Divisibility Properties of the Fibonacci Numbers

1, 1, 2, 3, **5**,
8, 13, 21, 34, **55**,
89, 144, 233, 377, **610**,
987, 1597, 2584, 4181, **6765**,
10946, 17711, 28657, 46368, **75025**,
121393, 196418, 317811, …

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21, 34, 55, 89, 144, 233, **377**, 610,
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75025, 121393, 196418, **317811**, …

$$F_{12} = 144 = 2\times 2\times 2\times 2\times 3\times 3$$

$$F_{18} = 2584 = 2\times 2\times 2\times 17\times 19$$

$\gcd(F_{12},F_{18})=F_{\gcd(12,18)}=F_6=8$

DEPT. OF COMP. SCIENCE &
MATHEMATICS
NIPISSING UNIVERSITY
100 COLLEGE DRIVE, BOX 5002,
NORTH BAY, ON, CANADA P1B 8L7
TEL: 705.474.3450
FAX: 705.474.1947
NUINFO@NIPISSINGU.CA

WWW.NIPISSINGU.CA/MATHEMATICS

Recall that Fibonacci numbers are the members of the sequence 1, 1, 2, 3, 5, 8, …, so that every next number is the sum of two previous numbers.

It is a part of the mathematical folklore that every 5th Fibonacci number is divisible by 5. It also happens that 5 is the 5th Fibonacci number.
In fact, a more general statement is true:
Every *n*-th Fibonacci number (i.e. number F_{nk} for some k) is divisible by F_n, the *n*-th Fibonacci number.
Among the different ways to prove that is to use the matrix

$$A=\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix},\quad A^n=\begin{pmatrix}F_{n-1} & F_n\\ F_n & F_{n+1}\end{pmatrix}.$$

Using the fact that $A^{m+n}=A^m\times A^n$ and comparing the (1,2) entries of the two matrices, we obtain a helpful representation of F_{m+n}:

$$F_{m+n}=F_{m-1}F_n+F_mF_{n+1}.$$

Assuming m=nk, mathematical induction implies that F_{nk} is divisible by F_n.

Interestingly, the same formula allows us to prove that:
F_n does not divide F_k, $k>n$, unless *n* divides *k*.

Recalling that two consecutive Fibonacci numbers are always relatively prime, we can establish the property of the Euclidean Algorithm, namely if $m>n$ are two integers such that $m=qn+r$, $0<r<n-1$, then

$$\gcd(F_m,F_n)=\gcd(F_n,F_r).$$

$$F_m=F_{qn+r}=F_{qn}F_{r+1}+F_{qn-1}F_r,$$

Therefore, the Euclidean Algorithm on the pair (F_m,F_n) has the same run as the Euclidean Algorithm on their indices (m,n) and

$$\mathbf{\gcd(F_m,F_n)=F_{\gcd(m,n)}}.$$

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