ACCESSIBLE POINTS OF PLANAR EMBEDDINGS OF TENT INVERSE LIMIT SPACES

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Abstract. In this paper we study a class of embeddings of tent inverse limit spaces. We introduce techniques relying on the symbolic description of tent inverse limit spaces and use them to study the sets of accessible points and prime ends of given embeddings. We find phenomena which do not occur in the standard embeddings arising from the Barge-Martin construction of global attractors. Standard embeddings are treated in detail at the end of the paper.

1. Introduction

A problem of classifying continua that can be embedded in the plane is of substantial interest in Continuum Theory, mainly because it is intrinsically related with the solution of the Fixed Point Property for planar non-separating continua. In the case when a continuum is chainable, *i.e.,* it admits an ε-mapping on the interval $[0,1]$ for every $\varepsilon > 0$, it follows from an old result of Bing [8] that the continuum can be embedded in the plane. Two embeddings of a continuum are said to be equivalent if there exists a planar homeomorphism mapping one onto the other. Therefore, it is natural to ask how many possible non-equivalent embeddings of a specific chainable continuum there exist and what these embeddings look like. The straightforward way to approach the description of embeddings is through their sets of accessible points or through their prime end structure.

Inverse limit spaces on intervals are chainable. In [2] Bruin and the authors showed that there exist uncountably many non-equivalent embeddings of tent map inverse limit spaces for all tent maps with slopes greater than $\sqrt{2}$, but they give no insight what the constructed embeddings look like. In this paper we study the class of planar embeddings from [2] in detail, focusing primarily on accessible sets and the prime end structure in the finite critical orbit case.

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The study of sets of accessible points of planar embeddings of the Knaster continuum was given by Mayer in [23], and the characterization of possible sets of accessible points of embeddings of Knaster continua was given by Dębski & Tymchatyn in [18]. The study of embeddings of unimodal inverse limit spaces appears in the literature in two forms; corresponding to attractors of orientation preserving (by Brucks & Diamond in [14]) and orientation reversing (by Bruin in [16]) planar homeomorphisms. We refer to those embeddings as *standard*. Generally, Barge and Martin showed in [6] that every inverse limit space with a single interval bonding map can be realized as an attractor of an orientation preserving planar homeomorphism which acts on the attractor in the same way as the natural shift homeomorphism acts on the inverse limit. Using the construction from [6], Boyland, de Carvalho and Hall recently gave in [10] the complete classification of the prime end structure and accessible sets of the Brucks-Diamond embedding of unimodal inverse limit spaces (satisfying certain regularity conditions valid for *e.g.* tent map inverse limits). For non-standard embeddings of tent inverse limit spaces constructed in [2], the natural shift homeomorphism cannot be extended to the plane as we show in Section 8. Thus, we lack dynamical techniques as used in [10]. Therefore, for the construction and study of embeddings we chose a symbolic approach emerging from the Milnor-Thurston kneading theory in [25] which was already used in constructions of embeddings by Brucks & Diamond [14], Bruin [16] and Bruin and the authors [2]. It turns out that such construction gives straightforward calculation techniques on the itineraries which we exploit throughout the paper.

By $\mathbb{N}$ we denote the set of natural numbers and let $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. The *Hilbert cube* is the space $[0,1]^{\mathbb{N}_0}$ equipped with the product metric

$$d(x,y) := \sum_{i\leq 0} 2^i |\pi_i(x) - \pi_i(y)|,$$

where $\pi_i\colon [0,1]^{\mathbb{N}_0} \to [0,1]$ denote the coordinate projections for $i \leq 0$.

The *tent map family* $T_s : [0,1] \to [0,1]$ is defined by $T_s := \min\{sx, s(1-x)\}$ where $x \in [0,1]$ and $s \in (0,2]$. Let $c = \frac{1}{2}$ denote the *critical point* of the map T_s. In the rest of the file we work with tent maps for slopes $s \in (\sqrt{2}, 2]$ and when there is no need to specify the slope we set for brevity $T := T_s$. The *inverse limit space with the bonding map T* is a subspace of the Hilbert cube defined by

$$X := \varprojlim([0,1], T) = \{x \in [0,1]^{\mathbb{N}_0} : T(\pi_i(x)) = \pi_{i+1}(x),\ i \leq 0\}.$$

The space X is a *continuum*, *i.e.,* compact and connected metric space. Define the *shift homeomorphism* as $\sigma\colon X \to X$, $\pi_i(\sigma(x)) := T(\pi_i(x))$ for every $i \leq 0$.

The space obtained by restricting the bonding map T to its dynamical core is called the *core* of X and will be denoted by X':

$$X' := \varprojlim([T^2(c), T(c)], T|_{[T^2(c),T(c)]}).$$

A continuum is *indecomposable* if it cannot be expressed as a union of two proper subcontinua. When $s \in (\sqrt{2}, 2]$, the core X' is indecomposable and by Bennett's theorem from [7], $X = X' \cup \mathcal{C}$, where $\mathcal{C}$ is a ray which contains the fixed point $(\ldots, 0, 0)$

and it compactifies on X' (for details see *e.g.* [22]). The ray $\mathcal{C}$ shields off some points of a continuum X and thus has an important effect on the set of accessible points in embeddings of X. However, the interesting phenomena regarding the structure of sets of accessible points occur in X' and thus we will mostly ignore $\mathcal{C}$ in the remainder of the paper. The structure of embedded X (including $\mathcal{C}$) will be briefly discussed in Section 5.

A *composant* of a point $x \in K$ is the union of all proper subcontinua in K that contain x. If a continuum is indecomposable it consists of uncountably many pairwise disjoint composants and every composant is dense in the continuum, see [26]. The arc-component $\mathcal{U}_x$ of a point $x \in K$ is the union of all arcs from K that contain point x.

A point $a \in K \subset \mathbb{R}^2$ from a continuum K is *accessible (i.e., from the complement of* K*)* if there exists an arc $A \subset \mathbb{R}^2$ such that $A \cap K = \{a\}$. We say that an arc-component $\mathcal{U}_x$ is *fully accessible*, if every point from $\mathcal{U}_x$ is accessible. Mainly we will be interested in embeddings of inverse limits of indecomposable cores of tent maps with finite critical orbit. In these cases every arc-component of a point corresponds to the composant of that point (see Proposition 3 from [13]).

We denote the class of embeddings of tent inverse limit spaces X and their cores X' constructed in [2] by $\mathcal{E}$ and refer to them as $\mathcal{E}$-embeddings. In [2], every $\mathcal{E}$-embedding of X is represented as a union of uncountably many horizontal segments (called basic arcs) which are aligned along vertically embedded Cantor set with prescribed identifications between some endpoints of basic arcs (see Section 2 of this paper and [2] for details). An $\mathcal{E}$-embedding of X is then uniquely determined by the left infinite itinerary $L = \dots l_2 l_1$, which is a symbolic description of the largest basic arcs among all basic arcs.

In Section 2, we give a short symbolic preliminaries and recap the construction of embeddings of tent inverse limit spaces as given in [2]. We give a symbolic characterization of arc-components in X, generalizing the result from the paper by Brucks& Diamond [14]. In Section 3, we characterize the possible sets of accessible points in an arc-component of any indecomposable plane non-separating continuum K. In Section 4 we briefly introduce Carathéodory's prime end theory and prove that there are no fourth kind prime ends associated to an indecomposable plane non-separating continuum whose only proper subcontinua are arcs (which occurs *e.g.* for tent map inverse limits with long-branched bonding maps). In Section 5, we begin our study of embeddings $\mathcal{E}$. We introduce the notion of cylinders of basic arcs and techniques to explicitly calculate their extrema. We show that two $\mathcal{E}$-embeddings of the same space X are equivalent when they are determined by eventually the same left infinite tail L. Given an $\mathcal{E}$-embedding of X, we prove that the arc-component of the top basic arc with symbolic description L (throughout the file this arc-component is denoted by $\mathcal{U}_L$) is fully accessible, if the top basic arc is not a spiral point (see Definition 2.7 and Figure 1). However, we also show that $\mathcal{U}_L$ is not necessarily the unique (fully) accessible arc-component. In the same section we briefly discuss $\mathcal{E}$-embeddings of decomposable continuum X and characterize the set of accessible points up to two points on the corresponding circle of prime ends.

From Section 6 onwards we study $\mathcal{E}$-embeddings of indecomposable continuum X'. In Section 6 we give sufficient conditions on itineraries of L and kneading sequences ν associated with X' so that the embeddings of X' allow more than one fully accessible arc-component and give some interesting examples of such embeddings.

We say that $x \in X$ is a *folding point* if for every $\varepsilon > 0$ there exists a neighbourhood U_ε of x, which is not homeomorphic to the $C \times (0,1)$, where C is the Cantor set. A point $x \in X$ is called an *endpoint* if for every two subcontinua $X_1, X_2 \subset X$ such that $x \in X_1 \cap X_2$, either $X_1 \subset X_2$ or $X_2 \subset X_1$. Note that endpoints are also folding points. In Section 7 we characterize accessible folding points of $\mathcal{E}$-embeddings when the critical orbit of the tent map is finite. Surprisingly, no endpoints will be accessible in any $\mathcal{E}$-embedding of X' with the exception of Brucks-Diamond embedding. Another surprising phenomenon is the occurrence of *Type 3* folding points (see Definition 7.19 and Figure 13) when the orbit of the third iterate of the critical point is periodic but the critical point itself is not periodic. Such a phenomenon does not occur in the standard (Brucks-Diamond or Bruin's) embedding of any tent map inverse limit space.

In Section 8 we prove that for every embedding constructed in [2] except for the ones constructed by Brucks & Diamond [14] and Bruin [16], natural shift homeomorphism can not be extended from the $\mathcal{E}$-embedding of X' to the whole plane. Showing that we answer on a question posed by Boyland, de Carvalho and Hall in the paper [10] on the page 4. In Section 9 we study special examples of embeddings of X'. We explicitly show that every X' can be embedded with at least two non-degenerate fully accessible arc-components. In a finite orbit case when we have exactly two fully accessible arc-component we show that there exists an embedding of X' with exactly two simple dense canals.

We conclude the paper with the complete characterization of sets of accessible points (and thus also the prime end structure of the corresponding circle of prime ends) of the standard two embeddings: the Bruin's embedding of X' (Section 10) and the Brucks-Diamond embedding of X' (Section 11) using symbolic dynamics. In Section 10 we show that for the Bruin's embedding of X' there is exactly one fully accessible non-degenerate arc-component and no other point from the embedding of X' is accessible. We show that if X' is not the Knaster continuum, then Bruin's embedding of X' has exactly one simple dense canal. In Section 11 we explicitly calculate the extrema of cylinders and neighbourhoods of folding points and obtain equivalent results as obtained recently by Boyland, de Carvalho and Hall in [10]. Moreover, since the symbolic description makes it possible to distinguish endpoints within the set of folding points, our results extend the classification given in [10].

2. Preliminaries on symbolic dynamics

In [2] uncountably many non-equivalent planar embeddings of indecomposable X' were constructed with the use of symbolic dynamics by making any given $x \in X'$ accessible.

We give a short overview of symbolic dynamics but we refer to [2] and [14] for the more complete picture.

The *kneading sequence* of a map T is a right-infinite sequence $\nu = c_1c_2\ldots \in \{0,1\}^\infty$, where

$$c_i = \begin{cases} 0, & T^i(c) \in [0,c], \\ 1, & T^i(c) \in [c,1], \end{cases}$$

for all $i \in \mathbb{N}$. If $c_n = c$ for some $n \in \mathbb{N}$, the critical point c is periodic and the ambiguity in the definition of ν is resolved by defining ν to be the smaller of $(c_1\ldots c_{n-1}0)^\infty$ and $(c_1\ldots c_{n-1}1)^\infty$ in the *parity-lexicographical ordering on* $\{0,1\}^\infty$ defined below.

By $\#_1(a_1\ldots a_n)$ we denote the number of ones in a finite word $a_1\ldots a_n \in \{0,1\}^n$; it can be either even or odd. Choose $t = t_1t_2\ldots \in \{0,1\}^\infty$ and $s = s_1s_2\ldots \in \{0,1\}^\infty$ such that $s \neq t$. Take the smallest $k \in \mathbb{N}$ such that $s_k \neq t_k$. Then the *parity-lexicographical ordering* is defined as

$$s \prec t \Leftrightarrow \begin{cases} s_k < t_k \text{ and } \#_1(s_1\ldots s_{k-1}) \text{ is even, or} \\ s_k > t_k \text{ and } \#_1(s_1\ldots s_{k-1}) \text{ is odd.} \end{cases}$$

Fix the kneading sequence $\nu = c_1c_2\ldots$. The finite word $a_1\ldots a_n \in \{0,1\}^n$ is called *admissible* if $c_2c_3\ldots \preceq a_i\ldots a_n \preceq c_1c_2\ldots$ for every $i \in \{1,\ldots,n\}$. Two-sided infinite sequence $\ldots s_{-2}s_{-1}.s_0s_1\ldots \in \{0,1\}^{\mathbb{Z}}$ is called *admissible* if every finite subword is admissible. Analogously we define an admissible left- or right-infinite sequence. Additionally, two-sided sequences $0^\infty s_ks_{k+1}\ldots$ will also be called admissible if $s_k = 1$ and every finite subword of the right-infinite sequence $s_ks_{k+1}\ldots$ is admissible. Denote the set of all admissible two-sided infinite sequences by Σ_{adm}.

The set $\Sigma_{adm} \subset \Sigma = \{0,1\}^{\mathbb{Z}}$ inherits the topology of Σ given by the metric

$$d((s_i)_{i\in\mathbb{Z}}, (t_i)_{i\in\mathbb{Z}}) := \sum_{i\in\mathbb{Z}} \frac{|s_i - t_i|}{2^{|i|}}.$$

Define the *shift homeomorphism on symbolic sequences* $\sigma_\Sigma \colon \Sigma \to \Sigma$ as

$$\sigma_\Sigma(\ldots s_{-2}s_{-1}.s_0s_1\ldots) := \ldots s_{-1}s_0.s_1s_2\ldots$$

The continuum X is homeomorphic to the space $\Sigma_{adm}/\!\sim$ (see Proposition 2 in [2]), where $\sim$ is the equivalence relation on Σ_{adm} given by

$$s \sim t \Leftrightarrow \begin{cases} \text{either } s_i = t_i \text{ for every } i \in \mathbb{Z}, \\ \text{or if there exists } k \in \mathbb{Z} \text{ such that } s_i = t_i \text{ for all } i \neq k \text{ but } s_k \neq t_k \\ \text{and } s_{k+1}s_{k+2}\ldots = t_{k+1}t_{k+2}\ldots = \nu. \end{cases}$$

Sequences of the form $0^\infty s_ks_{k+1}\ldots$, treated differently in the definitions above, correspond to the points from $\mathcal{C}$. By removing these sequences from the definition of Σ_{adm}, we get a space homeomorphic to the core X'. Shifts σ and σ_Σ are conjugated (see Theorem 2.5 in [14]). Thus we will from here onwards abuse the notation and denote both σ and σ_Σ by σ.

