Fuzzy Logic and Systems

COSC3007 Presentation

Fuzzy Systems

A control system that operates at least partially on fuzzy logic

If part of a system is fuzzy, all affected output from the system can be considered fuzzy

Comparable to Analog: Continuous input instead of 1 or 0

Fuzzy Logic

Many-Valued Logic

Deals with partial truths and falses, as well as traditional true-false logic

Deals with irrationality through “fuzzjective”

History

History

“Laws of Thought”

-Aristotle

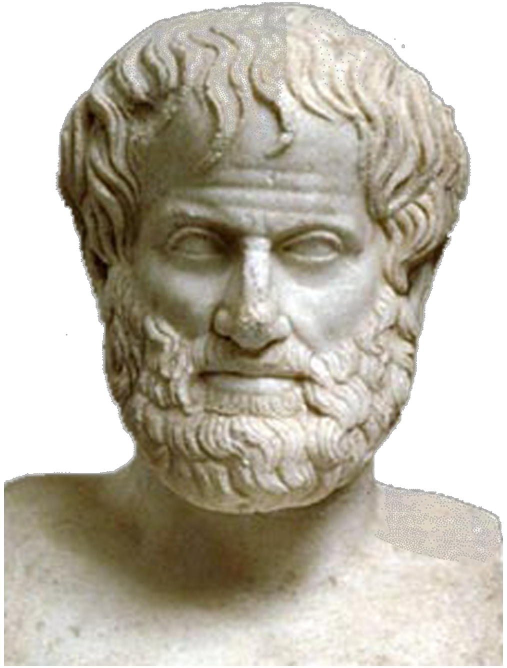
-Beginnings of logic

“Law of Excluded Middle”

-False!

Multiple states of being

-Plato



History

Term “Fuzzy Logic” introduced in 1965 in the proposal “fuzzy set theory” by Lofti A. Zadeh

Has been studied as far back as the 1920’s under the name infinite value logics

History

Originally just an abstract idea on how to deal with relative-ness of reality and non binary situations

Quickly found many uses in everything from camera focus technology to train managment

History

Recent History

-A tool for controlling and steering systems and complex industrial processes

3 Major phases of development

Phase 1(1965 - 1973)

Interpolating values over an interval

Membership functions

Logic and reasoning not center-stage

Phase 2(1973-1999)

Linguistic variables

-Hot, tall, big etc

-Computing with Words

If-then rules

Sendai Railway

Phase 3(1999-Now)

Widespread use

Mathematical Foundation

-fuzzy arithmetic, fuzzy calculus

Development of narrow and wide FL

Narrow FL

Classic Fuzzy Logic

Generalization of multivalued logic

“Simple” fuzzy logic

Wide FL

Mathematical System

- -fuzzy set theory
- -possibility theory
- -calculus of fuzzy if-then rules
- -fuzzy arithmetic

Attempts to encompass maths

How Fuzzy Logic Works

Fuzzy AI

Used to describe reactions based on a situation

Most any fuzzy system can be considered AI

Everything a matter of degree

Wide Fuzzy Logic

“FL has been usefully applied to a massive number of fields from possibility theory and possibilistic logic, knowledge representation, decision analysis, cluster analysis, pattern recognition, fuzzy arithmetic, fuzzy mathematical programming, fuzzy topology and, more generally, fuzzy mathematics”

“Computing with Words”

Precisiated Natural Language (PNL)

-Zadeh, Lotfi A, 2004

-System for describing perceptions

Search Engines

Google

Not so Fuzzy

Fuzziness is not imprecise or random! (necessarily)

There can be imprecise data

(But we typically know how imprecise it is)

Best thought of as probability

"it might rain tomorrow"

Maybe a Little Bit Fuzzy...

Can arise of ambiguity of language

- High building vs high score

Fuzzy is not random!

