DANIEL JARVIS

Mathematics and Visual Arts: *Exploring the Golden Ratio*

Geometry has two great treasures: one is the Theorem of Pythagoras: the other the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel.
—*Johannes Kepler (1571–1630)*

MATHEMATICS AND VISUAL ARTS HAVE long shared aspects of both form and function. One such ancient connection is found in the golden ratio. This article introduces the reader to this unique mathematical phenomenon in three separate contexts: (1) historical, (2) mathematical, and (3) pedagogical. A brief history of this fascinating number, various teaching strategies, and a project for middle school mathematics students will be explored. A list of resources is given for further study.

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Historical Context

THE GREEKS WERE THE FIRST TO ACCURATELY define the curious phenomenon known as an "extreme and mean ratio." Although Plato referred to this mathematical relationship as simply the "section," it was not until the nineteenth century that this famous proportion came to be known as the *golden section*. Read Walser's (2001) description of this unusual ratio:

> The Golden Section has turned up, since antiquity, in many aspects of geometry, architecture, music, art, and even philosophy; but it appears also in the newer domains of engineering and fractals. In this way the Golden Section is no isolated phenomenon but, in many cases, the first and, indeed, simplest example in the context of a sequence of generalizations of a common idea. (p. vii)

The Parthenon, supervised by the sculptor Phidias (Curlee 2004, p. 8), was built atop the Acropolis in Athens and is an example of, and long-lasting monument to, the ideals of the golden proportion (Newman and

$$\frac{AB}{BC} = \frac{BC}{AC}$$

Fig. 1 Extreme and mean ratio

Quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{for } ax^2 + bx + c = 0$$

$$\frac{1}{x} = \frac{x}{x+1}$$

$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$x \approx 1.61803...$$

Fig. 2 The algebraic derivation of the golden number phi (ϕ)

Boles 1992, pp. 64–69; Runion 1990, pp. 83–89). Frayling, Frayling, and Van Der Meer (1992) described some of the features within the Parthenon:

> Analysis of the proportions of the building has revealed many hidden Golden Section rectangles. The ground plan consists of two reciprocal golden rectangles, and the interior chamber is in golden proportion; the main façade fits into a single golden rectangle, and a comparison of the façade with the ground plan reveals the basic harmonic relationships we associate with golden proportion. All of these connections add a sense of order and continuity to the sequence of spaces and forms. (p. 11)

It was not unusual for the dimensions of Greek buildings and sculptures to be related proportionally to measurements of human beings. Vitruvius (AD 100), a Roman architect, concluded after much study that the "height of a well-proportioned man is the same as the span of his outstretched arms" (Frayling, Frayling, and Van Der Meer 1992, p. 11). To illustrate this relational system of measurement, he showed a man standing with outstretched arms and feet within a circle, the center of which is marked by his navel. Fourteen centuries later, Leonardo da Vinci, familiar with the work of Vitruvius, would create a similar drawing titled *The Proportions of the Human Figure*. Da Vinci also used the golden section to organize his two-dimensional compositions, believing that harmonic proportions could add profound depth and beauty to a work of art. The *Mona Lisa* is claimed to be an example of the use of this design philosophy.

The celebration and use of this mathematical phenomenon dates back to ancient Greece and extends to the present day. Examples of the golden section ratio can be seen throughout nature (e.g., in shells, plants, animals), art (e.g., in drawings and paintings by Seurat, da Vinci, Dali, Colville), and architecture (e.g., in the Parthenon of Phidias; in work by architect Le Corbusier). It is one of the strongest and oldest ties between the disciplines of mathematics and the visual arts. It can be said that this historical and universal pervasiveness qualifies the golden ratio to be part of the mathematics education curriculum.

Mathematical Context

IN MATHEMATICS, A RATIO IS DEFINED AS A COMparison of two numbers. A proportion is a comparison of two ratios. One of the most famous mathematical ratios and proportions is the golden section, or the divine proportion. It is derived from dividing a line segment into two segments, with the special property that the ratio of the shorter segment to the longer segment is the same as the ratio of the longer segment to the whole segment (see **fig. 1**). This division is known as "extreme and mean ratio." This ratio became designated by the Greek letter ϕ, or phi, to honor Phidias, architect of the Parthenon.

Given a line that has been divided into extreme and mean ratio, one can set up the mathematical proportion "small is to large as large is to whole" and then solve for x using the quadratic formula (see **fig. 2**). (Although the quadradic formula may not yet be familiar to middle school mathematics students, it can still be demonstrated in a cursory manner to clarify the origins of this famous ratio.) Therefore, an approximation for ϕ (phi) can be taken as 1.61803, to five decimal places. Practically speaking, this means that any given length of line can be multiplied or divided by ϕ (1.61803) to find an external (next largest) or an internal (next smallest) golden section length, respectively. As students begin to work with this ratio in constructing the golden section and the related golden rectangle, encourage them to carry at least five decimals in their calculations and to measure as precisely as possible. (The golden rectangle is formed with the special property that its length divided by its width equals ϕ, or approximately 1.61803. Interior "whirling squares" can be shown to fill the rectangle in a continual spiral formation that maintains its proportions.)