A point $x \in X$ is identified with a point in $\Sigma_{adm}/\!\sim$ by the equivalence class of $\bar{x} = \overleftarrow{x}.\overrightarrow{x} = (x_i)_{i\in\mathbb{Z}}$ according to the following rule:

$$x_i = \begin{cases} 0, & \pi_i(x) \in [0,c], \\ 1, & \pi_i(x) \in [c,1], \end{cases}$$

for $i \leq 0$ and

$$x_i = \begin{cases} 0, & T^i(\pi_0(x)) \in [0,c], \\ 1, & T^i(\pi_0(x)) \in [c,1], \end{cases}$$

for $i \in \mathbb{N}$. If the ambiguity in the definition of x_i happens more than once, then c is periodic and we study the itinerary of the modified kneading sequence instead. That way for every $x \in X$ there are at most two corresponding identified itineraries.

An *arc* is a homeomorphic image of an interval $[0,1] \subset \mathbb{R}$. A key fact for constructing embeddings in [2] is that X can be represented as the union of *basic arcs* defined below.

From now on, when we speak about left infinite sequences we omit minuses in indices and write $\overleftarrow{s} = \dots s_2 s_1$ for the sake of brevity.

Definition 2.1. *Let $\overleftarrow{s} = \dots s_2 s_1 \in \{0,1\}^\infty$ be an admissible left-infinite sequence. The set*

$$A(\overleftarrow{s}) := \{x \in X; \overleftarrow{x} = \overleftarrow{s}\} \subset X$$

is called a basic arc.

Remark 2.2. *Let $\overleftarrow{s} = \dots s_2 s_1 \in \{0,1\}^\infty$ be an admissible left-infinite sequence. There is a one-to-one correspondence between sequences $\overleftarrow{s}$ and basic arcs $A(\overleftarrow{s})$. When it is clear from the context that we refer to the basic arc $A(\overleftarrow{s})$ we abbreviate notation and write only $\overleftarrow{s}$.*

Note that $\pi_0 : A(\overleftarrow{s}) \to [0,1]$ is injective. In [16, Lemma 1] it was observed that $A(\overleftarrow{s})$ is either an arc or it is degenerate. For every basic arc we define two quantities as follows:

$$\begin{aligned} \tau_L(\overleftarrow{s}) &:= \sup\{n > 1 : s_{n-1}\dots s_1 = c_1c_2\dots c_{n-1}, \#_1(c_1\dots c_{n-1}) \text{ odd}\}, \\ \tau_R(\overleftarrow{s}) &:= \sup\{n \geq 1 : s_{n-1}\dots s_1 = c_1c_2\dots c_{n-1}, \#_1(c_1\dots c_{n-1}) \text{ even}\}. \end{aligned}$$

Lemma 2.3. [16, Lemma 2] *Let $\overleftarrow{s} \in \{0,1\}^\infty$ be an admissible left-infinite sequence such that $\tau_L(\overleftarrow{s}), \tau_R(\overleftarrow{s}) < \infty$. Then*

$$\pi_0(A(\overleftarrow{s})) = [T^{\tau_L(\overleftarrow{s})}(c), T^{\tau_R(\overleftarrow{s})}(c)].$$

If $\overleftarrow{t} \in \{0,1\}^\infty$ is another admissible left-infinite sequence such that $s_i = t_i$ for all $i > 0$ except for $i = \tau_R(\overleftarrow{s}) = \tau_R(\overleftarrow{t})$ (or $i = \tau_L(\overleftarrow{s}) = \tau_L(\overleftarrow{t})$), then $A(\overleftarrow{s})$ and $A(\overleftarrow{t})$ have a common boundary point.

Let $\omega(c)$ denote the set of all accumulation points of the forward orbit of the critical point c by the map T.

Proposition 2.4. [28, Theorem 2.2] *A point $x \in X$ is a folding point if and only if $\pi_n(x) \in \omega(c)$ for every $n \in \mathbb{N}$.*

We have the following symbolic characterization of endpoints in X.

Proposition 2.5. [16, Proposition 2] *A point $x \in X$ such that $\pi_i(x) \neq c$ for every $i < 0$ is an endpoint of X if and only if $\tau_L(\overleftarrow{x}) = \infty$ and $x_0 = \inf\pi_0(A(\overleftarrow{x}))$ or $\tau_R(\overleftarrow{x}) = \infty$ and $x_0 = \sup\pi_0(A(\overleftarrow{x}))$.*

If $\pi_i(x) = c$ for some $i < 0$, then x is an endpoint of X' if and only if $\sigma^i(x)$ is an endpoint. We can apply Proposition 2.5 to $\sigma^i(x)$ in this case.

Fix admissible left-infinite sequence $L = \dots l_2 l_1 \in \{0,1\}^\infty$. Space $\Sigma_{adm}/\sim$ is embedded in the plane with respect to any chosen L as a subset of $[0,1] \times C$, where $C \subset [0,1]$ denotes the Cantor set

$$C := [0,1] \setminus \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} (\frac{3k+1}{3^m}, \frac{3k+2}{3^m}).$$

Every basic arc $A(\overleftarrow{s})$ is embedded as the horizontal arc $\pi_0(A(\overleftarrow{s})) \times \{\psi_L(\overleftarrow{s})\}$, where

$$\psi_L(\overleftarrow{s}) := \sum_{i=1}^{\infty} (-1)^{\#_1(l_i \dots l_1) - \#_1(s_i \dots s_1)} 3^{-i} + \frac{1}{2},$$

This implies a linear order on the left-infinite sequences (=basic arcs) in which L is the largest. The precise definition is given by:

Definition 2.6. *Let $\overleftarrow{s}, \overleftarrow{t} \in \{0,1\}^\infty$ and let $k \in \mathbb{N}$ be the smallest natural number such that $s_k \neq t_k$. Then*

$$\overleftarrow{s} \prec_L \overleftarrow{t} \Leftrightarrow \begin{cases} t_k = l_k \text{ and } \#_1(s_{k-1} \dots s_1) - \#_1(l_{k-1} \dots l_1) \text{ even, or} \\ s_k = l_k \text{ and } \#_1(s_{k-1} \dots s_1) - \#_1(l_{k-1} \dots l_1) \text{ odd.} \end{cases} \tag{1}$$

If two basic arcs have a common boundary point, the embedded arcs are joined with a semi-circle on the left (right) if $\tau_L(\overleftarrow{s}) = \tau_L(\overleftarrow{t})$ ($\tau_R(\overleftarrow{s}) = \tau_R(\overleftarrow{t})$), see Lemma 2.3.

Throughout the paper, L will denote the left-infinite sequence of the largest basic arcs which determines the planar embedding φ_L of X by the rules in the equation (1). Let us fix the inverse limit space X. Denote by $\mathcal{E}$ the family of all embeddings of X constructed in [2], *i.e.,* with respect to all admissible tails L and refer to them as $\mathcal{E}$-embeddings. From now onwards we think of X as a planar continuum obtained by an $\mathcal{E}$-embedding of $\Sigma_{adm}/\sim$ described above.

We want to describe the sets of accessible points of embedded X, focusing primarily on the fully accessible arc-components. Since the approach in this study is mostly symbolic, we need to obtain a symbolic description of an arc-component in X. Recall that $\mathcal{U}_x$ denotes the arc-component of $x \in X$.

Definition 2.7. *We say that a point $x \in X$ is a* spiral point *if there exists a ray $R \subset X$ such that x is an endpoint of R and $[x, y] \subset R$ contains infinitely many basic arcs for every $x \neq y \in R$.*

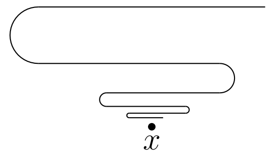

FIGURE 1. Point $x \in X$ is a spiral point.

Proposition 2.8. *If $x \in X$ is a spiral point, then $A(\overleftarrow{x})$ is degenerate and x is an endpoint of X.*

Proof. Assume that $A(\overleftarrow{x})$ is not degenerate. Note that x is not in the interior of $A(\overleftarrow{x})$ since then $R \cup A(\overleftarrow{x})$ is a triod. Assume wlog that x is the right endpoint of $A(\overleftarrow{x})$. If $\tau_R(A(\overleftarrow{x})) < \infty$, then by Lemma 2.3 there exists $y \in X$ such that $A(\overleftarrow{y})$ and $A(\overleftarrow{x})$ are connected by a semi-circle. If $A(\overleftarrow{y})$ is non-degenerate, then X again contains a triod. If $A(\overleftarrow{y})$ is degenerate, then $y = x$ is an endpoint of X, which is not possible since x is contained in the interior of an arc $A(\overleftarrow{x}) \cup R$. Therefore, $A(\overleftarrow{x})$ is degenerate.
Since $A(\overleftarrow{x})$ is degenerate it follows from Lemma 2.3 that $\tau_L(\overleftarrow{x}) = \infty$ or $\tau_R(\overleftarrow{x}) = \infty$. Thus, since $x_0 = \inf\pi_0(A(\overleftarrow{x})) = \sup\pi_0(A(\overleftarrow{x}))$, it follows by Proposition 2.5 that point x is an endpoint of X. $\square$

The following corollary follows directly from Proposition 2.8 since a spiral point cannot be contained in the interior of an arc.

Corollary 2.9. *Non-degenerate arc-components in X are:*

- *lines (*i.e., *continuous images of $\mathbb{R}$) with no spiral points,*
- *rays (continuous images of $\mathbb{R}^+$), where only the endpoint can be a spiral point,*
- *arcs, where only endpoints can be spiral points.*

Remark 2.10. *Let $y \neq z \in X$. By Lemma 2.3, $A(\overleftarrow{y})$ and $A(\overleftarrow{z})$ are connected by finitely many basic arcs if and only if there exists $k \in \mathbb{N}$ such that $\dots y_{k+1}y_k = \dots z_{k+1}z_k$. We say that y and z have the same* tail. *Thus every arc-component is determined by its tail with the exception of (one or two) spiral points with different tails. This generalizes the symbolic representation of arc-components for finite critical orbit c given in* [14] *on arbitrary tent inverse limit space X.*

3. General results about accessibility

Definition 3.1. *We say that a continuum $K \subset \mathbb{R}^2$* does not separate the plane *if $\mathbb{R}^2 \setminus K$ is connected.*

For $K \subset \mathbb{R}^2$ we denote by $\mathrm{Cl}(K)$ the closure of K in $\mathbb{R}^2$. The following proposition is a special case of Theorem 3.1. in [11].

Proposition 3.2. *Let $K \subset \mathbb{R}^2$ be a non-degenerate indecomposable continuum which does not separate the plane and let $A = [x, y] \subset K$ be an arc. If x and y are accessible, then A is fully accessible.*

Proof. Assume by contradiction that arc A is not fully accessible. Because $x, y \in K$ are both accessible there exists a point $z \in \mathbb{R}^2 \setminus K$ and arcs $A_x := [x, z], A_y := [y, z] \subset \mathbb{R}^2$ such that $(x, z], (y, z] \subset \mathbb{R}^2 \setminus K$.

Note that $A \cup A_x \cup A_y =: S$ is a simple closed curve in $\mathbb{R}^2$, see Figure 2. Thus $\mathbb{R}^2 \setminus S = S_1 \cup S_2$ where S_1 and S_2 are (open) sets in $\mathbb{R}^2$ such that $\partial S_1 = \partial S_2 = S$. Specifically S_1 contains no accumulation points of S_2 and vice versa. Denote by $K_1 := K \cap \mathrm{Cl}(S_1)$, $K_2 := K \cap \mathrm{Cl}(S_2)$. Note that K_1, K_2 are subcontinua of K and $K_1, K_2 \neq \emptyset$. Because A is not fully accessible it follows that $K_1, K_2 \neq K$. Furthermore $K_1 \cup K_2 = K$, which is a contradiction with K being indecomposable. □

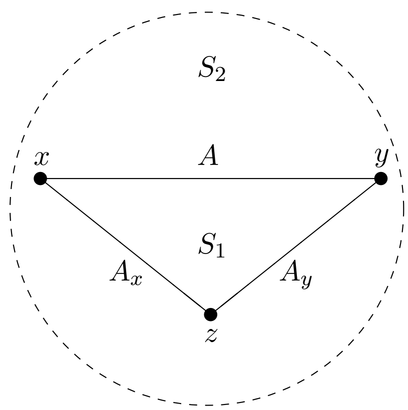

FIGURE 2. Simple closed curve from the proof of Theorem 3.2.

Corollary 3.3. *Let K be an indecomposable planar continuum which does not separate the plane and let $\mathcal{U}$ be an arc-component of K. There are four possibilities regarding the accessibility of $\mathcal{U}$:*

- *$\mathcal{U}$ is fully accessible.*
- *There exists an accessible point $u \in \mathcal{U}$ such that one component of $\mathcal{U} \setminus \{u\}$ is not accessible, and the other one is fully accessible.*
- *There exist two (not necessarily different) accessible points $u, v \in \mathcal{U}$ such that $\mathcal{U} \setminus [u, v]$ is not accessible and $[u, v] \subset \mathcal{U}$ is fully accessible.*
- *$\mathcal{U}$ is not accessible.*

Proof. By Proposition 3.2, the set of accessible points in $\mathcal{U}$ is connected. To see it is closed, take a sequence $(x_i)_{i \in \mathbb{N}}$ of accessible points in $\mathcal{U}$ such that $\lim_{i \to \infty} x_i =: x \in \mathcal{U}$. Let $z \in \mathbb{R}^2 \setminus K$ and let $A_i \subset \mathbb{R}^2$ be arcs with endpoints x_i and z and such that $A_i \cap K = x_i$ for every $i \in \mathbb{N}$. Denote by S_i the bounded open set in $\mathbb{R}^2$ with boundary $A_1 \cup A_i \cup [x_1, x_i]$, where $[x_1, x_i] \subset \mathcal{U}$. Note that $K \cap S_i = \emptyset$ for every $i \in \mathbb{N}$, since otherwise K is decomposable by arguments similar as in the proof of Proposition 3.2.

Then also $K \cap (\cup_{i\in\mathbb{N}} S_i) = \emptyset$. Since x is contained in the boundary of $\cup_{i\in\mathbb{N}} S_i$, which is arc-connected, we conclude that x can be accessed with a ray from $\cup_{i\in\mathbb{N}} S_i \subset \mathbb{R}^2 \setminus K$. $\square$

Remark 3.4. *Note that it follows from the third item of Corollary 3.3 that there can exists an endpoint $u = v \in \mathcal{U}$ which is accessible and every $x \in \mathcal{U} \setminus \{u\}$ is not accessible. For instance such embeddings are described in* [29] *for Knaster continuum where the endpoint is the only accessible point in the arc-component $\mathcal{C}$. In the course of the paper we show that all cases from Corollary 3.3 indeed occur in some embeddings of tent inverse limit spaces.*

4. Basic notions from the prime end theory

In this section we briefly recall the Carathéodory's prime end theory. Although the focus of this paper is not on the characterization of prime ends, we will include the study of prime ends of some interesting examples and in general study of standard planar embeddings at the end of the paper.