- Randomness is an objective statistic

Crisp Logic vs Fuzzy Logic

Crisp Logic is true/false only

- No exceptions

Fuzzy Logic bridges the qualitative and quantitative

Making Fuzzy Less Fuzzy

Fuzzy logic consists of qualitative linguistic rules

Fuzzy methodologies based on mathematical theory

Modeling a Situation

Tall vs not tall

- Crisp logic wouldn’t be realistic

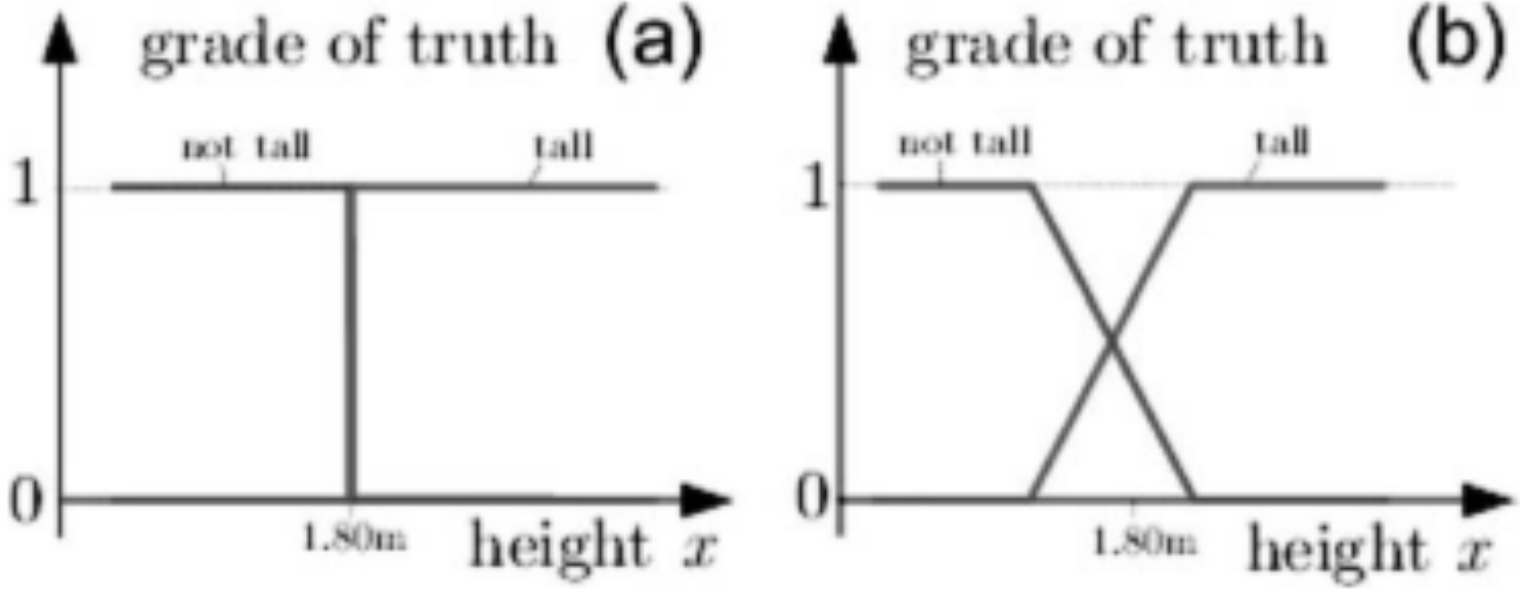
Need to produce smooth input

Modeling a Situation

Like crisp, use 2 sets

Sets can overlap

Small number of rules and interpolation to find a given value


grade of truth
(a)
not tall
tall
1
0
1.80m
height x
grade of truth
(b)
not tall
tall
1
0
1.80m
height x

Set Theory

Georg Cantor(1845 to 1918)

Created set theory

Initially met with MAJOR criticism…

...until a majority of mathematical concepts ended up being more easily modeled than with conventional methods

Set Theory

Conventional Set

"an item from a given universe"

Each item can exist in one set

Nearly anything called a set in ordinary conversation is acceptable in a mathematical sense

Fuzzy Sets

In fuzzy logic members of a set are graded with a “membership function”

- It assigns a value between 0 and 1

Often denoted by the greek letter μ (Mu)

Fuzzy Sets

Cantor’s set regarded as a special case

Crisp logic

Normal Fuzzy sets use a "membership function" which grades a value over an interval

