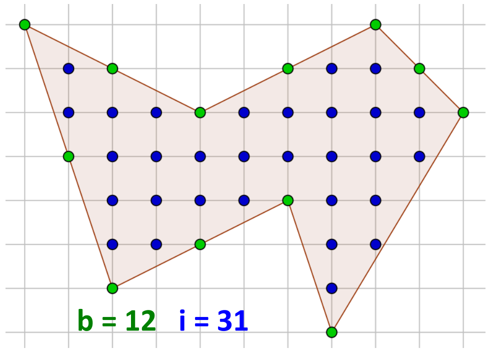
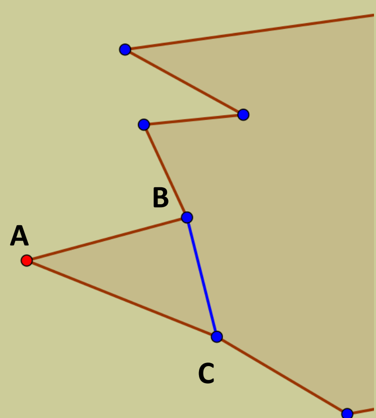
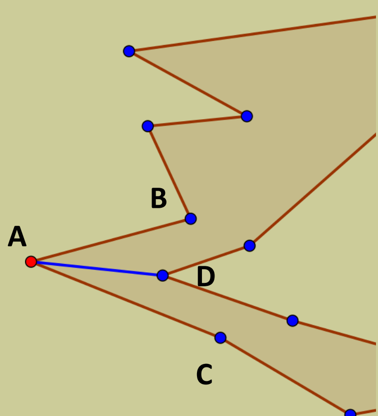
Pick's theorem and triangulation of polygons



Computation of areas and volumes is an important part of elementary and advanced geometry. Due to practical importance of such formulas, they have been studied since antiquity.
Often, such formulas involve various parameters of geometrical objects, such as lengths of sides and angle measurements. In the case of plane objects, such formulas exist for rectangles, triangles, and trapezoids. However, formulas for the area of general polygons involve coordinates of vertices and are quite complicated.
One general method to solve various problems about polygons is called **triangulation**, i.e. cutting a polygon into triangles. In the case of a convex polygon, it is easy to triangulate it using diagonals. Surprisingly, even a non-convex polygon can be cut into triangular pieces using diagonals. Indeed, let A be the leftmost vertex of a polygon P (see the diagrams on the left). Let B and C be the vertices adjacent to A. If triangle ABC does not contain any vertices of P, then BC is the diagonal lying entirely in the polygon and thus dividing it in two smaller parts. Otherwise, let D be the leftmost of those vertices that lie inside triangle ABC. Then AD is the required diagonal. Continuing in this way we eventually divide the whole polygon into triangles using diagonals.
This in turn allows us to establish another remarkable result showing that in some cases we can compute the area of a polygon simply by counting points.





Theorem (G.A. Pick, 1899). Consider a polygon P whose vertices have integer coordinates. If b is the number of points with integer coordinates lying on the boundary of P and i is the number of points with integer coordinates lying strictly inside P, then **Area of P = i + b/2 - 1**

For example, for the polygon on the picture above the area is 31+12/2 - 1 = 36. This theorem has an obvious generalization to the case of polygons whose vertices lie at the grid of equal-distanced points. To prove Pick's theorem one need first to triangulate P using its diagonals, keeping track of what happens with points inside and on the boundaries of respective triangles. The fact that the formula works for triangles is an easy exercise.

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