Definition 4.1. *Let $K \subset \mathbb{R}^2$ be a plane non-separating continuum. A* crosscut *of $\mathbb{R}^2 \setminus K$ is an arc $Q \subset \mathbb{R}^2$ which intersects K only in its endpoints. Note that $K \cup Q$ separates the plane into two components, one bounded and the other unbounded. Denote the bounded component by B_Q. A sequence $\{Q_i\}$ of crosscuts is called a* chain, *if the crosscuts are pairwise disjoint,* $\operatorname{diam} Q_i \to 0$ *and $B_{Q_{i+1}} \subset B_{Q_i}$ for every $i \in \mathbb{N}$. We say that two chains $\{Q_i\}$ and $\{R_i\}$ are* equivalent *if for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $B_{R_j} \subset B_{Q_i}$ and for every $j \in \mathbb{N}$ there exists $i' \in \mathbb{N}$ such that $B_{Q_{i'}} \subset B_{R_j}$. An equivalence class $[\{Q_i\}]$ is called a* prime end. *A basis for the natural topology on the set of all prime ends consists of sets $\{[\{R_i\}] : B_{R_i} \subset B_Q \text{ for all } i\}$ for all crosscuts Q. The set of prime ends equipped with the natural topology is a topological circle, called the* circle of prime ends, *see* e.g. *Section 2 in* [11].

Definition 4.2. *Let $P = [\{R_i\}]$ be a prime end. The* principal set of P *is* $\Pi(P) = \{\lim Q_i : \{Q_i\} \in P \text{ is convergent}\}$ *and the* impression of P *is* $I(P) = \cap_i \mathrm{Cl}(B_{R_i})$. *Note that both $\Pi(P)$ and $I(P)$ are subcontinua in X' and $\Pi(P) \subseteq I(P)$. We say that P is of the*

(1) first kind *if $\Pi(P) = I(P)$ is a point.*
(2) second kind *if $\Pi(P)$ is a point and $I(P)$ is non-degenerate.*
(3) third kind *if $\Pi(P) = I(P)$ is non-degenerate.*
(4) fourth kind *if $\Pi(P) \subsetneq I(P)$ are non-degenerate.*

Theorem 4.3 (Iliadis [21])**.** *Let K be a plane non-separating indecomposable continuum. The circle of prime ends corresponding to K can be decomposed into open intervals and their boundary points such that every open interval U uniquely corresponds to a composant of K which is accessible in more than one point and $I(e) \subsetneq K$ for every $e \in U$. For the boundary points e it holds that $I(e) = K$.*

Proposition 4.4. *Let K be a plane non-separating continuum such that every proper subcontinuum of K is an arc and such that every composant contains at most one folding* *point. Then $\Pi(P)$ is degenerate or equal to K for every prime end P. Specially, there exist no prime ends of the fourth kind.*

Proof. Assume there exists a prime end P such that $\Pi(P)$ is non-degenerate and not equal to K. Then $\Pi(P) = [a,b]$ is an arc in K. We claim that both a and b are folding points. Assume that there exists $\varepsilon > 0$ such that $B(a,\varepsilon) \cap K = C \times I$, where C is the Cantor set and $B(a,\varepsilon)$ denotes the open planar ball of radius ε around the point a. Since $a \in \Pi(P)$, there exist a chain of crosscuts $\{Q_i\} \in P$ such that $Q_i \to a$ as $i \to \infty$. Note that $Q_i \in B(a,\varepsilon)$ for large enough i, so the endpoints of Q_i are contained in $C \times I$ and the interior of Q_i does not intersect K. Therefore, it is possible to translate every Q_i along I and find a point $x \notin [a,b]$ for which there exists a chain of crosscuts $\{R_i\}$ equivalent to $\{Q_i\}$ such that $R_i \to x$ as $i \to \infty$, see Figure 3. This contradicts the assumption, *i.e.,* point a is a folding point. The proof for the point b is analogous. We conclude that there exists a composant with at least two folding points, which is a contradiction. □

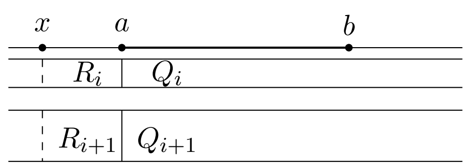

FIGURE 3. Translating the chain of crosscuts along I in Proposition 4.4.

Definition 4.5. *Let K be a plane non-separating continuum. A prime end P such that $\Pi(P)$ is non-degenerate but different than K is called an* infinite canal. *A third kind prime end P such that $\Pi(P) = I(P) = K$ is called a* simple dense canal.

We obtain the following corollary, which we use later in the paper for discussing the prime end structure of $\mathcal{E}$-embeddings of X when the critical orbit is finite.

Corollary 4.6. *Let K be an indecomposable plane non-separating continuum such that its every subcontinuum is an arc and every composant contains at most one folding point. Then the circle of prime ends corresponding to K can be partitioned into open intervals and their endpoints. Open intervals correspond to accessible open arcs in K. The endpoints of open intervals are the second or the third kind prime ends for which the impression is K. The second kind prime end corresponds to an accessible folding point in K and the third kind prime end corresponds to a simple dense canal in K.*

Question. If X' is a core of a tent map inverse limit, is there a planar embedding $\varphi\colon X' \to \mathbb{R}^2$ such that $\varphi(X')$ has fourth kind prime end?

5. An Introduction to the study of accessible points of $\mathcal{E}$-embeddings

By Corollary 3.3, if $x \in \mathcal{U}_x \subset X$ is accessible it does not a priori follow that every point from $\mathcal{U}_x$ is accessible, see *e.g.* Figure 4. Recall that $X = \mathcal{C} \cup X'$. In this paper we study the accessible sets of embeddings of either X or X' and the two cases substantially differ as we will see in this section. In the rest of the paper we are concerned only with embeddings of the cores X'.

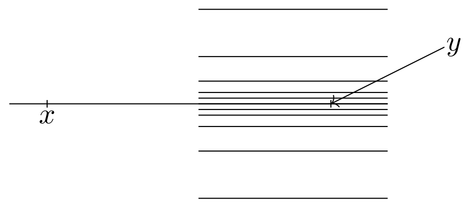

FIGURE 4. Point x is accessible from the complement while point y which has neighbourhood of Cantor set of arcs is not.

We will denote the smallest admissible left-infinite tail in X' with respect to $\prec_L$ by S. The arc-component of points from L (S) will be denoted from now onwards by $\mathcal{U}_L$ ($\mathcal{U}_S$). The following examples show that $\mathcal{U}_L$ and $\mathcal{U}_S$ do not necessarily coincide. Later in this section we will especially be concerned with the accessibility of $\mathcal{U}_L$ and $\mathcal{U}_S$.

Example 1. Assume that the kneading sequence is given by $\nu = (101)^\infty$. Embed X' in the plane according to the ordering in which $L = \dots(01)(01)(01)$ is the largest. Note that the smallest sequence is then $S = \dots(10)(10)(10) \not\subset \mathcal{U}_L$.

Example 2. Take the kneading sequence $\nu = 1001(101)^\infty$. Embed X' in the plane according to the ordering in which $L = \dots(001)(001101)(001)(001101)$ is the largest. The smallest is then $S = \dots(100)(101100)(100)(101100) \not\subset \mathcal{U}_L$. Note that in comparison with the previous example this time $S \neq \sigma^k(L)$ for every $k \in \mathbb{N}$.

Definition 5.1. *Let ν be a kneading sequence. For any admissible finite word $a_n \dots a_1 \in \{0,1\}^n$ define the* cylinder $[a_n \dots a_1]$ *as*

$$[a_n \dots a_1] := \{\overleftarrow{s} = \dots s_{n+2}s_{n+1}a_n \dots a_1 : \overleftarrow{s} \text{ is an admissible left infinite sequence}\}.$$

Lemma 5.2. *If $a_n \dots a_1$ is admissible, then $[a_n \dots a_1]$ is not an empty set.*

Proof. Say that $1a_n \dots a_1$ is not admissible. In that case $1a_n \dots a_1 \succ c_1 \dots c_{n+1}$, so $a_n \dots a_1 \prec c_2 \dots c_{n+1}$, which is a contradiction with $a_n \dots a_1$ being admissible. Note that the left infinite tail $1^\infty a_n \dots a_1$ is admissible, which concludes this proof. $\square$

Definition 5.3. *Assume X is embedded in the plane with respect to $L = \dots l_2 l_1$ and take an admissible finite word $a_n \dots a_1$. The* top *of the cylinder $[a_n \dots a_1]$ is the left infinite tail denoted by $L_{a_n \dots a_1} \in [a_n \dots a_1]$ such that $L_{a_n \dots a_1} \succeq_L \overleftarrow{s}$, for all $\overleftarrow{s} \in [a_n \dots a_1]$. Analogously we define the* bottom *of the cylinder $[a_n \dots a_1]$, denoted by $S_{a_n \dots a_1}$, as the smallest left infinite sequence in $[a_n \dots a_1]$ with respect to the order $\prec_L$.*

Remark 5.4. *Note that each cylinder is a compact set (as a subset of the plane). Thus for admissible finite words $a_n \dots a_1$ there always exist $L_{a_n \dots a_1}$ and $S_{a_n \dots a_1}$ (they can be equal).*

Lemma 5.5. *Assume X is embedded in the plane with respect to L. For every admissible finite word $a_n \dots a_1$ the arcs $A(L_{a_n \dots a_1})$ and $A(S_{a_n \dots a_1})$ are fully accessible.*

Proof. Take a point $x \in A(L_{a_n\dots a_1})$ and denote by $p_x = \psi(L_{a_n\dots a_1})$ the point in the Cantor set C corresponding to the y-coordinate of x. Then the arc

$$A = \left\{ \left(\pi_0(x), p_x + \frac{t}{2 \cdot 3^{n+1}} \right), t \in [0,1] \right\}$$

has the property that $A \cap X = \{x\}$, see Figure 5. When $x \in A(S_{a_n\dots a_1})$, we can analogously construct the arc A' such that $A' \cap X = \{x\}$ and conclude that x is accessible. $\square$

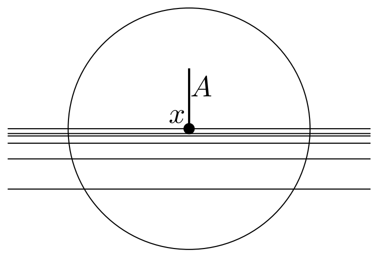

FIGURE 5. Point at the top of the cylinder $[a_n \dots a_1]$ is accessible by an arc A.

From Lemma 5.5 it follows specially that $A(L)$ and $A(S)$ in Example 1 and Example 2 are fully accessible as they are the largest and the smallest arcs respectively among all the arcs in embedding of X' determined by L.
The following proposition is the first step in determining the set of accessible points of $\mathcal{E}$-embeddings.

Proposition 5.6. *Take $L = \dots l_2 l_1$ and construct the embedding of X with respect to L. Then every point in X with the same symbolic tail as L is accessible. If $A(L)$ is not a spiral point, then $\mathcal{U}_L$ is fully accessible.*

Proof. Take a point $x \in X$, where $\overleftarrow{x} = \dots x_2 x_1$ and there exists $n > 0$ such that $\dots x_{n+2} x_{n+1} = \dots l_{n+2} l_{n+1}$. If $\#_1(x_n \dots x_1)$ and $\#_1(l_n \dots l_1)$ have the same parity, then $\dots l_{n+2} l_{n+1} x_n \dots x_1 = L_{x_n\dots x_1}$ and it is equal to the $S_{x_n\dots x_1}$ otherwise. Lemma 5.5, Corollary 3.3 and Remark 2.10 conclude the proof. $\square$

Definition 5.7. *Let $\varphi, \psi \colon K \to \mathbb{R}^2$ be two embeddings of a continuum K in the plane. We say that the embeddings are* equivalent *if the homeomorphism $\psi \circ \varphi^{-1} \colon \varphi(K) \to \psi(K)$ can be extended to a homeomorphism of the plane.*

By φ_L we denote the $\mathcal{E}$-embedding of X so that the arc $A(L)$ is the largest among all basic arcs. In the following proposition we observe that given L^1, L^2 with eventually the same tail, we get equivalent embeddings.

Proposition 5.8. *Let $L^1 = \dots l_2^1 l_1^1$ and $L^2 = \dots l_2^2 l_1^2$ be such that there exists $n \in \mathbb{N}$ so that for every $k > n$ it holds that $l_k^1 = l_k^2$. Then the embeddings φ_{L^1} and φ_{L^2} of X are equivalent.*

Proof. If $\#_1(l^1_n \dots l^1_1)$ and $\#_1(l^2_n \dots l^2_1)$ are of the same (different) parity, then for every admissible $\overleftarrow{x} = \dots x_2x_1$ and $\overleftarrow{y} = \dots y_2y_1$ such that $x_n \dots x_1 = y_n \dots y_1$ it follows that $\overleftarrow{x} \prec_{L^1} \overleftarrow{y}$ if and only if $\overleftarrow{x} \prec_{L^2} \overleftarrow{y}$ ($\overleftarrow{x} \succ_{L^2} \overleftarrow{y}$).
We conclude that $\varphi_{L^2} \circ \varphi^{-1}_{L^1} \colon \varphi_{L^1}(X) \to \varphi_{L^2}(X)$ preserves (reverses) the order in every n-cylinder $[a_n \dots a_1]$. There exists a planar homeomorphism h so that $h|_{\varphi_{L^1}(X)} = \varphi_{L^2}(X)$ and h permutes n-cylinders from the order determined by L^1 to the order determined by L^2, which concludes the proof. □

Now we briefly comment on $\mathcal{E}$-embeddings of X (including $\mathcal{C}$). For the rest of the section assume that X is not the Knaster continuum (since then $X = X'$, *i.e.*, $\mathcal{C}$ is contained in the core X'). Let X be embedded in the plane with respect to $L = \dots l_2l_1 \neq 0^\infty l_n \dots l_1$ for every $n \in \mathbb{N}$. The case when $\mathcal{E}$-embedding is equivalent to $L = 0^\infty$ (the Brucks-Diamond embedding from [14]) will be studied in Section 11.