This means items can be only partially in a set

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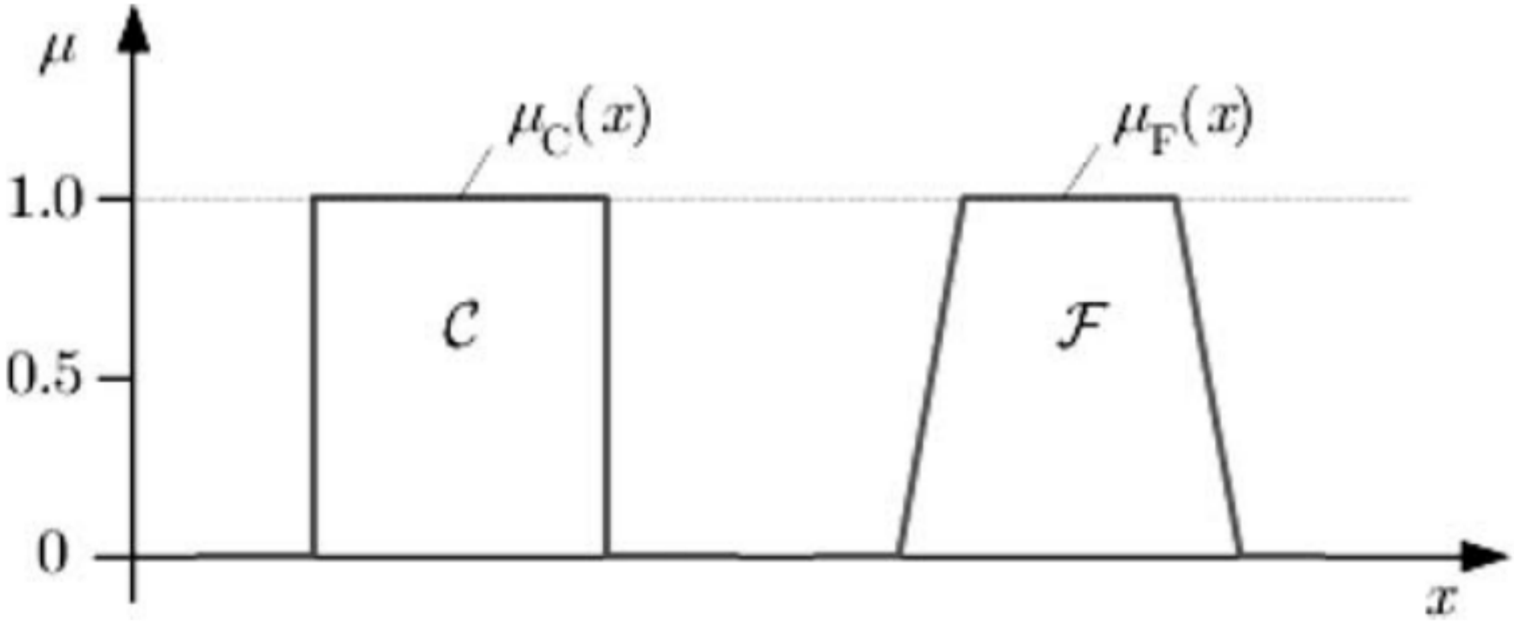


Fuzzy Sets

Base set “X”

