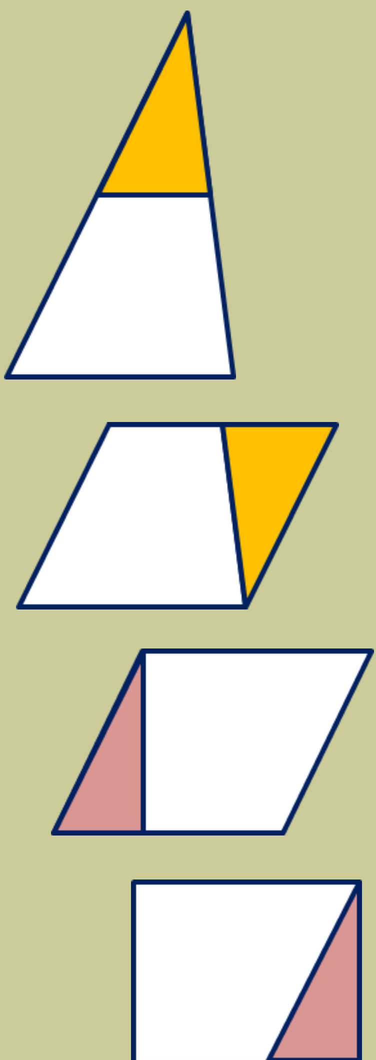
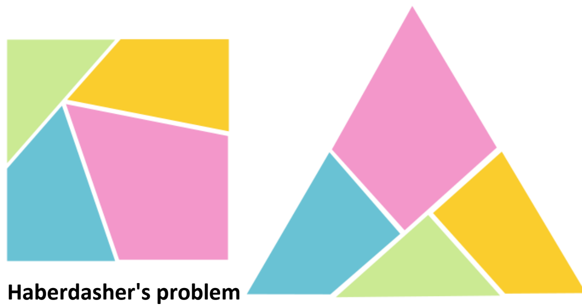
**Nipissing University - Mathematics**

# From cutting puzzles to Hilbert's Third Problem and beyond





Haberdasher's problem

Cutting (or dissection) puzzles, that require one to cut a shape into several pieces that satisfy certain conditions, have amused people for centuries. The most famous example is perhaps a proof of the Pythagoras Theorem using a dissection of a square. Another interesting example of dissection is the so-called haberdasher's problem, proposed in 1907 by Henry Dudeney: an equilateral triangle is cut into four pieces to make a square. It is not known if the number of pieces can be reduced to three.

Cut-and-paste techniques were also used to compute areas and volumes. For example, it is easy to find a formula for the area of a triangle by cutting a rectangle into triangular pieces. Similar formula for the volume of a pyramid has been known for ages, but all proofs relied on some type of approximation and limits techniques. Therefore, it is natural to ask whether it can be derived by cutting and pasting.

In the case of the plane, the Wallace–Bolyai–Gerwien theorem (proved around 1830) states that two polygons have the same area if and only if one of them can be cut into finitely many triangular pieces which can be reassembled to yield the second. The famous mathematician David Hilbert asked whether given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second. This problem is known as the Third problem on the famous list of Hilbert's mathematical problems, published in 1900. It was also the first to be solved. Hilbert's student, Max Dehn, introduced a special invariant $D$ (now called Dehn's invariant), with the property that if a polyhedron P is cut into two polyhedral pieces $Q$ and $R$ then $D(P) = D(Q) + D(R)$. He then showed that $D(\text{cube}) = 0$, while $D(\text{regular tetrahedron}) > 0$. Thus the answer to the Hilbert's question is "No". To develop his invariant, Dehn applied methods of algebra to a geometric problem. Such a synergy of algebraic, geometric, and topological methods flourished throughout the whole 20th century, and continues now, leading to wonderful results and new theories in mathematics.

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