Remark 5.9. *When the arc-component $\mathcal{C}$ is included, there exist cylinders $[a_n \dots a_1]$ where $a_n \dots a_1$ is not an admissible word, but there is $k \in \{1, \dots, n-1\}$ such that $a_k \dots a_1$ is admissible, $a_k = 1$ and $a_n \dots a_{k+1} = 0^{n-k}$. In that case, $[a_n \dots a_1]$ contains only one basic arc, that is $[a_n \dots a_1] = \{0^\infty a_n \dots a_1\}$ and $L_{a_n \dots a_1} = S_{a_n \dots a_1} = 0^\infty a_n \dots a_1$.*

Remark 5.10. *The arc-component $\mathcal{C}$ is isolated (when X is not the Knaster continuum), and thus it is fully accessible in any $\mathcal{E}$-embedding of X.*

Proposition 5.11. *Take an admissible left-infinite sequence $\overleftarrow{a} = \dots a_2a_1$ such that $A(\overleftarrow{a}) \not\subset \mathcal{C}$ and $a_n \neq l_n$ for infinitely many $n \in \mathbb{N}$. Then there exist sequences $(\overleftarrow{s_i})_{i\in\mathbb{N}}$ and $(\overleftarrow{t_i})_{i\in\mathbb{N}}$ such that $A(\overleftarrow{s_i}), A(\overleftarrow{t_i}) \subset \mathcal{C}$, $\overleftarrow{s_i}, \overleftarrow{t_i} \to \overleftarrow{a}$ as $i \to \infty$ and $\overleftarrow{s_i} \prec_L \overleftarrow{a} \prec_L \overleftarrow{t_i}$.*

Proof. First note that the assumption $A(\overleftarrow{a}) \not\subset \mathcal{C}$ is indeed needed since by Remark 5.10, $\mathcal{C}$ is isolated and thus the statement of the proposition does not hold for basic arcs from $\mathcal{C}$; thus assume $A(\overleftarrow{a}) \not\subset \mathcal{C}$.
Let $(N_i)_{i\in\mathbb{N}}$ be the sequence of natural numbers such that $a_n \neq l_n$ for $n \in \{N_i : i \in \mathbb{N}\}$. Since $a_n \neq l_n$ for infinitely many $n \in \mathbb{N}$ such sequence $(N_i)_{i\in\mathbb{N}}$ indeed exists. Denote by

$$\overleftarrow{t_i} := 0^\infty a^*_{N_{2i-1}} a_{N_{2i-1}-1} \dots a_1$$

$$\overleftarrow{s_i} := 0^\infty a^*_{N_{2i}} a_{N_{2i}-1} \dots a_1$$

for every $i \in \mathbb{N}$. By contradiction, if a sequence $\overleftarrow{t_i}$ is not admissible it holds that $1a_{N_{2i-1}-1} \dots a_1 \succ_L \nu$. Thus, $a_{N_{2i-1}-1} \dots a_1 \prec \overrightarrow{c_2}$ which is a contradiction with $a_{N_{2i-1}-1} \dots a_1$ being an admissible word. Thus $\overleftarrow{t_i}$ is admissible sequence and proof goes analogously for $\overleftarrow{s_i}$. Note that $A(\overleftarrow{t_i}), A(\overleftarrow{s_i}) \subset \mathcal{C}$ for every $i \in \mathbb{N}$.

Since $\#_1(a_{N_{2i-1}-1} \dots a_1)$ and $\#_1(l_{N_{2i-1}-1} \dots l_1)$ are of the same parity (the sequences differ on even number of entries) and $\#_1(a_{N_{2i}-1} \dots a_1)$ and $\#_1(l_{N_{2i}-1} \dots l_1)$ are of different parity (the sequences differ on odd number of entries), it holds that $\overleftarrow{s_i} \prec_L \overleftarrow{a} \prec_L \overleftarrow{t_i}$ for every $i \in \mathbb{N}$. □

Combining Proposition 5.6 with Proposition 5.11 we obtain that only basic arcs from $\mathcal{U}_L$ or $\mathcal{C}$ can be tops or bottoms of cylinders of $\mathcal{E}$-embeddings of X. Thus we obtain the following corollary.

Corollary 5.12. *If $A(L)$ is not a spiral point, then $\varphi_L(X)$ has exactly two fully accessible arc-components, namely $\mathcal{U}_L$ and $\mathcal{C}$ (for $s=2$, since $X_s = X'_s$ it is possible that $\mathcal{C} = \mathcal{U}_L$). If $A(L)$ is non-degenerate, there are two remaining points on the circle of prime ends and they correspond either to an infinite canal in X or to a folding point. If $A(L)$ is degenerate then there are no infinite canals in X.*

The following statements are going to be used often throughout the paper to determine that an arc-component is fully accessible.

Definition 5.13. *Let $\overleftarrow{s} = \dots s_2 s_1$ be an admissible left-infinite sequence. If $\tau_R(\overleftarrow{s}) < \infty$, the tail $\overleftarrow{r(s)} = \dots s_{\tau_R(\overleftarrow{s})+1} s^*_{\tau_R(\overleftarrow{s})} s_{\tau_R(\overleftarrow{s})-1} \dots s_1$ will be called* the right neighbour of *$\overleftarrow{s}$ and if $\tau_L(\overleftarrow{s}) < \infty$, the tail $\overleftarrow{l(s)} = \dots s_{\tau_L(\overleftarrow{s})+1} s^*_{\tau_L(\overleftarrow{s})} s_{\tau_L(\overleftarrow{s})-1} \dots s_1$ will be called* the left neighbour of *$\overleftarrow{s}$.*

Proposition 5.14. *Embed X' in the plane with respect to L. Assume $\overleftarrow{s}$ is at the bottom (top) of some cylinder. If $\overleftarrow{r(s)}$ is not the top (bottom) of any cylinder, then $A(\overleftarrow{r(s)})$ contains an accessible folding point, see Figure 6. Analogous statement holds for $\overleftarrow{l(s)}$.*

Proof. If $\overleftarrow{r(s)}$ is not the top of any cylinder, then there exist left-infinite admissible sequences $\overleftarrow{x_i} \succ_L \overleftarrow{r(s)}$ such that $\overleftarrow{x_i} \to \overleftarrow{r(s)}$ as $i \to \infty$. If $\tau_R(\overleftarrow{x_i}) = \infty$ for infinitely many $i \in \mathbb{N}$, we have found a folding point in $A(\overleftarrow{r(s)})$. So assume without the loss of generality that $\tau_R(\overleftarrow{x_i}) < \infty$ for all $i \in \mathbb{N}$. If $\overleftarrow{s} \succ_L \overleftarrow{r(x_i)}$ for infinitely many $i \in \mathbb{N}$ we get a contradiction with $\overleftarrow{s}$ being the top of some cylinder. But then $\overleftarrow{r(x_i)} \prec_L \overleftarrow{r(s)}$ for all but finitely many $i \in \mathbb{N}$ which gives a folding point in $A(\overleftarrow{r(s)})$ again. $\square$

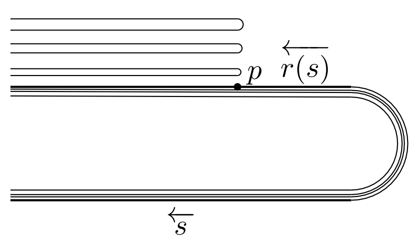

FIGURE 6. Setup of Proposition 5.14.

The following corollary follows directly from Proposition 5.14.

Corollary 5.15. *Let $\mathcal{U} \subset X'$ be an arc-component which contains no folding points and let X' be $\mathcal{E}$-embedded. If there exists a basic arc $A \subset \mathcal{U}$ that is fully accessible, then $\mathcal{U}$ is fully accessible.*

Remark 5.16. *When we embed only the core X', there can exist accessible points in $X' \setminus \mathcal{U}_L$, see* e.g. *Example 1 and Example 2. In these two examples $\mathcal{U}_S \neq \mathcal{U}_L$ and points from $A(S)$ are accessible. It can happen that $\mathcal{U}_S$ is fully accessible (see Lemma 9.1 from Section 9), but that is not always the case. In Section 7.2 we explicitly construct examples in which the arc-component $\mathcal{U}_S$ is only partially accessible.*

From Lemma 5.5 it follows that the points at top or bottom of cylinders are accessible. If a point which is not at the top or bottom of any cylinder has a neighbourhood homeomorphic to the Cantor set of arcs, we can conclude that is not accessible. However, the accessibility of folding points needs to be studied separately, since it is not straightforward to determine if they are accessible or not in a given embedding, see for example Figure 7. Thus we need to do a detailed study on conditions for a folding point to be accessible. For instance, in embeddings of the Knaster continuum in [29] the endpoint is always accessible.

Remark 5.17. *When the orbit of c is finite, with (pre)period $n \in \mathbb{N}$, there exist exactly n folding points (see* [15]*). They are contained in different arc-components which are permuted by the shift homeomorphism. If the orbit of c is periodic, the folding points are endpoints (see* [5]*).*

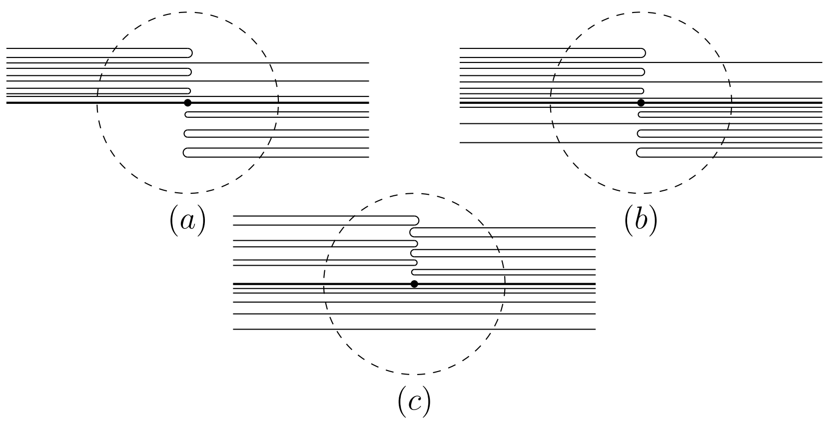

FIGURE 7. Neighbourhoods of folding points. In Case (a) and (c) folding point is accessible, while in Case (b) it is not.

6. Tops/bottoms of finite cylinders

In this section we study the symbolics of tops/bottoms of cylinders depending on an $\mathcal{E}$-embedding of X' and we restrict to cases where $L \neq 0^\infty l_n \dots l_1$ for all $n \in \mathbb{N}$.

For $t \in \{0,1\}$, we denote by $t^* = 1-t$. For $A = a_1 \dots a_n$ denote by ${}^*A = a_1^* a_2 \dots a_n$, $A^* = a_1 \dots a_{n-1} a_n^*$ and ${}^*A^* = a_1^* a_2 \dots a_{n-1} a_n^*$.

Definition 6.1. *Let ν be a kneading sequence. We say that a finite word $a_1 \dots a_n \in \{0,1\}^n$ is* irreducibly non-admissible *if it is not admissible and $a_2 \dots a_n$ is admissible.*

Definition 6.2. *Fix a kneading sequence ν. We say that a finite cylinder $B = [b_n \dots b_1]$ of length $n \in \mathbb{N}$* alters *$L = \dots l_2 l_1$, if there exist words $(A_i)_{i \in \mathbb{N}}$ such that $\dots A_3 A_2 A_1 = \dots l_{n+2} l_{n+1}$ and the words $A_1 B$ and ${}^*\!A^*{}_i {}^*\!A^*{}_{i-1} \dots {}^*\!A^*{}_2 {}^*\!A_1 B$ are irreducibly non-admissible for every $i \geq 2$.*

Proposition 6.3. *If a finite cylinder B alters the admissible sequence L then $L_B \not\subset \mathcal{U}_L$ or $S_B \not\subset \mathcal{U}_L$.*

Proof. Assume B alters L with words A_i as in the definition. If $\#_1(B) - \#_1(l_n \dots l_1)$ is even, then $L_B = \dots {}^*\!A^*{}_i {}^*\!A^*{}_{i-1} \dots {}^*\!A^*{}_2 {}^*\!A_1 B \not\subset \mathcal{U}_L$, since L_B differs from L on infinitely many places. If $\#_1(B) - \#_1(l_n \dots l_1)$ is odd, then $S_B = \dots {}^*\!A^*{}_i {}^*\!A^*{}_{i-1} \dots {}^*\!A^*{}_2 {}^*\!A_1 B \not\subset \mathcal{U}_L$. $\square$

The following example shows that there exist $\mathcal{E}$-embeddings of X' such that none of the extrema of certain cylinders are contained in $\mathcal{U}_L$.

Example 3. Let $\nu = (100111011)^\infty$ and $L = (001)^\infty 11$. Note that $S_{10} = (100)^\infty(101)10 \subset \mathcal{U}_{L_{10}}$ and $L_{10} = (100)^\infty 10 \subset \mathcal{U}_{L_{10}}$. Therefore, $L_{10}, S_{10} \not\subset \mathcal{U}_L$.

In Example 3 both extrema belong to the same arc-component. This is not necessarily always the case, see *e.g.* Example 4 below.

Proposition 6.4. *If B is such that $L_B \not\subset \mathcal{U}_L$ or $S_B \not\subset \mathcal{U}_L$, then there exists a finite word B' such that B' alters L.*

Proof. Assume $\#_1(B) - \#_1(l_n \dots l_1)$ is even and $L_B \not\subset \mathcal{U}_L$. Then obviously $B' = B$ alters L. Similarly, if $\#_1(B) - \#_1(l_n \dots l_1)$ is odd and $S_B \not\subset \mathcal{U}_L$. So assume $\#_1(B) - \#_1(l_n \dots l_1)$ is even and $S_B \not\subset \mathcal{U}_L$. Then $l^*_{n+1} B$ alters L, if $l^*_{n+1} B$ is admissible. If $l^*_{n+1} B$ is not admissible, there exists $i \in \mathbb{N}$ such that $l^*_{n+i} \dots l_n B$ is admissible, since otherwise $S_B = L_B$, which is a contradiction. Analogously, if $\#_1(B) - \#_1(l_n \dots l_1)$ is odd and $L_B \not\subset \mathcal{U}_L$. $\square$

Example 4. Let $\nu = 1001(101)^\infty$ and $L = \dots(001)(001101)(001)(001101)$. Then $S = S_0 = \dots(100)(101100)(100)(101100) \not\subset \mathcal{U}_L$. So $B = 0$ alters L and words A_i are divided by brackets.

Next we show there exist $\mathcal{E}$-embeddings with more than two accessible arc-components.

Proposition 6.5. *Assume that ν starts with some finite words $\nu = 1B \dots = 1ABA \dots$, where B^* and ABA^* are irreducibly non-admissible. The embedding of X' with respect to $L = \dots ABABABA$ contains at least three tails which are extrema of cylinders.*

Proof. Note that $S = \dots {}^*\!ABA^{**}\!B^{**}\!ABA^{**}\!B^{**}\!ABA^*$. Take any admissible word C such that $|C| = |A|$ and such that $\#_1(C) - \#_1(A)$ is even. Then $S_C = \dots A^{**}\!B^{**}\!ABA^{**}\!B^* C \not\subset \mathcal{U}_L \cup \mathcal{U}_S$ and therefore we found three different tails which are extrema of cylinders. $\square$

The following example shows that it is indeed possible to satisfy the conditions of Proposition 6.5.

Example 5. Take $\nu = 1001100100111\ldots$, $B = 001$, $A = 0011$ and $L = \ldots ABABA$. which is easily checked to be admissible. For C take *e.g.* $C = 1111$. Note that $S_{A^*} = \ldots\ldots {}^*ABA^*{}^*B^*{}^*ABA^*{}^*B^*{}^*ABA^*$ and $S_C = \ldots A^*{}^*B^*{}^*ABA^*{}^*B^*C$ and thus we obtain three accessible basic arcs with different tails. If we take *e.g.* $\nu = (10011001001111)^\infty$, since the only folding points are endpoints, by Lemma 5.14 it follows that there are three fully accessible non-degenerate dense arc-components. Moreover, none of those arc-components contains an endpoint so they are all lines. We will return to this particular example later in Example 9.