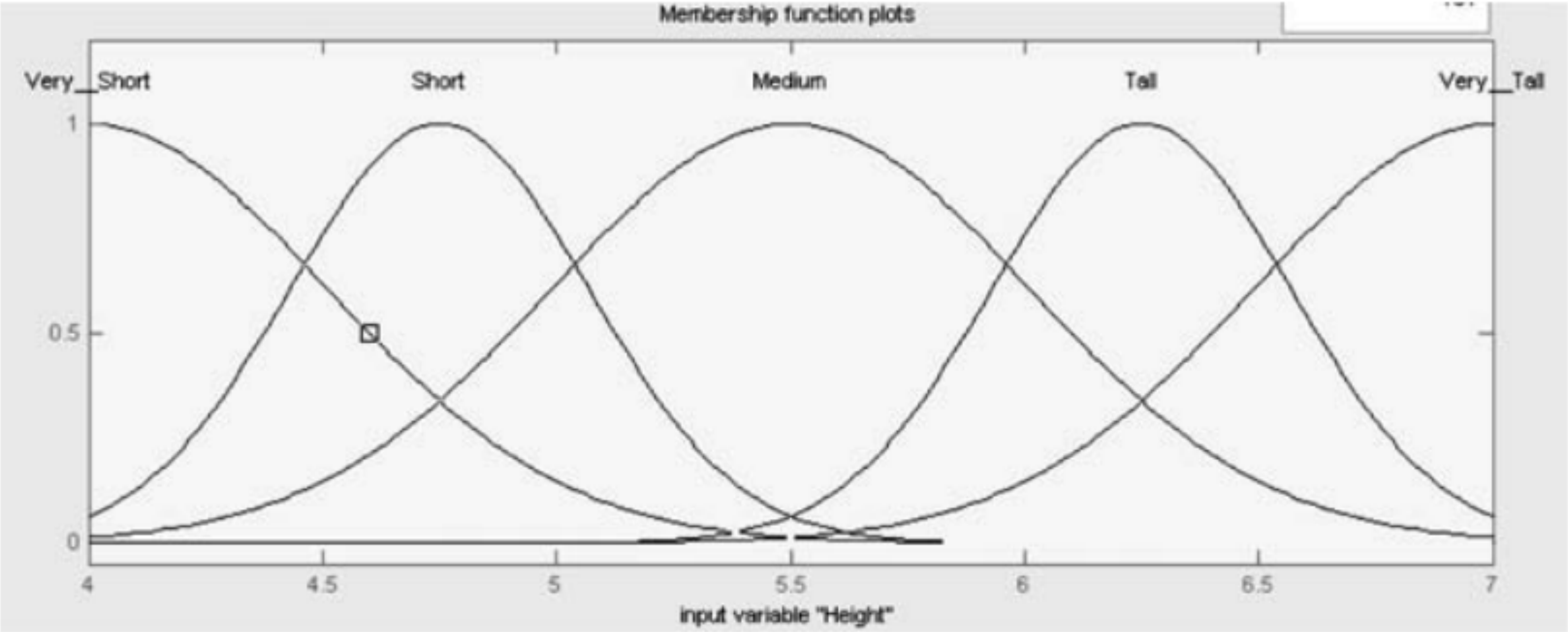
- Denotes a “universal” set

Everything is a subset of X

Fuzzy Sets

Since both our “crisp” and “fuzzy” sets use X, conversion between the two is possible

Allows for interpolation across our universal set based on the given rules


Membership function plots
Very__Short
Short
Medium
Tall
Very__Tall
1
0.5
0
4
4.5
5
5.5
6
6.5
7
input variable "Height"

Different Universes

What about non-numeric sets?

- Taste for example (sweet, bitter, etc)

Just needs a universal set

Called a psychological continuum

Membership Functions

Has the form $\mu_A(x)$

$x \in X$ on the fuzzy set A

$\mu_A(x)$ can be a number of classes of functions

Grade of an Element

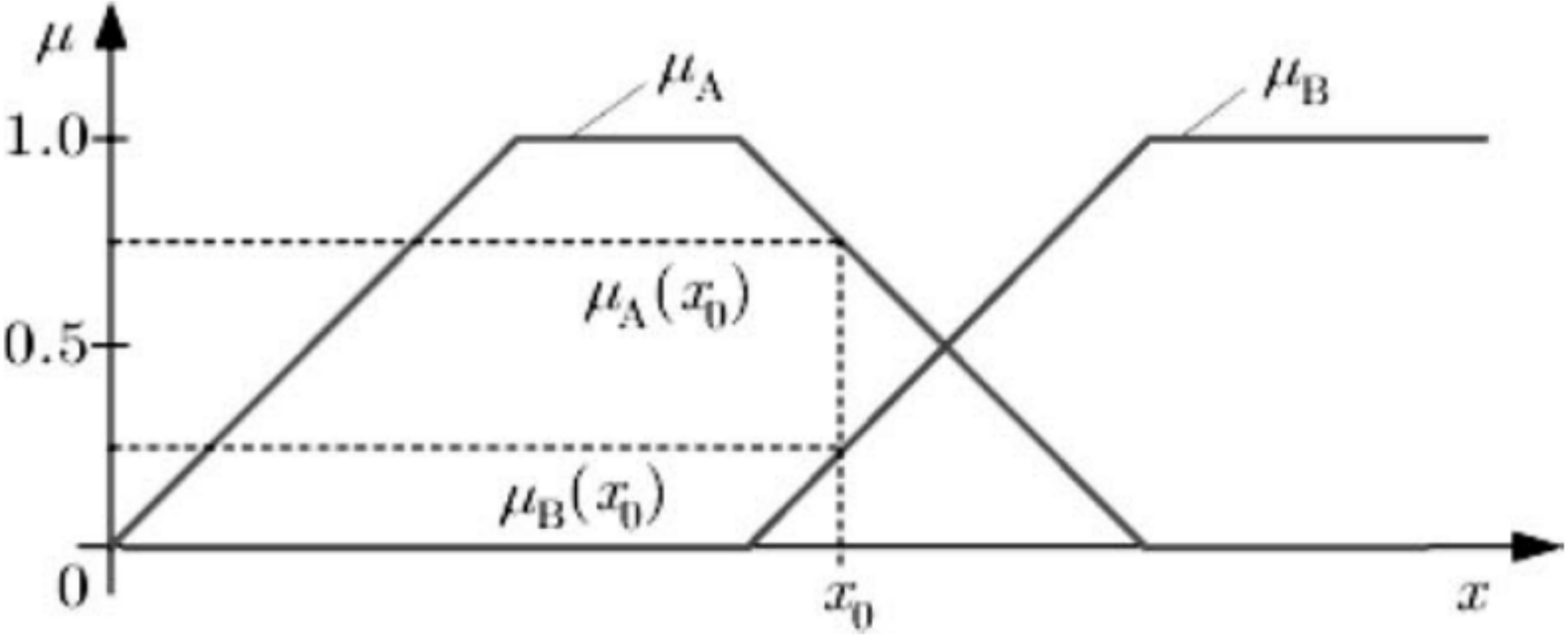
$\mu_A(x_0)$ represents the grade of x_0

