Prime numbers, arithmetic progressions, and the Green-Tao theorem

2,	3,	5,	7,	11,	13,
17,	19,	23,	29,	31,	37,
41,	43,	47,	53,	59,	61,
67,	71,	73,	79,	83,	89,
97,	101,	103,	107,	109,	113,
127,	131,	137,	139,	149,	151,
157,	163,	167,	173,	179,	181,
191,	193,	197,	199,	211,	223,
227,	229,	233,	239,	241,	251,
257,	263,	269,	271,	277,	281,
283,	293,	307,	311,	313,	317,
331,	337,	347,	349,	353,	359,
367,	373,	379,	383,	389,	397,
401,	409,	419,	421,	431,	433,
439,	443,	449,	457,	461,	463,
467,	479,	487,	491,	499,	503,
509,	521,	523,	541,	547,	557,
563,	569,	571,	577,	587,	593,
599,	601,	607,	613,	617,	619,
631,	641,	643,	647,	653,	659,
661,	673,	677,	683,	691,	701,
709,	719,	727,	733,	739,	743,
751,	757,	761,	769,	773,	787,
797,	809,	811,	821,	823,	827,
829,	839,	853,	857,	859,	863,
877,	881,	883,	887,	907,	911,
919,	929,	937,	941,	947,	953,
967,	971,	977,	983,	991,	997

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3, 5, 7

5, 11, 17, 23, 29

7, 37, 67, 97, 127, 157

What do these sequences have in common?
Yes, you are right – all these sequences are arithmetic progressions and all of them consist of prime numbers. Recall that an arithmetic progression is a finite or infinite sequence of numbers such that the difference between any two consecutive terms is constant. Thus, the n-th term of the arithmetic progression has the form $a_n = a_0 + nd$, where a_0 is the initial term. You can try to find longer arithmetic progressions that consist entirely of primes, but this is not an easy task. The question about distribution of prime numbers inside arithmetic progressions of natural numbers has long history. Perhaps the most famous statement about prime numbers and arithmetic progressions is the following theorem.

Theorem (Dirichlet, 1837). Let a and b be relatively prime natural numbers. Then the arithmetic progression a+nb contains infinitely many primes.

Recall that a and b are called relatively prime (or coprime) if they do not have any common divisors except for 1. The proof of Dirichlet theorem is complicated and requires advanced tools from algebra, calculus, and analytic number theory.
Here is another natural question:
How long can an arithmetic progression that consists of prime numbers be?
It is easy to see that such progression cannot be infinitely long. Indeed, if, in an arithmetic progression a+bn we let n=a, then the a-th term of this progression is a+ba=a(1+b) which is not prime as soon as a >1 (and if a=1 we can consider a+b instead of a and let n=a+b+1). It is, however, extremely surprising that such progression can be of any finite length. The next theorem is a corollary of even more general result, recently proved by Ben Green and Terence Tao.

Theorem (Green-Tao, 2004). For any natural number n there exists an arithmetic progression of length n consisting entirely of prime numbers.

It should be noted that the statement of this theorem has been known as a conjecture for over 200 years, but the proof required several deep results which culminated in the work of Green and Tao.