7. Accessible folding points

In this section we study accessibility of folding points which are not at the top or the bottom of any cylinder.

7.1. Accessible endpoints.

Let us fix X' and the $\mathcal{E}$-embedding depending on L. Recall that we denote by $\mathcal{U}_L$ the arc-component of $x \in A(L) \subset X'$. By Proposition 5.6, every point with the same symbolic tail as L is accessible.

The following remark is a direct consequence of Proposition 2.5.

Remark 7.1. *If $e \in X'$ is an endpoint, then there exists a strictly increasing sequence $(m_i)_{i\in\mathbb{N}}$ such that $\bar{e} = \ldots e_{m_i+1}c_1 \ldots c_{m_i}.c_{m_i+1}\ldots = \ldots e_{m_i+1}\nu$ for every $i \in \mathbb{N}$.*

In this section we work with the concept of an endpoint being capped which is defined below. See Figure 8.

Definition 7.2. *Let $e \in X'$ be an endpoint with $\tau_L(\overleftarrow{e}) = \infty$ ($\tau_R(\overleftarrow{e}) = \infty$). We say that a point e is* capped from the left (right)*, if there exist sequences of admissible itineraries $(\overleftarrow{y}^i)_{i\in\mathbb{N}}, (\overleftarrow{z}^i)_{i\in\mathbb{N}} \subset \{0,1\}^\infty$ such that $\overleftarrow{y}^i, \overleftarrow{z}^i \to \overleftarrow{e}$ as $i \to \infty$, $\overleftarrow{y}^i \prec_L \overleftarrow{e} \prec_L \overleftarrow{z}^i$ for every $i \in \mathbb{N}$ and arcs $A(\overleftarrow{y}^i)$ and $A(\overleftarrow{z}^i)$ are joined on the left (right).*

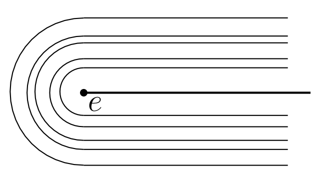

Figure 8. Endpoint e is capped from the left.

Remark 7.3. *If $e \in X'$ is a right (left) endpoint which is not capped from the right (left), then e is accessible by a horizontal arc in the plane. Note that if $\overleftarrow{e}$ lies on an extremum of a cylinder (which holds if* e.g. *e has the same symbolic tail as L), then e is not capped.*

Remark 7.4. *Let $\nu = 10^\infty$,* i.e., *$X = X'$ is a Knaster continuum and let L be arbitrary. Note that any two points $x, y \in X'$ that are $\varepsilon > 0$ close to the point $\bar{0}$ and are identified have the form $x_k x_{k-1}\ldots x_1 = y_k y_{k-1}\ldots y_1 = 10^{k-1}$ for some $k \in \mathbb{N}$. It follows that* *either* $\overleftarrow{x}, \overleftarrow{y} \prec_L \overleftarrow{0}$ *or* $\overleftarrow{x}, \overleftarrow{y} \succ_L \overleftarrow{0}$, *depending on the parity of* $\#_1(l_{k-1} \dots l_1)$. *Then endpoint* $\bar{0} \in X'$ *is not capped and thus always accessible in* $\mathcal{E}$*-embeddings, see Figure 9.*

From now on we assume in this subsection that X' is not the Knaster continuum and thus $\nu \neq 10^\infty$.

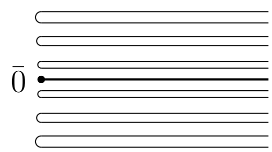

FIGURE 9. Neighbourhood of the end-point $\bar{0}$ of the Knaster continuum ($\nu = 10^\infty$) in an $\mathcal{E}$-embedding.

It is well known (see *e.g.* [5]) that X' contains endpoints if and only if the critical point c of map T is recurrent (*i.e.*, $T^n(c)$ get arbitrary close to c as $n \to \infty$).

Definition 7.5. *Fix a kneading sequence* ν *and let* $e \in X'$ *be an endpoint and thus* $\tau_L(\overleftarrow{e}) = \infty$ $(\tau_R(\overleftarrow{e}) = \infty)$. *A sequence* $(m_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ *is called the* complete sequence for e, *if for every* $n \in \mathbb{N}$ *such that* $e_n \dots e_1 = c_1c_2 \dots c_n$ *and* $\#_1(c_1c_2\dots c_n)$ *is odd (even) there exist* $i \in \mathbb{N}$ *such that* $m_i = n$.

From $\tau_L(\overleftarrow{e}) = \infty$ (or $\tau_R(\overleftarrow{e}) = \infty$) it follows that the sequence $(m_i)_{i\in\mathbb{N}}$ indeed exists. The main result in this subsection is that every endpoint of X' (where X' is not the Knaster continuum) which is not contained in $\mathcal{U}_L$ is capped in an $\mathcal{E}$-embedding of X' which is non-equivalent to Brucks-Diamond embedding from [14]. In the proof of Theorem 7.12 we construct an increasing subsequence $(n_i)_{i\in\mathbb{N}} \subset (m_i)_{i\in\mathbb{N}}$ and basic arcs $A(\overleftarrow{x}^{O(i)}), A(\overleftarrow{x}^{I(i)}) \subset \mathcal{R} \subset X'$ such that

$$\overleftarrow{x}^{O(i)} = 1^\infty a^i_k \dots a^i_1 0 c_1c_2\dots c_{n_i} \qquad \overleftarrow{x}^{I(i)} = 1^\infty a^i_k \dots a^i_1 1 c_1c_2\dots c_{n_i}. \tag{2}$$

and $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{O(i)}$ for some admissible word $a^i_k \dots a^i_1 \in \{0,1\}^k$. Note that the arcs $A(\overleftarrow{x}^{O(i)})$ and $A(\overleftarrow{x}^{I(i)})$ are joined by left (right) semi-circle. Here $\mathcal{R}$ denotes the arc-component of the fixed point $(\dots,1,1)$ which is a dense line in X' independently on the choice of ν (see Proposition 1 in [13]).

Remark 7.6. *Let* $e \in X'$ *be an endpoint and thus* $\tau_L(\overleftarrow{e}) = \infty$ $(\tau_R(\overleftarrow{e}) = \infty)$. *Then* $\#_1(c_1\dots c_{m_i})$ *is odd (even) and* $\#_1(c_1 \dots c_{m_{i+1}-m_i})$ *is even (even) for every* $i \in \mathbb{N}$.

Definition 7.7. *For* $\nu = c_1c_2\dots$ *we define*

$$\kappa := \min\{i - 2 : i \geq 3, c_i = 1\}.$$

Remark 7.8. *Definition 7.7 says that the beginning of the kneading sequence is* $\nu = 10^\kappa 1\dots$. *If* $\kappa = 1$, *since we restrict to non-renormalizable case for* T, *we can conclude even more, namely that* $\nu = 10(11)^n 0\dots$, *for some* $n \in \mathbb{N}$.

Remark 7.9. *Fix the kneading sequence* ν. *Assume that* $a_{n-1}\dots a_1 \in \{0,1\}^n$ *is admissible but* $a_n \dots a_1 \in \{0,1\}^n$ *is not. Then* $a_n \dots a_1 \prec c_2 \dots c_{n+1}$.

Lemma 7.10. *Let ν be an admissible kneading sequence. A word $c_2 \dots c_n^*$ is not admissible if and only if either $\#_1(c_2 \dots c_n)$ is odd or there exists $k \in \{3, \dots, n\}$ such that $c_k \dots c_n = c_2 \dots c_{n-k+2}$ and $\#_1(c_k \dots c_n)$ is odd.*

Proof. Assume that $c_2 \dots c_n^*$ is not admissible, so there exists $i \in \{2, \dots, n\}$ such that $c_i \dots c_n^*$ is not admissible. Take the largest such index i and note that $c_i \dots c_n = c_2 \dots c_{n-i+2}$ and $c_2 \dots c_{n-i+2}^* \prec c_2 \dots c_{n-i+2}$. Let us assume by contradiction that $\#_1(c_2 \dots c_{n-i+2})$ is even. If $c_{n-i+2} = 0$ ($c_{n-i+2} = 1$) it follows that $\#_1(c_2 \dots c_{n-i+1})$ is even (odd) and in both cases $c_2 \dots c_{n-i+2}^* \succ c_2 \dots c_{n-i+2}$ and thus $c_2 \dots c_{n-i+2}^*$ is admissible, a contradiction. □

Lemma 7.11. *Let ν be an admissible kneading sequence and let $(m_i)_{i \in \mathbb{N}}$ be the complete sequence for an endpoint $e \in X'$. Then for every $k \geq 3$ and $j \in \{0, \dots, m_i\}$, the word $c_k \dots c_{m_{i+1}-m_i}^* c_1 c_2 \dots c_j$ is admissible for every $i \in \mathbb{N}$. In specific, if $j = 0$, we set $c_1 \dots c_j = \emptyset$.*

Proof. Assume by contradiction that there exists $k \geq 3$ and $j \in \mathbb{N}_0$ such that the word $c_k \dots c_{m_{i+1}-m_i}^* c_1 c_2 \dots c_j$ is not admissible and assume that k is the largest and j is the smallest such index. By the choice of k and j every proper subword of $c_k \dots c_{m_{i+1}-m_i}^* c_1 c_2 \dots c_j$ is admissible. Thus $c_k \dots c_{m_{i+1}-m_i}^* c_1 c_2 \dots c_j = c_2 c_3 \dots c_{m_{i+1}-m_i-k}$ $c_{m_{i+1}-m_i-k+1} \dots c_{m_{i+1}-m_i-k+j+1}^*$ and $\#_1(c_2 c_3 \dots c_{m_{i+1}-m_i-k+j+1}^*)$ is even by Lemma 7.10. Furthermore, Lemma 7.10 implies that $\#_1(c_k \dots c_{m_{i+1}-m_i}^* = c_2 \dots c_{m_{i+1}-m_i-k-1})$ is even.

If $j = 1$, then both $\#_1(c_k \dots c_{m_{i+1}-m_i}^*)$ and $\#_1(c_k \dots c_{m_{i+1}-m_i}^* c_1)$ are even, which is impossible.

If $j \geq 2$, it follows by Lemma 7.10 that $\#_1(c_2 \dots c_j)$ is odd. Thus $c_2 \dots c_j^* = c_{m_{i+1}-m_i-k+1} \dots c_{m_{i+1}-m_i-k+j+1}$ is not admissible, which is a contradiction, since $c_2 c_3 \dots c_j^* \subset \nu$.

Let $c_1 \dots c_j$ be an empty word. Then $c_k \dots c_{m_{i+1}-m_i} = c_2 c_3 \dots c_{m_{i+1}-m_i-k} c_{m_{i+1}-m_i-k+1}$ and $\#_1(c_2 c_3 \dots c_{m_{i+1}-m_i-k} c_{m_{i+1}-m_i-k+1})$ is odd. Let l be the maximal natural number such that $c_{m_{i+1}-m_i+1} \dots c_{m_{i+1}-m_i+l} = c_{m_{i+1}-m_i-k+2} \dots c_{m_{i+1}-m_i-k+l+1}$, *i.e.*,

$$c_k \dots c_{m_{i+1}-m_i+l} = c_2 c_3 \dots c_{m_{i+1}-m_i-k+l+1}$$

and $c_{m_{i+1}-m_i+l+1} \neq c_{m_{i+1}-m_i-k+l+2}$. Such l indeed exists since (m_i) is complete. Note that $c_{m_{i+1}-m_i-k+2} \dots c_{m_{i+1}-m_i-k+l+1} = c_1 \dots c_l$ and $\#_1(c_1 \dots c_{l+1})$ is odd by Lemma 7.10. Thus $\#_1(c_1 \dots c_l c_{l+1}^*)$ is even and we conclude that $\#_1(c_2 \dots c_{m_{i+1}-m_i-k+l+2})$ is odd. Since $c_2 \dots c_{m_{i+1}-m_i-k+l+2}^* = c_k \dots c_{m_{i+1}-m_i+l+1}$ is admissible, we get a contradiction. □

The main idea of the proof of the following theorem is illustrated in the Example 6.

Theorem 7.12. *Let $e \in X'$ be an endpoint such that $\tau_R(\overleftarrow{e}) = \infty$ ($\tau_L(\overleftarrow{e}) = \infty$) and let $L = \dots l_2 l_1 \neq 0^\infty l_n \dots l_1$ be admissible and $\nu \neq 10^\infty$. If L and $\overleftarrow{e}$ have different tails, then e is capped from the right (left).*

Proof. Let $(m_i)_{i\in\mathbb{N}}\subset\mathbb{N}$ be the complete sequence for an endpoint e where $\tau_R(\overleftarrow{e})=\infty$. The proof works analogously if $\tau_L(\overleftarrow{e})=\infty$. We will find infinitely many $i\in\mathbb{N}$ such that $\overleftarrow{x}^{O(i)}\succ_L\overleftarrow{e}\succ_L\overleftarrow{x}^{I(i)}$ (or with reversed inequalities) and arcs $\overleftarrow{x}^{O(i)}$ and $\overleftarrow{x}^{I(i)}$ are joined by a semi-circle on the right.

Fix some $i\in\mathbb{N}$ and let $M_1(i)>m_i+1$ be the smallest natural number such that $l_{M_1(i)}=e^*_{M_1(i)}$. Note that such $M_1(i)$ exists, otherwise $\overleftarrow{e}$ and L have the same tail.

Assume that $M_1(i)=m_{i+1}$. Note that then $e_{M_1(i)-1}\dots e_{m_i+1}=c_2\dots c_k=0^{\kappa}1c_{\kappa+2}\dots c_k$ ($c_{\kappa+2}\dots c_k$ can be empty) and $e_{M_1(i)}=1$. Then $l_{M_1(i)}\dots l_{m_i+1}=0^{\kappa+1}1c_{\kappa+3}\dots c_k$, which is not admissible.

Assume that $M_1(i)=m_{i+1}+1$. By the paragraph above $M_1(i+1)\neq m_{i+2}$. If $M_1(i+1)=m_{i+2}+1$, then $l_{m_{i+2}}\dots l_{m_{i+1}+2}l_{m_{i+1}+1}=c_1\dots c_{m_{i+2}-m_{i+1}+1}c^*_{m_{i+2}-m_{i+1}}$ which is not admissible since $c_1\dots c_{m_{i+2}-m_{i+1}}$ is even by Remark 7.6. So either $M_1(i)\in\{m_i+2,\dots m_{i+1}-1\}$ or there is $k\in\mathbb{N}$ such that $M_1(i+k)=M_1(i)$. Note that there is infinitely many $i\in\mathbb{N}$ such that $M_1(i)\in\{m_i+2,\dots m_{i+1}-1\}$ and from now on we work with such $i\in\mathbb{N}$.