Value between 1 and 0

Determines how much the value is an element of A

x_0 could also be a member of B; $\mu_B(x_0)$

[redacted]



Membership Types

-Triangular Function

-Γ(gamma) Function

-S Function

-Trapezoidal Function

-Gaussian Function

-Exponential Function

Fuzzy Logic Example

```
IF object IS big AND basket IS small
THEN basket size IS increased
```

Adapts to conditions: big, small are relative

“Defuzzification”: Final answer is changed into exact values

Triangular Function

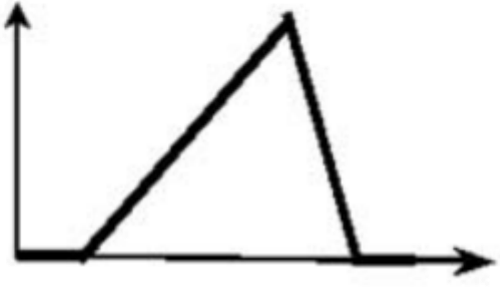
a,b = lower and upper bounds

C = non zero value

$$A(x) = \begin{cases} 0, & if\ x \leq a \\ \frac{x-a}{b-a}, & if\ x\ \epsilon[a,b] \\ \frac{c-x}{c-b}, & if\ x\ \epsilon[b,c] \\ 0, & if\ x\ \geq a \end{cases}$$

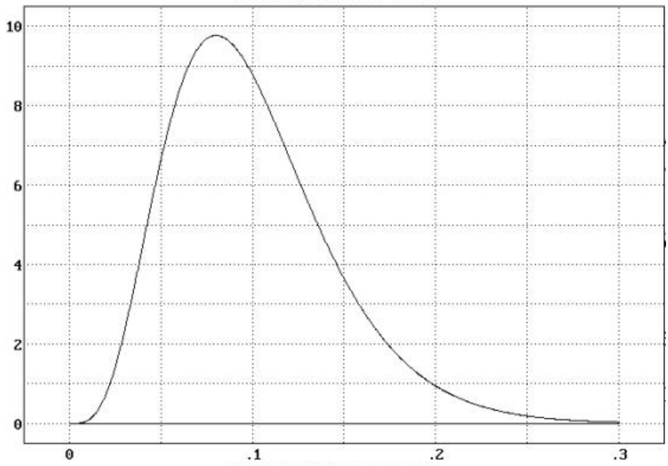
Triangular MF:

f(x; a, b, c) = max(min((x-a)/(b-a), (c-x)/(c-b)), 0)



Γ Function

$$A(x) = \begin{cases} 0, & if\ x \leq a \\ 1 - e_{-k(x-a)^2} & if\ x{<}a \end{cases}$$



or

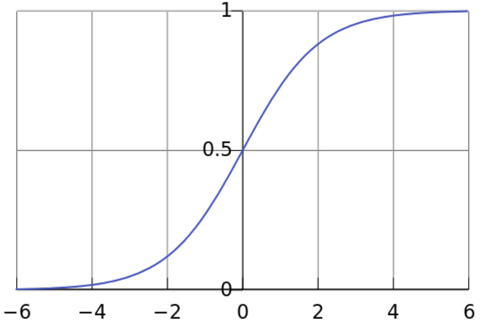
$$A(x) = 0 \begin{cases} 0, & if x \leq a \\ \frac{k(x-a)^2}{1+k(x-a)^2}, & if\ x{>}a \end{cases}$$

where k>0.

