Multivariable Calculus

Chapter 12

Multivariable Calculus

- Rather than having single variable equations like

  $$y = f(x) \qquad p = g(q) \qquad \text{etc.}$$

  we could have functions that depend on several variables

  $$z = h(x, y)$$

$$z = 2x^2 + 3y + 4$$

$$z = 8x^2 + 2x + 4y^2 - 13y + 3$$

Multivariable Calculus

- Given a function of several variables, we can still apply all of our previous calculus rules
- When finding the derivatives, we will find the "partial" derivative, i.e. the rate of change with respect to one variable
- In doing so, we will treat the other variables as constants

Multivariable Differentiation

- Ex. $$z = 2x^2 + 3y + 4$$

Notice how we use a different "d" to denote partial derivative

$$\frac{\partial z}{\partial x} = 4x \qquad \frac{\partial z}{\partial y} = 3$$

When finding the partial derivative of z with respect to x, we only differentiate the terms that have an x in them

When finding the partial derivative of z with respect to y, we only differentiate the terms that have a y in them

Multivariable Differentiation

- If it is instead written $f(x,y)=2x^2+3y+4$ then the partial derivatives are written as

$$f'_x(x,y)=4x$$

First Partial Derivative with respect to x

$$f'_y(x,y)=3$$

First Partial Derivative with respect to y

Ex.

$$z = 8x^2 + 2x + 4y^2 - 13y + 3$$

Ex.

$$f(x, y) = 3x^2 + 2xy + y^2$$

Ex.

$$f(x,y)=xy\left(x^2+y^2\right)^{-2}$$

Ex.

$$f(x, y) = e^x + ye^x$$

Ex.

$$f(x, y, z) = 4x^2 + 2xy + xyz^2 - \ln y + 3e^{2z}$$

Problems

- Page 866 (846/885) # 4 (4/2), 8 (8/6), 14 (14/12), 17 (17/15)
- (note: corresponding page numbers and problems from the 5th / 4th edition appear in brackets)

Second-Order Partial Derivatives

- Once we have differentiated with respect to one variable, we can take that derivative and differentiate it again with respect to the same variable or other variables
- The First Derivative still provides us with the slope of the function and the Second Derivative still tells us how that slope is changing; however, we are now simply working in greater dimensions

Second-Order Partial Derivatives

f''_{xx} First differentiate with respect to x, then differentiate with respect to x

f''_{xy} First differentiate with respect to x, then differentiate with respect to y

f''_{yx} First differentiate with respect to y, then differentiate with respect to x

f''_{yy} First differentiate with respect to y, then differentiate with respect to y

Ex.

Find all Second Order Partial Derivatives

$$f(x, y) = 8x^2 + 2x + 4y^2 - 13y + 3$$

Ex.

Find all Second Order Partial Derivatives

$$f(x, y) = 3x^2 + 2xy + y^2$$

f_{xy}'' vs. f_{yx}''

- Notice that in both of the previous examples $$f_{xy}'' = f_{yx}''$$
- It is always the case that $f_{xy}'' = f_{yx}''$ so it doesn't matter which variable you use to differentiate first

Problems

- Page 866 (846/885) # 35 (35/33), 38 (38/36)

- (note: corresponding page numbers and problems from the 5th / 4th edition appear in brackets)

Maxima and Minima of Multivariable Functions

- Just like with single-variable functions, multivariable functions may have relative minima and/or maxima
- For any multivariable function $f(x,y)$ we can find relative extrema using the following rules:

Maxima and Minima of Multivariable Functions

1. Find the critical points of the function using

$$f'_x = 0$$

$$f'_y = 0$$

i.e. BOTH partial derivatives must show a minimum/maximum

Note: You may need to use some algebra and substitution to arrive at a final answer

Maxima and Minima of Multivariable Functions

2. Use the Second Derivative Test

$$D(x,y) = f''_{xx} f''_{yy} - \left(f''_{xy}\right)^2$$

If D > 0 AND $f''_{xx} < 0$ then $f(x,y)$ is a relative maximum

If D > 0 AND $f''_{xx} > 0$ then $f(x,y)$ is a relative minimum

If D < 0 then $f(x,y)$ is neither a maximum nor a minimum

If D = 0 then the test is inconclusive

Ex.

Find the relative extrema of the function

$$f\left(x,y\right)=8x^{2}+2x+4y^{2}-13y+3$$

Ex.

The demand functions for two products are given by

$$p_x = 12 - 2x \quad \text{and} \quad p_y = 20 - y$$

The joint cost function is

$$C(x, y) = x^2 + 2xy + 2y^2$$

Prices are in dollars and Quantities are in thousands of units. Determine the prices and quantities that will maximize profit.

Ex.