If both of the following sequences are admissible, we set:

$$\overleftarrow{x}^{O(i)}=1^{\infty}e^*_{M_1(i)}e_{M_1(i)-1}\dots e_{m_i+2}0e_{m_i}\dots e_1,$$

$$\overleftarrow{x}^{I(i)}=1^{\infty}e^*_{M_1(i)}e_{M_1(i)-1}\dots e_{m_i+2}1e_{m_i}\dots e_1.$$

a) Assume that $\#_1(e_{m_i}\dots e_1)$ and $\#_1(l_{m_i}\dots l_1)$ have the same parity and e_{m_i+1} $=l_{m_i+1}=0$.
Then it follows that $\overleftarrow{e}\succ_L\overleftarrow{x}^{I(i)}$. Because $l_{M_1(i)-1}\dots l_{m_i+2}=e_{M_1(i)-1}\dots e_{m_i+2}$ the parities of $\#_1(e_{M_1(i)-1}\dots e_1)$ and $\#_1(l_{M_1(i)-1}\dots l_1)$ are the same and because $l_{M_1(i)}=e^*_{M_1(i)}$ it follows that $\overleftarrow{x}^{O(i)}\succ_L\overleftarrow{e}$.

b) Assume that $\#_1(e_{m_i}\dots e_1)$ and $\#_1(l_{m_i}\dots l_1)$ have the same parity and $e_{m_i+1}=l_{m_i+1}=1$.
Then it follows that $\overleftarrow{e}\succ_L\overleftarrow{x}^{O(i)}$. Because $l_{M_1(i)-1}\dots l_{m_i+2}=e_{M_1(i)-1}\dots e_{m_i+2}$ the parities of $\#_1(e_{M_1(i)-1}\dots e_1)$ and $\#_1(l_{M_1(i)-1}\dots l_1)$ are the same and because $l_{M_1(i)}=e^*_{M_1(i)}$ it follows that $\overleftarrow{x}^{I(i)}\succ_L\overleftarrow{e}$.

c) Assume that $\#_1(e_{m_i}\dots e_1)$ and $\#_1(l_{m_i}\dots l_1)$ have the same parity and $e_{m_i+1}=1\neq 0=l_{m_i+1}$.
Then $\overleftarrow{x}^{O(i)}\succ_L\overleftarrow{e}$. Since the parities of $\#_1(e_{M_1(i)-1}\dots e_1)$ and $\#_1(l_{M_1(i)-1}\dots l_1)$ are different and $l_{M_1(i)}=e^*_{M_1(i)}$, it follows that $\overleftarrow{e}\succ_L\overleftarrow{x}^{I(i)}$.

d) Assume that $\#_1(e_{m_i}\dots e_1)$ and $\#_1(l_{m_i}\dots l_1)$ have the same parity and $e_{m_i+1}=0\neq 1=l_{m_i+1}$.
Then $\overleftarrow{x}^{I(i)}\succ_L\overleftarrow{e}$. Since the parities of $\#_1(e_{M_1(i)-1}\dots e_1)$ and $\#_1(l_{M_1(i)-1}\dots l_1)$ are different and $l_{M_1(i)}=e^*_{M_1(i)}$, it follows that $\overleftarrow{e}\succ_L\overleftarrow{x}^{O(i)}$.

Note that if $\#_1(e_{m_i}\dots e_1)$ and $\#_1(l_{m_i}\dots l_1)$ are of different parities, then all the inequalities in cases a), b), c) and d) are reversed and we use analogous arguments to conclude that either $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{O(i)}$.

Now assume that one of $e^*_{M_1(i)}e_{M_1(i)-1}\dots e_{m_i+2}e^{(*)}_{m_i+1}\dots e_1$ is not admissible (where $s^{(*)}$ means s^* or s). Then we set $x^{O(i)}_{M_1(i)} = x^{I(i)}_{M_1(i)} = e_{M_1(i)}$. If $e_{M_1(i)+1} = l_{M_1(i)+1}$, then we set $x^{O(i)}_{M_1(i)+1} = x^{I(i)}_{M_1(i)+1} = e^*_{M_1(i)+1}$ and we argue that $e^*_{M_1(i)+1}e_{M_1(i)}\dots e_{m_i+2}e^{(*)}_{m_i+1}\dots e_1 = e^*_{M_1(i)+1}10^{\kappa-1}1\dots e_1$ are admissible words. Indeed, word $e_{M_1(i)}\dots e_{m_i+2}e^{(*)}_{m_i+1}\dots e_1$ is admissible by Lemma 7.11. If $e^*_{M_1(i)+1}10^{\kappa-1}1\dots$ were not admissible, then $T^3(c) > T^4(c)$ which is a contradiction with T being non-renormalizable. So the following sequences are admissible:

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{M_1(i)+1}e_{M_1(i)}\dots e_{m_i+2}0e_{m_i}\dots e_1,$$
$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{M_1(i)+1}e_{M_1(i)}\dots e_{m_i+2}1e_{m_i}\dots e_1,$$

and $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$.

Assume that $e^*_{M_1(i)+1} = l_{M_1(i)+1}$. Set $x^{O(i)}_{M_1(i)+1} = x^{I(i)}_{M_1(i)+1} = e_{M_1(i)+1}$. Then the words $e_{M_1(i)+1}e_{M_1(i)}\dots e^{(*)}_{m_i+1}e_{m_i}\dots e_1$ are admissible by Lemma 7.11, if $M_1(i)+1 \neq m_{i+1}-1$. Now say that $M_1(i) = m_{i+1}-2$. By the assumption in the beginning of this paragraph, at least one of the words $e^*_{m_{i+1}-2}e_{m_{i+1}-3}\dots e_{m_i+1}e^{(*)}_{m_i+1}\dots e_1$ is not admissible.
a) Say $\nu = 10^\kappa 1\dots$, where $\kappa > 1$. By Lemma 7.11, $e^*_{m_{i+1}-2}e_{m_{i+1}-3}\dots = c_3^*c_4c_5\dots = 10^{\kappa-2}1\dots$ is always admissible, a contradiction.
b) Say that $\nu = 10(11)^n0\dots$. Then $e^*_{m_{i+1}-2}e_{m_{i+1}-3}\dots = 0(11)^{n-1}10\dots$ is again always admissible, because $\#_1(0(11)^{n-1}1)$ is odd, a contradiction.

Thus caps have been constructed except in the following case:
(one of) $e^*_{M_1(i)}e_{M_1(i)-1}\dots e_{m_i+2}e^{(*)}_{m_i+1}\dots e_1$ is not admissible and $e^*_{M_1(i)+1} = l_{M_1(i)+1}$.

For $j > 1$ denote by $M_j(i)$ the smallest $k \in \mathbb{N}$ such that $k > M_{j-1}(i)$ and $e^*_k = l_k$. By the previous paragraph, it follows that $M_2(i) < m_{i+1}-1$. Take the largest $N \in \mathbb{N}$ such that $M_N(i) < m_{i+1}-1$. Note that for odd $j \in \{1,\dots N\}$ and

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{M_j(i)}e_{M_j(i)-1}\dots e_{m_i+2}0e_{m_i}\dots e_1,$$
$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{M_j(i)}e_{M_j(i)-1}\dots e_{m_i+2}1e_{m_i}\dots e_1,$$

if follows that $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$. The conclusion follows from the fact that $\#_1(l_{M_j(i)-1}\dots l_{m_i+2})$ and $\#_1(e_{M_j(i)-1}\dots e_{m_i+2})$ are of the same parity since j is odd.

Assume that for every odd $j \in \{1,\dots,N\}$ we have that $1^\infty e^*_{M_j(i)}e_{M_j(i)-1}\dots e_{m_i+2}e^{(*)}_{m_i+1}e_{m_i}\dots e_1$ are not admissible. If $M_{j+1}(i) > M_j(i)+1$, we set:

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{M_j(i)+1}e_{M_j(i)}\dots e_{m_i+2}0e_{m_i}\dots e_1,$$
$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{M_j(i)+1}e_{M_j(i)}\dots e_{m_i+2}1e_{m_i}\dots e_1,$$

and argue that both are admissible as in preceding paragraphs. Calculations as above give $\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{O(i)}$.

The situation left to consider is when $1^\infty e^*_{M_j(i)} e_{M_j(i)-1} \dots e_{m_i+2} e^{(*)}_{m_i+1} e_{m_i} \dots e_1$ are not admissible and $M_{j+1}(i) = M_j(i)+1$ for every odd $j \in \{1, \dots, N\}$. Note that N must be even. Otherwise $1^\infty e^*_{m_{i+1}-2} e_{m_{i+1}-3} \dots e_{m_i+2} e^{(*)}_{m_i+1} e_{m_i} \dots e_1$ are not admissible and we have already argued that this is not possible.

Thus we conclude that L is of the form:

$$\dots e_{M_N(i)+1} e^*_{M_N(i)} e^*_{M_N(i)-1} e_{M_N(i)-2} \dots e_{M_1(i)+2} e^*_{M_1(i)+1} e^*_{M_1(i)} e_{M_1(i)-1} \dots e_{m_i+2} l_{m_i+1} \dots l_1.$$

Note that $\#_1(e^*_{M_N(i)} e^*_{M_N(i)-1} e_{M_N(i)-2} \dots e_{M_1(i)+2} e^*_{M_1(i)+1} e^*_{M_1(i)} e_{M_1(i)-1} \dots e_{m_i+2})$ is of the same parity as $\#_1(e_{M_N(i)} e_{M_N(i)-1} e_{M_N(i)-2} \dots e_{M_1(i)+2} e_{M_1(i)+1} e_{M_1(i)} e_{M_1(i)-1} \dots e_{m_i+2})$, because changes in L compared with $\overleftarrow{e}$ always appear in pairs (as two consecutive letters). We set

$$\overleftarrow{x}^{O(i)} = 1^\infty e^*_{m_{i+1}-1} e_{m_{i+1}-2} \dots e_{m_i+2} 0 e_{m_i} \dots e_1,$$

$$\overleftarrow{x}^{I(i)} = 1^\infty e^*_{m_{i+1}-1} e_{m_{i+1}-2} \dots e_{m_i+2} 1 e_{m_i} \dots e_1,$$

and note that

$\overleftarrow{x}^{O(i)} \succ_L \overleftarrow{e} \succ_L \overleftarrow{x}^{I(i)}$ (or with reversed inequalities). Also note that $\overleftarrow{x}^{I(i)}$ and $\overleftarrow{x}^{O(i)}$ set in such a way are always admissible by Lemma 7.11 and since $e_{m_{i+1}-1} = c_2 = 0$.

We have constructed the sequence corresponding to basic arc with the following properties: $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{I(i)}$ or $\overleftarrow{x}^{I(i)} \prec_L \overleftarrow{e} \prec_L \overleftarrow{x}^{O(i)}$, $\overleftarrow{x}^{O(i)}$ and $\overleftarrow{x}^{I(i)}$ are joined on the right and $\overleftarrow{x}^{O(i)}, \overleftarrow{x}^{I(i)} \to \overleftarrow{e}$ as $i \to \infty$. Since that can be done for infinitely many $i \in \mathbb{N}$, this concludes the proof. $\square$

Example 6. Let X' be the inverse limit space with the corresponding kneading sequence $\nu = (100111101011010111)^\infty$. Note first that $\#_1(100111101011010111)$ is even and every subword of ν is admissible; thus ν is an admissible kneading sequence. Let us study the cappedness of the endpoint $e \in X'$ with the itinerary $\bar{e} = (100111101011010111)^\infty.(100111101011010111)^\infty$ in an embedding determined by $L = (010111110011100111)^\infty$. It follows that $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e}$, because $\#_1(100111101011010111)$ and $\#_1(010111110011100111)$ are both even. Note that $M_1(i) := m_i + 5$ is the smallest index strictly greater than m_i+1 such that $e^*_{M_1(i)} = l_{M_1(i)}$. We obtain the following situation:

$$\dots(1\mathbf{0}01111\mathbf{01}011\mathbf{01}0111)(100111101011010111)^i = \overleftarrow{e}$$

$$\dots(0\mathbf{1}01111\mathbf{10}011\mathbf{10}0111)(010111110011100111)^i = L$$

$$1^\infty(1\mathbf{1}01111\mathbf{01}011\mathbf{01}0110)(100111101011010111)^i = \overleftarrow{x}^{O(i)}$$

$$1^\infty(1\mathbf{1}01111\mathbf{01}011\mathbf{01}0111)(100111101011010111)^i = \overleftarrow{x}^{I(i)}$$

where we denoted with bold the letters of $\overleftarrow{e}$ and L which differ for indices larger than m_i. Note that $M_3(i) = m_i + 10$ but the word $00110 = e^*_{m_i+10}e_{m_i+9}\dots e_{m_i+6}$ is not admissible and thus we need to set $x^{O(i)}_{M_3(i)} = x^{I(i)}_{M_3(i)} = e_{M_3(i)} = 1$. Note that $M_5(i) = m_i+17 = m_{i+1}-1$. Thus we set $x^{O(i)}_{M_5(i)} = x^{I(i)}_{M_5(i)} = e^*_{M_5(i)}$. Because $\#_1(e_{m_i+16}\dots e_1)$ and $\#_1(l_{m_i+16}\dots l_1)$ are of the same parity we obtain that $\overleftarrow{e} \prec_L \overleftarrow{x}^{I(i)}$. Lemma 7.11 again ensures that every subword of $\overleftarrow{x}^{O(i)}$ is admissible. Therefore points $x^{O(i)}, x^{I(i)} \in X'$ cap the point e from the right.

If an endpoint e is capped, we still cannot conclude that it is not accessible, see *e.g.* Figure 10. However, if we know that the length of basic arcs arbitrary close to $\overleftarrow{e}$ has a lower bound, the conclusion follows. Thus we introduce the notion of long-branchness in the following definition.

Definition 7.13. *Let $T\colon I \to I$ be a continuous map. The* lap *of T is a maximal interval of monotonicity of T and a* branch *of T is an image of a lap. We say that T is* long-branched, *if there exists $\delta > 0$ such that the length of all branches of T^n is larger than δ for all $n \in \mathbb{N}$.*

Remark 7.14. *Note that if critical point of T is periodic or non-recurrent, then T is long-branched.*

Corollary 7.15. *Assume $T \neq T_2$ is long-branched and let $e \in X'$ be an endpoint of X'. Assume X' is embedded in the plane with respect to L where $A(L) \not\subset \mathcal{C}$. If $\overleftarrow{e}$ and L have different tails, then e is not accessible.*

Proof. By the long-branchedness of the bonding map T it holds that in a sufficiently small neighbourhood of endpoint e every point $e \neq x \in A(\overleftarrow{e})$ has a neighbourhood homeomorphic to the Cantor set of arcs. Since $e \in X'$ is capped by Theorem 7.12 the proof follows. □

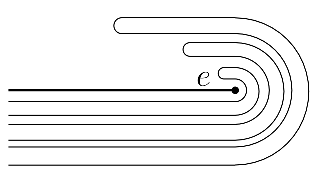

FIGURE 10. Neighbourhood of an endpoint e. Note that e is capped but also accessible.

We merge the knowledge from this and preceding section and give some interesting examples of embeddings of X'.