S Function

aka Sigmoid Function

$$A(x) = \begin{cases} \mathbf{0}, & if x \leq a \\ \mathbf{2}(\frac{x-a}{b-a})^2, & if\ x\ \epsilon[a, m] \\ \mathbf{1\text{-}2}(\frac{x-a}{b-a})^2, & if\ x\ \epsilon[m, b] \\ \mathbf{1}, & if\ x > b \end{cases}$$



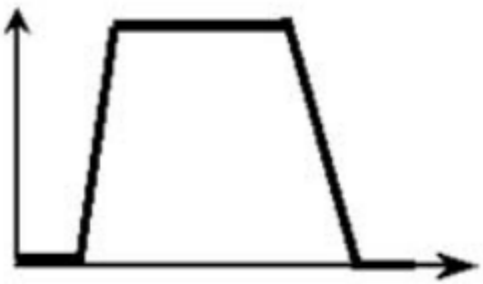
The point m=a+b/2 is known as the crossover of the S-function.

Trapezoidal Function

$$A(x) = \begin{cases} \mathbf{0}, & if x \leq a \\ \frac{x-a}{b-a}, & if\ x\ \epsilon[a, b] \\ \mathbf{1}, & if\ x\ \epsilon[b, c] \\ \frac{d-x}{d-c}, & if\ x\ \epsilon[c, d] \\ \mathbf{0}, & if\ x \geq b \end{cases}$$

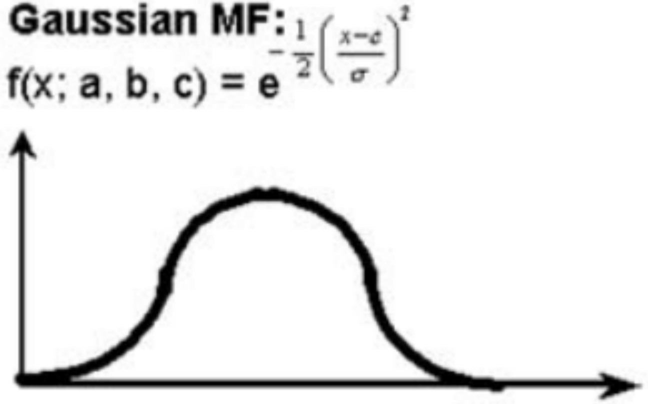
Trapezoidal MF:

f(x; a, b, c, d) = max(min((x-a)/(b-a), 1, (d-x)/(d-c)), 0)



Gaussian Function

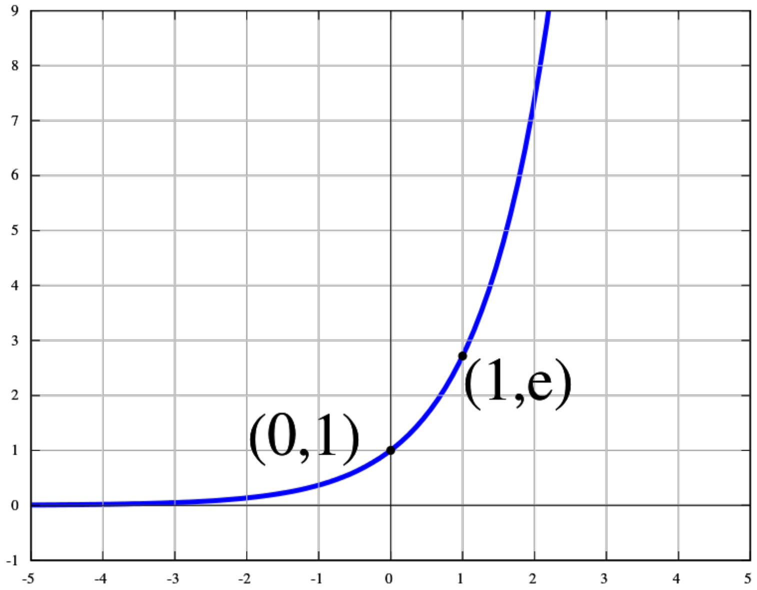
$$A(x) = -e^{\frac{(x-c)^2}{2\sigma^2}}, where\ 2\sigma^2 > 0.$$