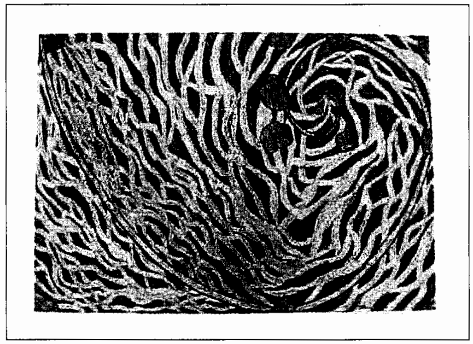
The golden section and the golden rectangle can also be constructed using a compass and pencil, as the ancient Greek mathematicians would have done. Instructions for these geometric constructions can be found within the resources section at the end of this article.

Pedagogical Context

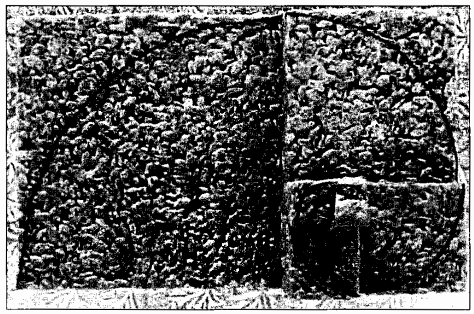
TO THE TEACHER WHO WISHES TO TRANSLATE this mathematical phenomenon into a challenging assignment, I offer the following ideas:

- Introduce the concept of the golden ratio with slides of famous art and architecture and former student work; provide a historical overview handout.
- Teach a short lesson on finding the golden section of a line (i.e., dividing any given length by ϕ, which is approximately 1.61803, to find the internal section, and multiplying any given length by ϕ to find the external section); extend this knowledge to constructing the corresponding golden section rectangle.
- Have students analyze and record findings in a chart for a variety of objects in the classroom or at home to determine if they are golden (i.e., measure the dimensions and divide each length by each width to see if the answer is close to 1.61803).
- Have students construct a golden section rectangle as either a series of diminishing ("whirling") squares to be decorated or as a design element (i.e., blueprint) for a sketch or painting. Encourage students to (a) use a light pencil to construct the original template and (b) measure very carefully, always keeping four or five decimal places on the calculator; small errors in measurement can quickly lead to an inaccurate internal rectangle. Errors can be fixed by adjusting the golden section markings and redrawing the rectangle lines until squares repeat in the proper pattern.
- Some students may wish to further investigate constructions such as the golden triangle, the golden spiral, the golden ellipse, the golden cuboid, the pentagram, the lute of Pythagoras, the Fibonacci number sequence, polyhedra, root rectangles, fractals, dynamic symmetry, and so on.
- Have students communicate their final works to their peers and participate in a classroom, school, or Web site exhibition of their projects.

The **Student Activity Sheets** explore a mathematics project that was completed by students in Ontario, Canada, as part of a research study (Jarvis 2001). These extended projects involved elements of mathematical research (definitions), problem solving and inquiry (both at school and at home),



Golden Spaghetti



Rice Krispies

Golden Cuboid Mobile

Fig. 3 Students' golden SMARTWORK projects

creative applications of mathematical concepts (golden section ratios incorporated into their projects), and mathematical communication (both written and verbal). Students had approximately four weeks to complete this assignment, both in and out of class, and were able to seek assistance from peers, the instructor, or others at home. The project required students to use the golden section ratio accurately within a creative piece known as SMARTWORK, which stands for "syncretized math and artwork." Students completed three SMARTWORK projects throughout their mathematics course. Selected samples of student projects featuring golden rectangle design elements are shown in **figure 3**.

Conclusion

THE GOLDEN SECTION FORMS A POWERFUL CONnection between what has been traditionally perceived as the polar disciplines of mathematics and visual art. In a broader sense, it is an intriguing mathematical phenomenon that has surfaced in art, architecture, music, science, and philosophy. This "precious jewel" of Kepler's mathematical universe is an exciting springboard for interdisciplinary learning. In the words of Albert Einstein, "It is the supreme art of the teacher to awaken joy in creative expression and knowledge." These two essential elements, *creative expression* and *knowledge*, can be easily awakened through both the *mystery* and *mastery* of the golden ratio. Teaching mathematics through the arts in the middle school offers the mathematics educator a rich and meaningful approach to classroom learning and communication.