Example 7. Let $\nu = (101)^\infty$ and let $L = (01^k)^\infty$ for any $k \geq 2$. Set $B = a_n \dots a_1 \in \{0,1\}^n$ for some $n \in \mathbb{N}$. If $l_{n+1}B$ is not admissible, then $\dots l_{n+3}l^*_{n+2}l^*_{n+1}B$ is admissible by the choice of k and since every non-admissible word for $\nu = (101)^\infty$ contains 00. Tail L is thus not altered by B for every finite admissible word B (recall Definition 6.2). Therefore, it follows that $L_{a_n\dots a_1} = S_{a_n\dots a_1} \subset \mathcal{U}_L$. We conclude that $\mathcal{U}_L$ is fully accessible and it is the only non-degenerate accessible set. By Corollary 7.15, endpoints of X' are not accessible. The remaining point on the circle of prime ends corresponds to the simple dense canal.

Example 8. Let $\nu = (101)^\infty$ and let $L = (01)^\infty$. Note that $S = (10)^\infty \not\subset \mathcal{U}_L$ and $S = S_0$. Thus, $B = 0$ alters L (recall Definition 6.2; here $A_1 = 0$, $A_i = 01$ for all $i \geq 2$). Since ν is periodic, it follows from Corollary 5.15 that both $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible. As in the example above (see also Lemma 9.1) one can show that no other point from X' is accessible. We conclude that there are two simple dense canals with shores $\mathcal{U}_L$ and $\mathcal{U}_S$.

Example 9. Take $\nu = (10011001001111)^\infty$, $B = 001$, $A = 0011$, $C = 1111$ and $L = \dots ABABA$ as in Example 5. Recall that at least three arc-components (which are dense lines) are fully accessible. Further calculations show that no other tail can be the top or the bottom of a cylinder. By Corollary 7.15 endpoints from X' are not accessible. So the remaining three points on the circle of prime ends correspond to three simple dense canals with shores from pairwise different fully accessible arc-components which are lines. In comparison, the kneading sequence from this example has height 2/7 (see the Definition 11.1) and belongs to the rational interior case, so in Brucks-Diamond embedding X' contains 7 fully accessible arc-components which are shores of 7 simple dense canals (see Section 11 in this paper or [10]).

7.2. **Accessible folding points when ν is preperiodic.** In this subsection we assume that $\nu = c_1 \dots c_k (c_{k+1} \dots c_{k+n})^\infty$ and that $c_k \neq c_{k+n}$, since otherwise also $\nu = c_1 \dots c_{k-1}(c_k \dots c_{k+n-1})^\infty$. By Remark 5.17 X' contains n folding points which are not endpoints with symbolic descriptions:

$$\sigma^i((c_{k+1} \dots c_{k+n})^\infty . (c_{k+1} \dots c_{k+n})^\infty)$$

for $i \in \{1, \dots n\}$. In this subsection we study the accessibility of folding points that are not contained in extrema of cylinders in $\mathcal{E}$-embeddings of X' when ν is preperiodic.

Let $Q \subset \mathbb{R}^2$ be an arc. From now onwards let $\mathrm{Int}(Q)$ denote the points from Q, which are not endpoints of Q.

Remark 7.16. *Let $\nu = c_1 \dots c_k (c_{k+1} \dots c_{k+n})^\infty$ and let $p \in X'$ be a folding point. Then an arc-component of p can contain at most one folding point. Also, since $c_k \neq c_{n+k}$ it holds that $p \in \mathrm{Int}(A(\overleftarrow{p}))$.*

The following lemma restricts the search for accessible folding points which are not tops/bottoms of cylinders to the case where $\nu = 10(c_3 \dots c_{n+2})^\infty$, *i.e.*, $k = 2$.

Proposition 7.17. *Assume c is preperiodic and such that $T^3(c)$ is not periodic. Embed X' in the plane with respect to $L \neq 0^\infty l_n \dots l_1$. A folding point $p \in X'$ is accessible if and only if the basic arc $A(\overleftarrow{p})$ is top or bottom of a finite cylinder.*

Proof. Note that $\nu = c_1 \dots c_k (c_{k+1} \dots c_{k+n})^\infty$ where $k > 2$. Take a folding point $p \in X'$ with the symbolic description

$$\bar{p} = (c_{k+1} \dots c_{k+n})^\infty c_{k+1} \dots c_{k+i} . c_{k+i+1} \dots c_{k+n} (c_{k+1} \dots c_{k+n})^\infty$$

and assume it is not on the top or bottom of any cylinder in X'. Denote $\pi_0(A(\overleftarrow{p})) =: [T^l(c), T^r(c)]$. By Remark 7.16 it holds that $\pi_0(p) \in (T^l(c), T^r(c))$.
Denote by $(p^M)_{M\in\mathbb{N}} \subset X'$ the points with the symbolic description

$$\bar{p}^M := 1^\infty c_1 \dots c_k (c_{k+1} \dots c_{k+n})^M c_{k+1} \dots c_{k+i} . c_{k+i+1} \dots c_{k+n} (c_{k+1} \dots c_{k+n})^\infty$$

Note that points p^M converge to p as $M \to \infty$ and the corresponding basic arcs $A(\overleftarrow{p}^M)$ project to $[T^l(c), T^{k+i+1}(c)]$ (we refer to them as *left*) or $[T^{k+i+1}(c), T^r(c)]$ (referred to as *right*) depending on the parity of M. We will find long basic arcs (*i.e.,* arcs projecting with π_0 also to $[T^l(c), T^r(c)]$) converging to $A(\overleftarrow{p})$ from both sides. Since c is preperiodic there exists a neighbourhood U of $A(\overleftarrow{p})$ which contains only basic arcs which project to $[T^l(c), T^r(c)]$, $[T^l(c), T^{k+i+1}(c)]$ or $[T^{k+i+1}(c), T^r(c)]$ (*i.e.,* only long or left/right arcs).

Assume that all but finitely many long arcs in U are greater than $A(\overleftarrow{p})$. Since $k > 2$, note that for every $M > 0$ basic arcs $1^\infty c_k (c_{k+1} \dots c_{k+n})^M c_{k+1} \dots c_{k+i}$ are long. Since $c_k \neq c_{k+n}$ it holds that both $1^\infty c_k (c_{k+1} \dots c_{k+n})^M c_{k+1} \dots c_{k+i} \succ_L \overleftarrow{p}$ and $\overleftarrow{p}^M \succ_L \overleftarrow{p}$. Thus, it follows that $A(\overleftarrow{p})$ is at the bottom of some cylinder, a contradiction. The proof goes analogously if all but finitely many long arcs are smaller than $A(\overleftarrow{p})$. $\square$

Therefore, by Proposition 7.17, if we want to find accessible folding points which are not at the top/bottom of any cylinder it is enough to study cases $\nu = 10(c_3 \dots c_{n+2})^\infty$ where $c_{n+2} = 1$.

Remark 7.18. *Assume c is preperiodic and p is an accessible folding point of an embedding of X'. By Corollary 3.3 and since every arc-component contains at most one folding point, only the following three cases can occur:*

1. $\overleftarrow{p}$ *is the top or the bottom of some cylinder; then $\mathcal{U}_p$ is fully accessible.*
2. $\overleftarrow{p}$ *is not the top or the bottom of any cylinder, but $\overleftarrow{r(p)}$ or $\overleftarrow{l(p)}$ is; then one component of $\mathcal{U}_p$ is fully accessible, and the other is not accessible. See Figure 6.*
3. $\overleftarrow{p}$, $\overleftarrow{r(p)}$ *and $\overleftarrow{l(p)}$ are not extrema of any cylinder; then c is order reversing and p is the only accessible point of $\mathcal{U}_p$. See Figure 7(c).*

Definition 7.19. *We say that an accessible folding point p is accessible of* Type i *if it satisfies the condition i from Remark 7.18 for $i \in \{1, 2, 3\}$.*

As it turns out, all Types of accessible folding points can occur in $\mathcal{E}$-embeddings. In the following subsections we describe how they can be constructed in preperiodic orbit case (when $T^3(c)$ is periodic) and give examples of such constructions. We will see that the standard Brucks-Diamond embedding does not allow Type 3 folding points for any X' (see Section 11).

7.2.1. *Type 2.* First we give examples of X' which cannot be $\mathcal{E}$-embedded with Type 2 folding points. Then we show in general how to construct a Type 2 accessible folding point and give an example of such construction in both the order preserving and the order reversing case.

Lemma 7.20. *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$ and assume that $c_i^* c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M$ is admissible for all $i \in \{3, \dots, n+1\}$ and for all but finitely many $M \in \mathbb{N}$. Then no folding point is Type 2 in any $\mathcal{E}$-embedding of X' which is non-equivalent to the Brucks-Diamond ($L = 0^\infty 1$) embedding.*

Proof. Take a folding point $p \in X'$ with symbolic description $\bar{p} = (c_3 \dots c_{n+2})^\infty.(c_3 \dots c_{n+2})^\infty$. We will try to reconstruct L which embeds p as Type 2 and see that this is not possible.

Assume first that $\#_1(c_3 \dots c_{n+2})$ is odd and for some natural number M we have (the following (or with reversed inequalities) needs to be satisfied in order for p to be a Type 2 folding point, see Figure 12):

$$\begin{aligned} \dots 0(c_3 \dots c_{n+2})^M &\succ_L \dots 1(c_3 \dots c_{n+2})^M \\ \dots c_i^* c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M &\prec_L \dots c_i c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M \\ \dots 0(c_3 \dots c_{n+2})^{M+k} &\prec_L \dots 1(c_3 \dots c_{n+2})^{M+k} \\ \dots c_i^* c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+k} &\prec_L \dots c_i c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+k} \\ \dots 0(c_3 \dots c_{n+2})^{M+N} &\succ_L \dots 1(c_3 \dots c_{n+2})^{M+N} \\ \dots c_{n+1}^* c_{n+2}(c_3 \dots c_{n+2})^{M+N} &\prec_L \dots c_{n+1} c_{n+2}(c_3 \dots c_{n+2})^{M+N} \end{aligned}$$

for all $i \in \{3, \dots, n+1\}$ and all $k \in \{1, \dots, N-1\}$, where natural number $N > 1$ is even.

If $\#_1((c_3 \dots c_{n+2})^M)$ is of the same parity as $\#_1(l_{Mn} \dots l_1)$, then $l_{Mn+1} = 0$, and if $\#_1((c_3 \dots c_{n+2})^M)$ is of different parity as $\#_1(l_{Mn} \dots l_1)$, then $l_{Mn+1} = 1$. In any case, $\#_1(c_{n+2}(c_3 \dots c_{n+2})^M)$ is of different parity as $\#_1(l_{Mn+1} \dots l_1)$ so $l_{Mn+2} = c_{n+1}^*$. So $\#_1(c_{n+1}c_{n+2}(c_3 \dots c_{n+2})^M)$ is of the same parity as $\#_1(l_{Mn+2}l_{Mn+1} \dots l_1)$ and thus $l_{Mn+3} = c_n$. Continuing further, we get

$$l_{(M+N)n+2} \dots l_{Mn+2} = c_{n+1}^* c_{n+2}^* (c_3 \dots c_{n+2})^{N-1} c_3 \dots c_n c_{n+1}^*.$$

Thus $c_{n+1}^* = 1$, $\#_1(c_3 \dots c_n)$ is even and the word on the right side of the last equation above is equal to $10(c_3 \dots c_{n+2})^{N-1} c_3 \dots c_n c_{n+1}^*$. Note that $10(c_3 \dots c_{n+2})^{N-1} c_3 \dots c_n c_{n+1}$ is even and thus $10(c_3 \dots c_{n+2})^{N-1} c_3 \dots c_n c_{n+1}^*$ is not admissible by Lemma 7.10, a contradiction.

Assume that $\#_1(c_3 \dots c_{n+2})$ is even. Note that in this case N is not necessarily even, but now the conclusion $c_{n+1} = 0$ implies that $c_3 \dots c_n$ is odd. We continue with arguments as in the paragraphs above. Since $\#_1(c_3 \dots c_{n+2})$ is even the word $10(c_3 \dots c_{n+2})^{N-1} c_3 \dots c_n c_{n+1}$ is even and thus $10(c_3 \dots c_{n+2})^{N-1} c_3 \dots c_n c_{n+1}^*$ is again not admissible by Lemma 7.10, a contradiction.

Note that the proof works analogously for other folding points from the space X'. □

Next we give examples of preperiodic ν where no folding point can be $\mathcal{E}$-embedded as Type 2.

Example 10. The assumptions from Lemma 7.20 hold for *e.g.* $\nu = 10(0^\alpha 1^\beta)$ for all $\alpha, \beta \in \mathbb{N}$.

The proof of the following lemma follows directly, see Figure 6.

Lemma 7.21 (Order preserving case)**.** *Let* $\nu = 10(c_3 \dots c_{n+2})^\infty$*,* $c_{n+2} = 1$*, and* $\#_1(c_3 \dots c_{n+2})$ *even. Let* $\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$ *be a symbolic description of a folding point* $p \in X'$*. Then* p *is a Type 2 folding point if and only if there exists a natural number* M *such that*

$$\dots c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i \succ_L \dots c_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i,$$

for all $N \in \mathbb{N}$ *and all* $j \in \{3, \dots, 1+n\}$ *for which* $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i$ *is admissible, and*

$$\dots 0(c_3 \dots c_{n+2})^{M+N'} c_3 \dots c_i \prec_L \dots 1(c_3 \dots c_{n+2})^{M+N'} c_3 \dots c_i,$$

for infinitely many $N' \in \mathbb{N}$*, or with reversed inequalities. See* e.g. *Figure 6.*

We give an example that satisfies the assumptions of Lemma 7.21.

Example 11 (Type 2, order preserving case)**.** Take $\nu = 10(01101001)^\infty$, $L = (10100101\ 11001001)^\infty$ and

$$\bar{p} = (01101001)^\infty 01.101001(01101001)^\infty.$$

Then $\overleftarrow{r(p)}$ is the smallest left-infinite tail so it is the smallest in cylinder [0]. As the calculations below show, all long basic arcs in small neighbourhood of $A(\overleftarrow{p})$ are below $A(\overleftarrow{p})$ and left arcs are both above and below $A(\overleftarrow{p})$, depending on the parity of period which corresponds with $\overleftarrow{p}$ in the left infinite description of basic arcs, see Figure 11.

$$\begin{aligned}
\dots 0(01101001)^{2N}01 &\succ_L \overleftarrow{p},\\
\dots 0(01101001)^{2N+1}01 &\prec_L \overleftarrow{p},\\
\dots 11(01101001)^{N}01 &\prec_L \overleftarrow{p},\\
\dots 101(01101001)^{N}01 &\prec_L \overleftarrow{p},\\
\dots 11001(01101001)^{N}01 &\prec_L \overleftarrow{p},\\
\dots 001001(01101001)^{N}01 &\prec_L \overleftarrow{p},\\
\dots 0101001(01101001)^{N}01 &\prec_L \overleftarrow{p},\\
\dots 11101001(01101001)^{N}01 &\prec_L \overleftarrow{p},
\end{aligned}$$

for all $N \in \mathbb{N}$. Further calculations show that only tails of L and S can appear as the extrema of cylinders. By Proposition 5.6, the arc-component $\mathcal{U}_L$ is fully accessible and since $\mathcal{U}_L$ contains no folding points, it corresponds to an open interval on the circle of prime ends. The accessible part of $\mathcal{U}_S$ corresponds to a half-open interval on the circle of prime ends, where the endpoint of the half-open interval corresponds to accessible folding point p. By further calculations we obtain that other folding points are not accessible, so the remaining point on the circle of prime ends corresponds to a simple dense canal with shores from $\mathcal{U}_L$ and $\mathcal{U}_S$.