Exponential Function

$$A(x) = \frac{1}{1+(k-m)^2}, k > 1$$

$$A(x) = \frac{(k-m)^2}{1+(k-m)^2}, k > 0$$



Linguistic Variables

Linguistic Variable

Algebraic values take numbers

Linguistic values take words or sentences as values

Hierarchy:
linguistic variable -> fuzzy variable -> base variable

Let x be a linguistic variable with the label "*Age*". Terms of this linguistic variable, which are fuzzy sets, could be "*old*", "*young*", "*very young*" from the term set.

T = Old, Very Old, Not So Old, More or Less Young, Quite Young, Very Young

Each term is a fuzzy variable defined on the base variable, which might be the scale from 0 to 100 years.

Operations on a Fuzzy Set

Generalizations of crisp logic operations

Set of standard fuzzy set operations

- 3 basic

Fuzzy Complement

Fuzzy set on U for which $A'(x) = 1 - A(x)$ for every x in U

Fuzzy Intersection

Fuzzy set on U for which (A B)(x) = min [A(x), B(x)] for every x in U

t-norm

Fuzzy Union

fuzzy set on U for which (A Y B)(x) = max [A(x), B(x)] for every x in U

t-conorm

Operations on a Fuzzy Set

Largest fuzzy set with intersection

Smallest with union

x E X where A(x) = A ˆ (x), x = equilibrium point

Other forms of union not associative

Smaller than intersection, greater than union

Intersection example

B = A I C

The intersection between young and old is middle aged

Sigma Count

“Scalar Cardinality”

|A|= sum (x) [A(x)] for all x in X

Properties of fuzzy sets

(i) a+A is a subset of aA

(ii) a ≤ b implies that bA is a subset of aA and b+A is a subset of

a+A

(iii) a(A I B) = aA I aB and a(AYB) = aA YaB

(iv) a+(A I B) = a+A I a+B and a+(A Y B) = a+A Y a+B

Properties Continued

(v) $a(Aˆ) = (1-a)+Aˆ$

This means that the alpha-cut of the compliment of A is the (1-a) strong alpha-cut of A complemented.

$a(Aˆ)$ is not equal to $aAˆ$

$a+(Aˆ)$ is not equal to $a+Aˆ$

Fuzzy Complement

c(A(x)) = cA(x)

Axioms for fuzzy complements

Axiom c1. c(0)=1 and c(1)=0 (boundary conditions)
Axiom c2. for all a,b in [0,1], if a<=b then c(a)>=c(b) (monotonicity)
Axiom c3. c is a continuous function (continuity)
Axiom c4. c is "involutive", which means that c(c(a))=a for all a in [0,1] (Involution)

Combining Operations

DeMorgan's Law

- The complement of the intersection of A and B equals the union of the complement of A and the complement of B.
- The complement of the union of A and B equals the intersection of the complement of A and the complement of B.

Fuzzy Arithmetic

Fuzzy Numbers

(i) A must be a normal fuzzy set'

(ii) a (alpha-cut of A; {x |A(x) ≥ a}) must be a closed interval for

every a in (0,1];

(iii) the support of A, 0+A (strong 0-cut of A; {x |A(x) >0 }), must be bounded

Arithmetic Operations on Intervals

1. Each fuzzy set, and thus each fuzzy number, can fully and uniquely be represented by its alpha-cuts.

2. Alpha-cuts of each fuzzy number are closed intervals of real numbers for all alpha in [0,1]. Therefore, arithmetic operations on fuzzy numbers can be defined in terms of arithmetic operations on their alpha-cuts (i.e., arithmetic operations on closed intervals), which is treated in the field of “interval analysis”.