Solutions to Student Activity Sheet 1

1. A comparison of two quantities (numbers) with the same units.
2. An equation that states that two ratios are equal.
3. The division of a segment into extreme and mean ratio; the name given to a proportion, which has often been thought to possess some aesthetic virtue in itself, some hidden harmonic proportion; a line divided in such a way that the smaller part is to the larger, as the larger is to the whole.
4. The number phi is approximately 1.61803 and is found by dividing a line segment into two segments so that the ratio of the shorter segment to the longer segment is the same as the ratio of the longer segment to the whole segment.
5. A classical Greek sculptor who used the golden ratio extensively in his work; the symbol ϕ was chosen in his honor; created frieze panels on Parthenon (525 BC) and the huge statue of Athena that was housed inside the temple.
6. A rectangle in which the lengths of the length and the width are in the golden ratio.
7. An isosceles triangle in which the lengths of the longer sides and the smaller side are the golden ratio.
8. The sequence of numbers in which each term is created by adding together the two preceding terms (e.g., 1, 1, 2, 3, 5, 8, 13, 21, . . .); the ratios of consecutive pairs converge to ϕ.
9. Ancient Greek temple, located at Athens on the Acropolis, which was constructed using the golden section proportions.
10. A polygon with five equal sides, internal angles of 72 degrees, and which encompasses numerous golden triangles when internal lines are drawn connecting the five vertices (forms pentagram).
11. $110/75 \approx 1.466 \neq \phi$; therefore, not golden; $110/1.61803 \approx 67.98$; use 110×68.0 inches.
12. $96/1.61803 \approx 59.33$; golden dimensions would be 96×59.33 inches; perimeter of stretcher is $2(96 + 59.33) = 310.66$ inches.
13. Answers will vary.

Solutions to Student Activity Sheet 2

1. Segment AB is 5.25 inches; $5.25/1.61803 \approx 3.24$ inches; the golden section (GS) label or mark should occur at 3.24 inches from point A.
2. Side BC is 2 3/16 inches, or 2.1875 inches; $2.1875 \times 1.61803 \approx 3.5394$ inches; the side of the rectangle should be 3.5 inches; the golden rectangle can also be constructed using compass and midpoint.

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Golden Ratio Resources

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(*Worksheets follow*)

Student Activity Sheet 1

NAME ______________________________

Mathematics Knowledge and Research

Define the following terms.

1. Ratio

2. Proportion

3. The golden ratio (also known as the golden section, the golden mean, the golden cut, the divine proportion, and the golden proportion)

4. ϕ (Greek letter and numerical approximation)

5. Phidias

6. The golden rectangle

7. The golden triangle

8. The Fibonacci sequence

9. The Parthenon

10. A regular pentagon

Solve or complete the following questions.

11. *The Golden Proportion:* Is a rectangle with dimensions 110 inches × 75 inches a golden rectangle (show your calculations)? If not, determine what width would make it one.

12. *The Artist's Stretcher:* An artist wants to design a canvas that is in the shape of the golden section rectangle for her next large oil painting. If the maximum length possible for the stretcher is 96 inches, find the corresponding width (to two decimals). What will be the total perimeter of the stretcher?

13. *The Golden Hunt:* Analyze 7 common rectangular objects around the house to determine if they are "golden."

Object	Length (B)	Width (A)	Ratio (B/A)	Close to ϕ?
One dollar bill				

Student Activity Sheet 2

NAME ____________________

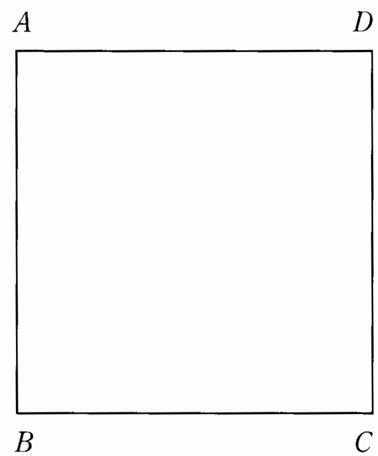
Mathematics Application of Concepts

Create the following items.

1. *A Golden Section:* On the given segment *AB*, label the golden section (GS) with a point.

A ____ *B*

2. *A Golden Rectangle:* From the given square *ABCD*, construct and label a golden section rectangle.



3. *A Golden SMARTWORK Project:* Construct a SMARTWORK project that features the golden ratio in some form. This project allows for great freedom of expression. Your piece may be two- or three-dimensional. It may be narrative (tells a story), literary (quotes or an illustrated poem), historical (perhaps a tie to mathematics history), abstract (no recognizable objects), a collage or a sculpture; your imagination is the limit. You may draw a golden rectangle or incorporate more complex ideas from the golden triangle, the golden spiral, the pentagram, the lute of Pythagoras, and/or the Fibonacci number sequence. The final SMARTWORK piece *must* include a measurable golden ratio somewhere in the composition and will also be assessed for creativity and complexity.