The Labour Cost of a firm is given by the function

$$C(x, y) = x^2 + y^3 - 6xy + 3x + 6y - 5$$

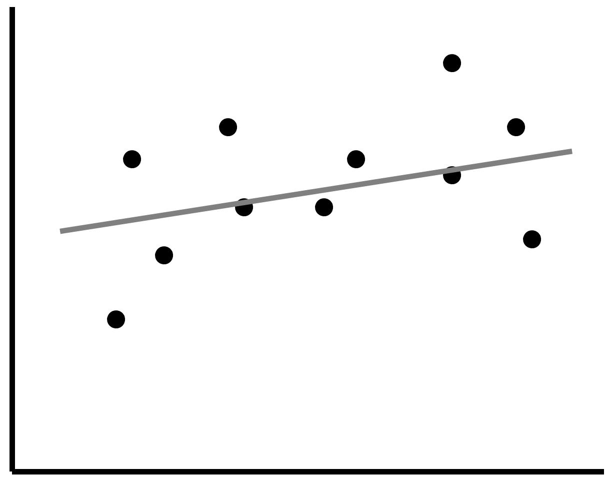
where x is the labour hours for skilled workers and y is the labour hours for semi-skilled workers. Find the cost minimizing combination of labour.

Problems

- Page 877 (857/897) # 9, 22

- (note: corresponding page numbers and problems from the 5^{th} / 4^{th} edition appear in brackets)

The Method of Least Squares



- Given a Scatter Plot of data, we can determine a line of best fit
- To do so, we want to minimize the distance between our points and the line; however, since any negative deviation would cancel a positive one, we have a problem

Developing a Line of Best Fit

- The solution lies in squaring the distance (or errors) from the line (that way all errors are positive and bigger deviations have more weight)
- We will develop a line such that

$$\hat{y} = mx + b \quad \text{given} \quad \min \sum_{i=1}^{n} \left(y_i - \hat{y}_i \right)^2$$

slope (m) intercept (b)

Fitted (or predicted) Value ($\hat{y}$)

Difference between each point and the line ($y_i - \hat{y}_i$)

Developing a Line of Best Fit

- How do we do it?
- There are two methods

1. Use Calculus (since this is a minimization problem) (but it is really hard)
2. Use Matrix Algebra

Reenter the Matrix – Line of Best Fit

- We can write all of our information into Matrix format

Let $X = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}$ $\beta = \begin{bmatrix} m \\ b \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Reenter the Matrix – Line of Best Fit

- Using Matrix Algebra we obtain

$$X\beta = Y$$

$$X^{T}X\beta = X^{T}Y$$

so that m and b must satisfy the equations

$$(x_1 + x_2 + \ldots + x_n)m + nb = (y_1 + y_2 + \ldots + y_n)$$

and

$$(x_1^2 + x_2^2 + \ldots + x_n^2)m + (x_1 + x_2 + \ldots + x_n)b = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

Line of Best Fit

- These are sometimes called the "normal" equations and can also be written as

1. $$\sum_{i=1}^{n} x_i m + nb = \sum_{i=1}^{n} y_i$$

2. $$\sum_{i=1}^{n} x_i^2 m + \sum_{i=1}^{n} xb = \sum_{i=1}^{n} xy$$

Ex.

- Not surprisingly, a student's homework grade tends to be an indicator of how well a student does on the term test. Using the sample of 5 students below, find the line of best fit. What would we predict a student with a homework grade of 75 would get on the test? What about a homework grade of 90?

	1	2	3	4	5
Homework Grade	45	85	65	99	87
Test Grade	55	83	73	100	85

Ex. Advertising vs. Sales

- A business would like to determine the relationship between the amount of money spent on advertising and its total weekly sales. Over a period of 5 weeks it gathered the following data. What will sales be if advertising is $15,000?

	1	2	3	4	5
Advertising ($ 000s)	10	17	11	18	21
Sales Volume ($ 000s)	50	61	55	60	70

Constrained Optimization

- In previous weeks (and earlier in today's lecture) we found unconstrained minima and maxima by differentiation
- However, in the real world, we may be faced with problems that have certain limits
  - i.e. maximize a consumer's utility/satisfaction but stay within his or her budget
  - maximize output while keeping cost within a certain level

Constrained Optimization

- These limitations on our activity are called "constraints"
- Whenever we have constraints, we will need to build them into our function

The Method of Lagrange Multipliers

- To find the relative extrema of the function $f(x,y)$ subject to the constraint $g(x,y) = 0$

1. Form a new function

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

called the Lagrangian (L) where λ is the Lagrange Multiplier

2. Solve $L'_x = 0$ $L'_y = 0$ $L'_\lambda = 0$
3. The solutions found in Step 2 will be candidates; however, if there is more than one, we will have to evaluate them individually. There is no Second-Derivative test for Lagrange functions

Ex.

$\max z = xy$ subject to $x + y - 16 = 0$

Ex.

A manufacturer produces two types of engine, x units of Type 1 and y units of Type 2. The joint profit function is

$$\pi = x^2 + 3xy - 6y$$

How many units of each should they produce if they only have enough parts to build 42 engines in total?

Ex.

A consumer has a utility function

$$u(x, y, z) = 2\ln x + \ln y + 3\ln z$$

If the consumer's income is \$450, the price of x is \$10, the price of y is \$3, and the price of z is \$5; how much of each good should the consumer buy?

Problems

- Supplement Page 576 # 10, 15
- Supplement Page 590 # 7, 24