Arithmatic Operators

Basic operators

+ $[a,b] + [d,e] = [a+d,b+e]$

- $[a,b] - [d,e] = [a-e,b-d]$

* $[a,b] * [d,e] = [\min(ad,ae,bd,be),\max(ad,ae,bd,be)]$

/ $[a,b] / [d,e] = [a,b] * [1/e,1/d] =$

$[\min(a/d,a/e,b/d,b/e),\max(a/d,a/e,b/d,b/e)]$

PROVIDED d and e does not contain 0

Arithmetic on Fuzzy Numbers

A#B = Union(all a in [0,1]) a(A#B)

where # is an operation

In other words an operator acting on a fuzzy set acts on all the fuzzy numbers in a set as a union

Putting it all Together

Fuzzy Washing Machine

X = dirty, moderate dirty, not dirty

Y = high, medium, low

Rules

(1) IF the type of dirt is dirty THEN the washing time is high.

(2) IF the type of dirt is moderate dirty THEN the washing time is medium.

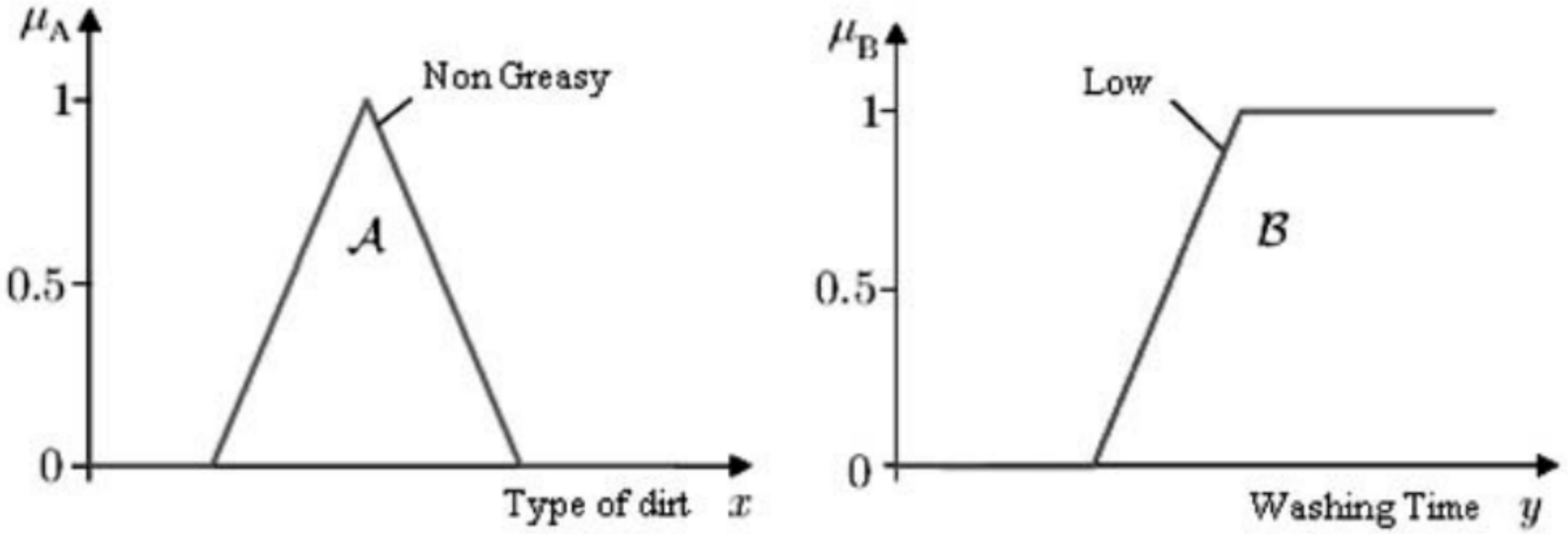
(3) IF the type of dirt is not dirty THEN the washing time is low.

Crisp Values

TABLE 6.1: Relation between the two crisp sets

	High	*Medium*	*Low*
Greasy	1	0	0
Moderate Greasy	0	1	0
Not Greasy	0	0	1

Functions Used



Mathematical Notation

$\mu R (x,y) = \min\{\mu A (x), \mu B (y)\}$

or

$\mu R (x,y) = \mu A (x), \mu B (y)$

Resultant Fuzzy Values

TABLE 6.2: Modified form of the Table 6.1

	High	*Medium*	*Low*
Greasy	1	0.5	0
Moderate Greasy	0.3	1	0.4
Not Greasy	0	0.2	1