Use mathematical language to communicate in the following forms.

4. *Written Description of Your SMARTWORK:* Describe in detail your finished project. For example, you should explain exactly where the golden section(s) is located, the reason you chose this particular format, style, and/or color scheme, and so on. Try to use as much mathematical language as possible (i.e., ratio, length, width, proportion).

5. *Verbal Presentation:* In a brief presentation to the class (3–5 minutes), simply display and describe your SMARTWORK project. You may discuss a few points from the planning stages (show preliminary sketches/ideas) or from the process (changes made, new ideas).

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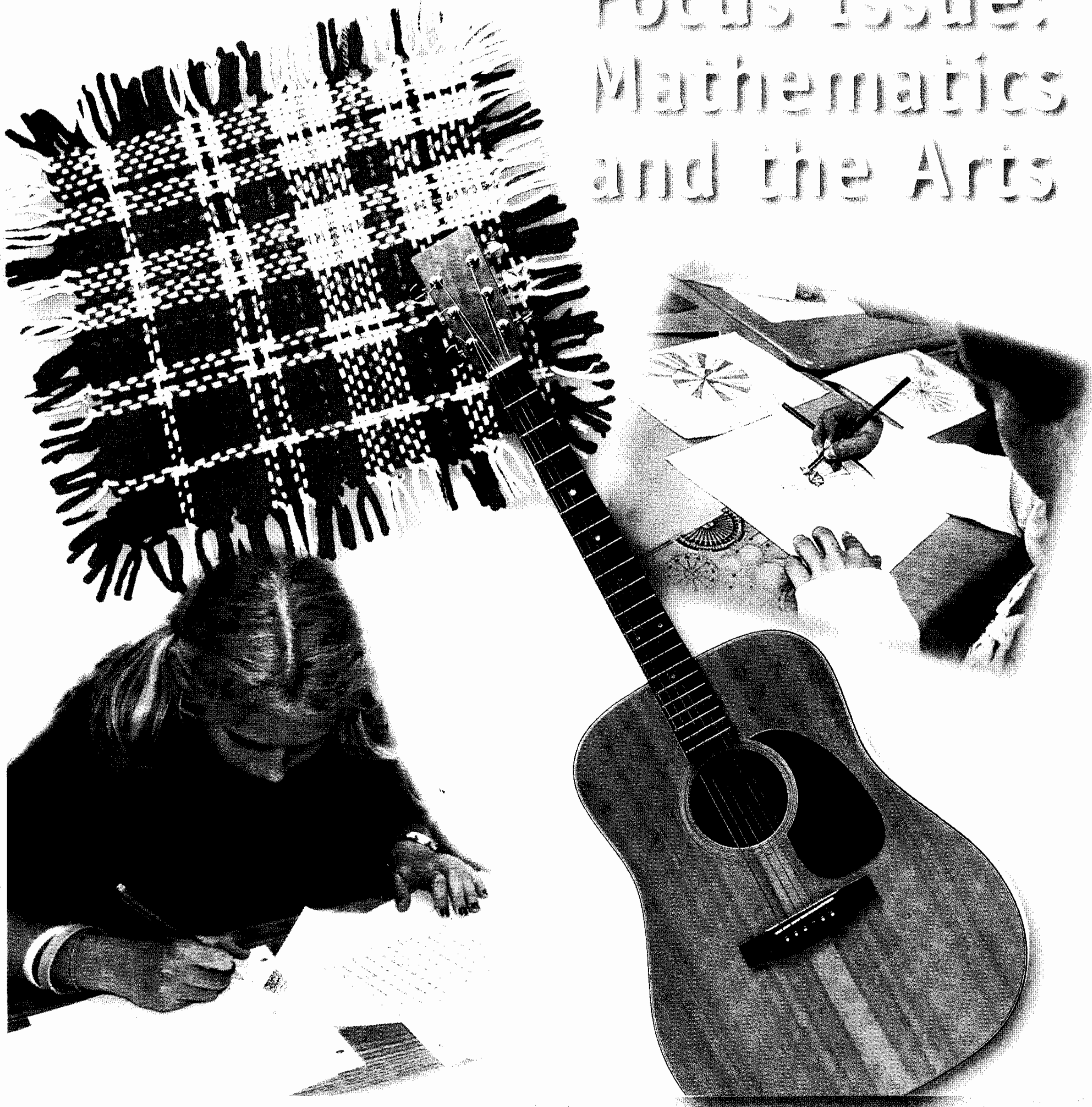

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