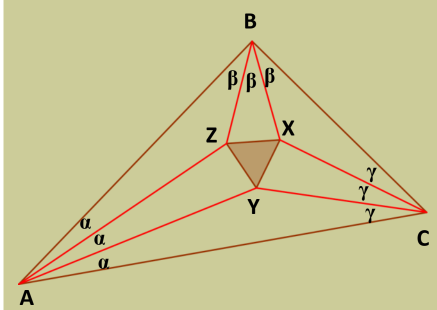
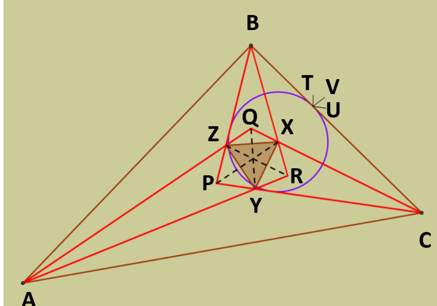
Nipissing University - Mathematics

Morley's trisector theorem



First proof



Second proof

Morley's trisector theorem is named after the English-born American mathematician Frank Morley, who proved it in 1899. It is a surprisingly beautiful result in plane geometry:

The points of intersection of the adjacent angle trisectors of a triangle form an equilateral triangle.

First proof is the so-called backward proof. We start with the equilateral triangle ΔXYZ and take its sides to have unit length. Our goal is to construct the triangle ΔABC having angles $3\alpha, 3\beta, 3\gamma$ where α, β, γ are any positive angle measures such that $\alpha + \beta + \gamma = 60°$. Thus ΔXYZ is the Morley triangle of ΔABC.

First we construct the triangle ΔAZY such that $\measuredangle ZAY = \alpha, \measuredangle AYZ = 60° + \gamma, \measuredangle AZY = 60° + \beta$.
From the law of sines for ΔAZY, we have $AY = \frac{AY}{YZ} = \frac{\sin(60°+\beta)}{\sin\alpha}$. Similarly, constructing ΔCXY such that $\measuredangle XCY = \gamma, \measuredangle CYX = 60° + \alpha, \measuredangle CXY = 60° + \beta$, we obtain $CY = \frac{CY}{YX} = \frac{\sin(60°+\beta)}{\sin\gamma}$. Thus, in ΔACY we have $\frac{AY}{CY} = \frac{\sin\gamma}{\sin\alpha}$. Also, $\measuredangle AYC = 360° - \measuredangle XYZ - \measuredangle AYZ - \measuredangle CYX = 360° - 60° - (60° + \gamma) - (60° + \alpha) = 180° - \alpha - \gamma = 120° + \beta$. Therefore, in ΔACY $\measuredangle CAY + \measuredangle ACY = \alpha + \gamma$ and $\frac{\sin\measuredangle ACY}{\sin\measuredangle CAY} = \frac{\sin\gamma}{\sin\alpha}$. Given that $\alpha + \gamma < 60°$, we must have $\measuredangle CAY = \alpha, \measuredangle ACY = \gamma$. Similar considerations establish the angles in ΔABZ and ΔBCX. In triangle ΔABC the lines AZ, AY, BX, BZ, CY, CX are the trisectors of the angles $\measuredangle CAB, \measuredangle ABC, \measuredangle BCA$, respectively, as we intended to show.

Second proof does not use trigonometry and is as follows. Starting with the equilateral ΔXYZ and the angle measures $\alpha + \beta + \gamma = 60°$, we proceed as follows. Point P is constructed on the altitude/median/angle bisector through vertex X in ΔXYZ, outside of the triangle so that $\measuredangle YPZ = 60° + 2\alpha$. Similarly, we construct the points Q opposite Y with $\measuredangle ZQX = 60° + 2\beta$ and R opposite Z with $\measuredangle XRY = 60° + 2\gamma$. Let the lines PZ and RZ intersect at point B. Using the fact that the angle measures of the interior angles in a quadrilateral add up to 360°, we obtain $\measuredangle PBR = \beta$ since $\measuredangle PYZ = 60° - \alpha, \measuredangle RYX = 60° - \gamma, \measuredangle XYZ = 60° \Rightarrow \measuredangle PYR = 180° - \alpha - \gamma$. Similarly, the lines PY and QX intersect at point C such that $\measuredangle PCQ = \gamma$. The point X lies, by construction on the angle bisector of $\measuredangle CPB$. Construct a circle, centered at X that touches the lines PB and PC. Let the tangents to this circle from points B and C touch the circle at points T and U, and intersect at point V. Then $\measuredangle TBX = \measuredangle PBX = \beta$ and $\measuredangle UCX = \measuredangle PCX = \gamma$. Thus, in the quadrilateral $CPBV$ we have $\measuredangle CPB = 60° + 2\alpha, \measuredangle PBV = 2\beta, \measuredangle VCP = 2\gamma$. Therefore, $\measuredangle BVC = 180°$ and the line BC is tangent to the circle (points T, U, V coincide). The point A is the intersection of the lines QZ and RY, and the tangencies in ΔARB and ΔAQC follow by symmetry.

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