	$-$	$+$	$-$	$+$
y	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$

  The function decreases on $(-\infty, -1/\sqrt{2})$ and $(0, 1/\sqrt{2})$ and increases on $(-1/\sqrt{2}, 0)$ and $(1/\sqrt{2}, \infty)$. Therefore 0 is a point of local maximum and $\pm 1/\sqrt{2}$ are points of local (and absolute) minimum.

- Concavity and inflection points:

  $y'' = 12x^2 - 2 = 2(6x^2 - 1) = 0$ if and only if $x = \pm 1/\sqrt{6}$. Note that $y'' > 0$ on $(-\infty, -1/\sqrt{6})$ and $(1/\sqrt{6}, \infty)$, and $y'' < 0$ on $(-1/\sqrt{6}, 1/\sqrt{6})$. Therefore the graph is concave upward on $(-\infty, -1/\sqrt{6})$ and $(1/\sqrt{6}, \infty)$ and concave downward on $(-1/\sqrt{6}, 1/\sqrt{6})$, and hence points $(\pm 1/\sqrt{6}, (1/\sqrt{6})^4 - (1/\sqrt{6})^2) = (\pm 1/\sqrt{6}, -5/36)$ are inflection points.

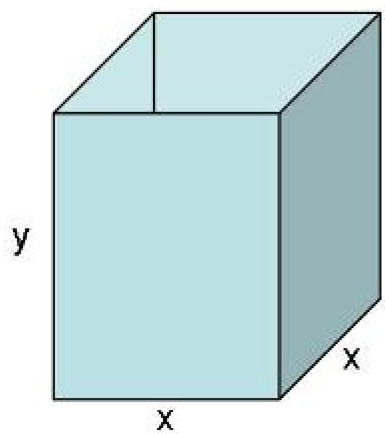
- Summarizing the above information, we sketch the graph:



12. If $1200\,\text{cm}^2$ of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Solution

Let x be the length of the side of the base and y be the height of the box (see the picture).



Then the total area of the surface is $x^2 + 4xy$ and the volume of the box is x^2y. Thus, we want to maximize $V = x^2y$ given that $x^2 + 4xy = 1200$. Solving the last equation for y, we get $y = \dfrac{1200 - x^2}{4x}$. Substituting this expression in the formula for V, we obtain:

$$V(x) = x^2\left(\frac{1200 - x^2}{4x}\right) = \frac{1}{4}x(1200 - x^2) = \frac{1}{4}(1200x - x^3)$$

Thus we need to find the absolute maximum of the function $V(x) = \frac{1}{4}(1200x - x^3)$ on the interval $[0, \sqrt{1200}] = [0, 20\sqrt{3}]$ (indeed $x \geq 0$ as a length and $x^2 \leq 1200$ since 1200 is the total surface area). Find critical numbers:

$$V'(x) = \frac{1}{4}(1200 - 3x^2) = 0 \Leftrightarrow 1200 - 3x^2 = 0 \Leftrightarrow x^2 = 400 \Leftrightarrow x = \pm 20$$

Only $x = 20$ belongs to $[0, 20\sqrt{3}]$. It is easy to verify that $V(x)$ increases on $[0, 20]$ and decreases on $[20, 20\sqrt{3}]$ and hence $x = 20$ is the point of (absolute) maximum. The corresponding value of y is $y = \dfrac{1200 - 20^2}{4 \cdot 20} = 10$. Thus the largest possible volume of the box is $V = 20^2 \cdot 10 = 4000$ cm^3.

Answer: 4000 cm^3

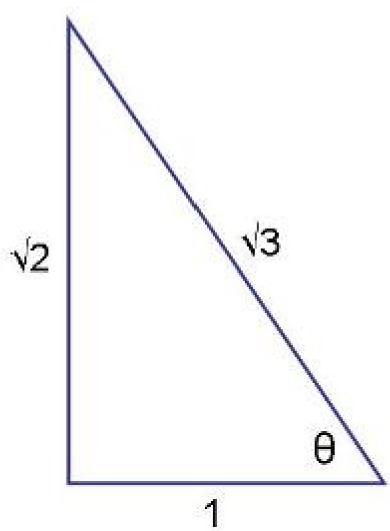
13. Find the exact value of the expression $\sin(2\tan^{-1}\sqrt{2})$. Do not use calculator!
Solution Let $\theta = \tan^{-1}\sqrt{2}$. Then, by the definition of $\tan^{-1}$, we have $-\pi/2 < \theta < \pi/2$, and

$$\tan\theta = \tan(\tan^{-1}\sqrt{2}) = \sqrt{2}.$$

We need to find $\sin(2\tan^{-1}\sqrt{2}) = \sin 2\theta$. Note that the double-angle formula gives us

$$\sin 2\theta = 2\,\sin\theta\,\cos\theta$$

and thus we need to find $\sin\theta$ and $\cos\theta$, given that $\tan\theta = \sqrt{2}$ and $-\pi/2 < \theta < \pi/2$. Since $\tan\theta > 0$ we may also assume that $0 < \theta < \pi/2$. Therefore θ can be viewed as an angle in a right triangle. Since $\tan\theta = \sqrt{2} = \dfrac{\sqrt{2}}{1}$, we may consider a triangle with sides 1 and $\sqrt{2}$. The hypothenuse of the triangle, by Pythagoras theorem, is $\sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$ (see the picture).



Thus we see that $\sin\theta = \sqrt{2}/\sqrt{3}$ and $\cos\theta = 1\sqrt{3}$, and therefore

$$\sin(2\tan^{-1}\sqrt{2}) = \sin 2\theta = 2\,\sin\theta\,\cos\theta = 2 \cdot \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{2\sqrt{2}}{3}$$

Answer: $\frac{2\sqrt{2}}{3}$

14. Find the limit $\lim_{x\to 0} \frac{\sin x - x}{x^3}$.

Solution Note that

$$\lim_{x\to 0}(\sin x - x) = \lim_{x\to 0} x^3 = 0$$

Therefore we have indeterminate form of type $\frac{0}{0}$. Using L'Hospital's Rule, we obtain:

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{(\sin x - x)'}{(x^3)'} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2}$$

It is not hard to see that still we have $\frac{0}{0}$, so we use L'Hospital's Rule again:

$$\lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim_{x\to 0} \frac{(\cos x - 1)'}{(3x^2)'} = \lim_{x\to 0} \frac{-\sin x}{6x}$$

...and again...

$$\lim_{x\to 0} \frac{-\sin x}{6x} = \lim_{x\to 0} \frac{(-\sin x)'}{(6x)'} = \lim_{x\to 0} \frac{-\cos x}{6} = \frac{-\cos 0}{6} = -1/6$$

Answer: $-1/6$

The End