Computers vs. Proof: Polya's conjecture and Euler's conjecture

k	1	2	3	4	5
pf	0	1	1	2	1
k	6	7	8	9	10
pf	2	1	3	2	2
k	11	12	13	14	15
pf	1	3	1	2	2
k	16	17	18	19	20
pf	4	1	3	1	3

Number of prime factors for the first 20 positive integers

Many famous conjectures in mathematics can be verified using computers for very large groups of objects they deal with. Examples include Fermat's Last Theorem (proved by Andrew Wiles in 1993) and (still unproved) Goldbach's conjecture stating that every integer greater than 2 is a sum of two primes. There are other important conjectures that can be checked in many special cases. The natural question that arises is: do we still need to prove a conjecture if it can be verified (by computers or otherwise) for very large numbers or for many possible cases? Here are some (counter)examples that hopefully will convince you that the answer is "Yes".

Polya's conjecture

It is well-known that every integer can be factored into a product of prime numbers. For example, 12 = 2x2x3 and thus have 3 prime factors, while 225 = 3x3x5x5 have 4 prime factors. George Pólya conjectured in 1919 that most of the natural number have an odd number of prime factors. More precisely, for any natural number n, at least 50% of natural numbers greater than 1 and not exceeding n have and odd number of prime factors. The table for numbers not exceeding 20 is on the left. Pólya's conjecture can be verified for very large numbers. However in 1958 C. Brian Haselgrove proved that there exists a counterexample (the estimation at that time would exceed 10^{300}). A first explicit counterexample, n = 906,180,359 was given in 1960. It is shown that the smallest counterexample is n = 906,150,257.

Euler's sum of powers conjecture – a generalization of Fermat's Last Theorem

Recall that Fermat's Last Theorem states that the equation $a^n+b^n=c^n$ has no solutions for all n>2 except when at least one of a, b, c is zero. Leonhard Euler conjectured in 1769 that the equation $a^n+b^n+c^n= d^n$ has no non-zero integral solutions for all n>3, $a^n+b^n+c^n+ d^n = e^n$ has no non-zero integral solutions for all n>4, and so on. However, a first counterexample $27^5 + 84^5 + 110^5 + 133^5 = 144^5$ was discovered in 1966, and another one,
$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$,
was found in 1986.

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