The proof of the following lemma follows directly from assumptions (see Figure 12).

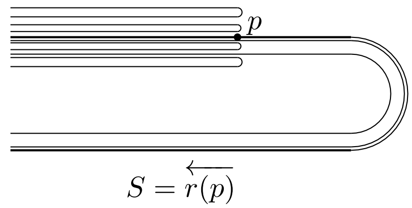

FIGURE 11. Type 2 folding point from Example 11.

Lemma 7.22 (Order reversing case)**.** *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$, $c_{n+2} = 1$, and $\#_1(c_3 \dots c_{n+2})$ odd. Let $\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$ be a symbolic description of a folding point $p \in X'$. Then p is a Type 2 folding point if and only if there exists a natural number M such that*

$$\dots c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i \succ_L \dots c_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i,$$

for all $N \in \mathbb{N}$ and all $j \in \{3, \dots, 1+n\}$ for which $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i$ is admissible, and

$$\dots 0(c_3 \dots c_{n+2})^{M+2N'} c_3 \dots c_i \prec_L \dots 1(c_3 \dots c_{n+2})^{M+N'} c_3 \dots c_i,$$

and

$$\dots 0(c_3 \dots c_{n+2})^{M+2N''+1} c_3 \dots c_i \succ_L \dots 1(c_3 \dots c_{n+2})^{M+N''} c_3 \dots c_i,$$

for infinitely many $N' \in \mathbb{N}$ and all but finitely many $N'' \in \mathbb{N}$, or with reversed inequalities.

We give an example that satisfies the assumptions of Lemma 7.22.

Example 12 (Type 2, order reversing case)**.** Take $\nu = 10(011101001)^\infty$, $L = (011101001011110010)^\infty$ and $\overleftarrow{p} = (011101001)^\infty$. What follows is an easy computation:

$$\begin{aligned}
\dots 0(011101001)^{2M+1} &\prec_L \overleftarrow{p}, \\
\dots 0(011101001)^{2M} &\succ_L \overleftarrow{p}, \\
\dots 11(011101001)^{M} &\prec_L \overleftarrow{p}, \\
\dots 101(011101001)^{M} &\prec_L \overleftarrow{p}, \\
\dots 11001(011101001)^{M} &\prec_L \overleftarrow{p}, \\
\dots 001001(011101001)^{M} &\prec_L \overleftarrow{p}, \\
\dots 0101001(011101001)^{M} &\prec_L \overleftarrow{p}, \\
\dots 01101001(011101001)^{M} &\prec_L \overleftarrow{p}, \\
\dots 111101001(011101001)^{M} &\prec_L \overleftarrow{p},
\end{aligned}$$

for every $M \in \mathbb{N}$. So p is accessible folding point of Type 2. Note that $\overleftarrow{l(p)} = (010010111)^\infty 01011 = S_{1011}$, see Figure 12. By further symbolic calculations we again conclude that there is one simple dense canal for this embedding of X'.

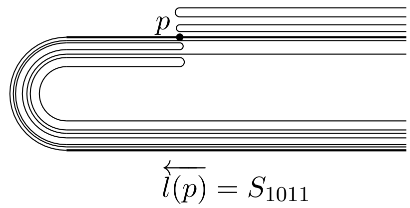

FIGURE 12. Type 2 folding point from Example 12.

7.2.2. *Type 3.* From now onwards we study folding points of Type 3, see Figure 13.

Remark 7.23. *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$ be such that $\#_1(c_3 \dots c_{n+2})$ is even and $c_{n+2} = 1$. Then X' does not contain folding points of Type 3.*

The following lemma gives necessary and sufficient symbolic conditions for a folding point to be $\mathcal{E}$-embedded as Type 3, the proof of it again follows directly from its assumptions.

Lemma 7.24 (Type 3)**.** *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$, $c_{n+2} = 1$, and $\#_1(c_3 \dots c_{n+2})$ odd. Let $\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$ be the symbolic description of a folding point $p \in X'$. Then p is a Type 3 folding point if and only if there exists $M > 0$ such that*

$$\dots c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i \prec_L \overleftarrow{p},$$

for all $N \in \mathbb{N}$ and all $j \in \{3, \dots, n+1\}$ for which $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^{M+N} c_3 \dots c_i$ is admissible, and

$$\dots 0(c_3 \dots c_{n+2})^{M+N'} c_3 \dots c_i \succ_L \overleftarrow{p},$$

for infinitely many $N' \in \mathbb{N}$, or with reversed inequalities. See Figure 13.

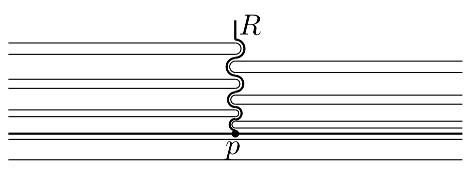

FIGURE 13. Type 3 folding point. Folding point p is accessible from the complement by an arc $R \cup \{p\} \subset \mathbb{R}^2$, where R is a ray.

The following lemma gives conditions on preperiodic order reversing ν such that no folding point can be $\mathcal{E}$-embedded as Type 3 folding point (except possibly with the Brucks-Diamond embedding studied in detail in Section 11).

Lemma 7.25. *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$ be such that $\#_1(c_3 \dots c_{n+2})$ is odd, $c_{n+2} = 1$ and let $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M c_3 \dots c_i$ be admissible for every $j \in \{3, \dots 1+n\}$ and all $M \in \mathbb{N}$. If $c_{n+1} = 1$ then there exists no L such that folding point $p \in X'$ is of Type 3.*

Proof. Take a folding point $p \in X'$ with the symbolic description

$$\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$$

for some $i \in \{3, \dots n+2\}$ and assume that $A(\overleftarrow{p})$ is not at the top or bottom of any cylinder in X'. Since $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^M c_3 \dots c_i$ is admissible for every $j \in \{3, \dots n+1\}$ and all $M \in \mathbb{N}$, the same calculations as in the proof of Lemma 7.20 imply that the only L which satisfies all the conditions from Lemma 7.24 is

$$L = \dots (c_3 \dots c_n 00)^\infty l_k \dots l_1,$$

for some $l_k \dots l_1$. However, the word $00c_3 \dots c_n$ is not admissible, a contradiction. □

Example 13 (No Type 3 folding point)**.** Note that $\nu = 10(0^\alpha 1^\beta)^\infty$ for $\beta \geq 2$ satisfies the assumptions of Lemma 7.25. Thus no folding point from the corresponding X' can be embedded as Type 3 folding point using $\mathcal{E}$-embeddings (except maybe Brucks-Diamond). Note that this example also satisfies the assumptions of Lemma 7.20, so no folding point can be $\mathcal{E}$-embedded as Type 2 either. Thus in these cases a point from X' is accessible if and only if it is on the top or the bottom of some cylinder. So there are $m \in \mathbb{N}$ simple dense canals, where m is the number of fully accessible arc-components for some $\mathcal{E}$-embedding of X'.

The following lemma gives sufficient symbolic conditions on a preperiodic ν such that every folding point can be $\mathcal{E}$-embedded as accessible folding point of Type 3.

Lemma 7.26. *Let $\nu = 10(c_3 \dots c_{n+2})^\infty$ be such that $\#_1(c_3 \dots c_{n+2})$ is odd and $c_{n+2} = 1$. Assume that $c_{n+1} = 0$ and the tail $(10c_3 \dots c_n)^\infty$ is admissible. For every folding point $p \in X'$ there exists L such that p is of Type 3 in $\varphi_L(X')$.*

Proof. Take a folding point $p \in X'$ with the symbolic description

$$\bar{p} = (c_3 \dots c_{n+2})^\infty c_3 \dots c_i . c_{i+1} \dots c_{n+2}(c_3 \dots c_{n+2})^\infty$$

for some $i \in \{3, \dots n+2\}$ and assume that $A(\overleftarrow{p})$ is not on the top or bottom of any cylinder in X'. Denote by $\pi_0(A(\overleftarrow{p})) =: [T^l(c), T^r(c)]$ for some $l, r \in \mathbb{N}$.

Let $L = (c_3 \dots c_n c_{n+1}^* c_{n+2}^*)^\infty c_3 \dots c_i$. Then

$$\dots 0(c_3 \dots c_{n+2})^m c_3 \dots c_i \succ_L \dots 1(c_3 \dots c_{n+2})^m c_3 \dots c_i,$$

$$\dots c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^m c_3 \dots c_i \prec_L \dots c_j c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^m c_3 \dots c_i,$$

for every $m \in \mathbb{N}$, every $j \in \{3, \dots n+1\}$ and all admissible $c_j^* c_{j+1} \dots c_{n+2}(c_3 \dots c_{n+2})^m c_3 \dots c_i$, see Figure 13 to visualize the construction. By the assumptions we conclude that $L = (10c_3 \dots c_n)^\infty 10c_3 \dots c_i$ is indeed admissible. Since $\#_1(c_3 \dots c_{n+2})$ is odd we get pairs of basic arcs joined at a point which project with π_0 to $\pi_0(p)$, approaching to $A(\overleftarrow{p})$ from above from both left and right side of p, exactly as on Figure 13. □

Example 14 (Type 3 folding point)**.** Take $\nu = 10(01101)^\infty$. If we embed X' with respect to admissible $L = (01110)^\infty$, then $\bar{p} = (01101)^\infty.(01101)^\infty$ is an accessible folding point of Type 3, since it satisfies the conditions of Lemma 7.26. Note that only $\mathcal{U}_L$ can be the extremum of a cylinder and it corresponds to the circle of prime ends minus a point. The remaining point is the second kind prime end corresponding to the accessible folding point p of Type 3. Specifically, there are no simple dense canals.

8. Extendability of σ-homeomorphism for $\mathcal{E}$-embeddings

Embeddings from [16] ($L = 1^\infty$) and [14] ($L = 0^\infty 1$) of X make $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{C}$ respectively fully accessible as can be deduced from Proposition 5.6 and Remark 5.10 (denote the two special embeddings from now onwards by $\varphi_{\mathcal{R}}$ and $\varphi_{\mathcal{C}}$ respectively). Additionally it can be deduced from Proposition 5.11 that only remaining accessible points (if existent) need to be folding points. The embeddings of unimodal inverse limit spaces $\varphi_{\mathcal{C}}$ follow Barge-Martin construction from [6] of attractors of orientation preserving planar homeomorphisms so σ is extendable to $\mathbb{R}^2$ for these embeddings. Bruin directly showed in [16] that σ-homeomorphism can be extended to the plane for embeddings $\varphi_{\mathcal{R}}$. Now we show that except for mentioned standard embedding, σ is not extendable for any $\mathcal{E}$-embedding of X'.

Note that if $\sigma : \varphi_L(X) \to \varphi_L(X)$ is extendable to $\mathbb{R}^2$, then $\sigma|_{\varphi_L(X')} : \varphi_L(X') \to \varphi_L(X')$ is also extendable to $\mathbb{R}^2$.

The following theorem answers the question weather for non-standard $\mathcal{E}$-embeddings σ-homeomorphism is extendable to the whole plane which was posed by Boyland, de Carvalho and Hall in [10].

Theorem 8.1. *If X' is embedded in the plane with respect to L where $A(L) \not\subset \mathcal{C}, \mathcal{R}$, then the shift homeomorphism $\sigma\colon \varphi_L(X') \to \varphi_L(X')$ cannot be extended to a homeomorphism of the plane.*

Proof. Let $\nu = c_1c_2\dots$ be a kneading sequence and $A(L) \not\subset \mathcal{C}, \mathcal{R}$ and assume by contradiction that $\sigma\colon \varphi_L(X') \to \varphi_L(X')$ is extendable to $\mathbb{R}^2$. Let (n_i) be an increasing sequence in $\mathbb{N}$ such that $l_{n_i+3}l_{n_i+2} = 01$. Since $A(L) \not\subset \mathcal{C}, \mathcal{R}$, the sequence (n_i) is indeed well defined. For $i \in \mathbb{N}$ define admissible tails

$$\overleftarrow{x_i} = 1^\infty 1011^{n_i},$$
$$\overleftarrow{y_i} = 1^\infty 0111^{n_i},$$
$$\overleftarrow{z_i} = 1^\infty 1101^{n_i}.$$

Note that $\overleftarrow{x_i}$ is between $\overleftarrow{y_i}$ and $\overleftarrow{z_i}$ and $\overleftarrow{x_i}1$ is the largest or the smallest among the admissible sequences $\overleftarrow{x_i}1$, $\overleftarrow{y_i}1$ and $\overleftarrow{z_i}1$ because of the chosen $l_{n_i+3}l_{n_i+2} = 01$.
For i large enough, note that $\pi_0(\overleftarrow{x_i}1) = [T^2(c), T(c)]$ so $A(\overleftarrow{x_i}1)$ is a horizontal arc in the plane of length $T(c) - T^2(c) =: \delta$. Note also that $\pi_0(\overleftarrow{x_i}) = \pi_0(\overleftarrow{y_i}) = \pi_0(\overleftarrow{z_i}) = [T^2(c), T(c)]$ for i large enough. Let $\overleftarrow{x_i}' = \pi_0^{-1}([c, T(c)]) \cap \overleftarrow{x_i}$, $\overleftarrow{y_i}' = \pi_0^{-1}([c, T(c)]) \cap \overleftarrow{y_i}$ and $\overleftarrow{z_i}' = \pi_0^{-1}([c, T(c)]) \cap \overleftarrow{z_i}$, see Figure 14, left picture. Denote by $A_i \subset \mathbb{R}^2$ ($B_i \subset \mathbb{R}^2$) the vertical segment which joins the left (right) endpoints of $\overleftarrow{y_i}'$ and $\overleftarrow{z_i}'$. Note that $\operatorname{diam}(A_i), \operatorname{diam}(B_i) \to 0$ as $i \to \infty$. Also $D = A_i \cup \overleftarrow{y_i}' \cup B_i \cup \overleftarrow{z_i}'$ separates the plane, denote the bounded component of $\mathbb{R}^2 \setminus D$ by $U \subset \mathbb{R}^2$. Note that $\operatorname{Int} \overleftarrow{x_i}' \subset U$.
Now note that $\sigma(\overleftarrow{x_i}') = \overleftarrow{x_i}1$ and similarly for $\overleftarrow{y_i}', \overleftarrow{z_i}'$. Since $\overleftarrow{x_i}1$ is the smallest or the

largest among $\overleftarrow{x_i}1, \overleftarrow{y_i}1, \overleftarrow{z_i}1$ and σ is extendable, at least one $\sigma(A_i)$ or $\sigma(B_i)$ has length greater than δ, see Figure 14. This contradicts the continuity of σ. $\square$

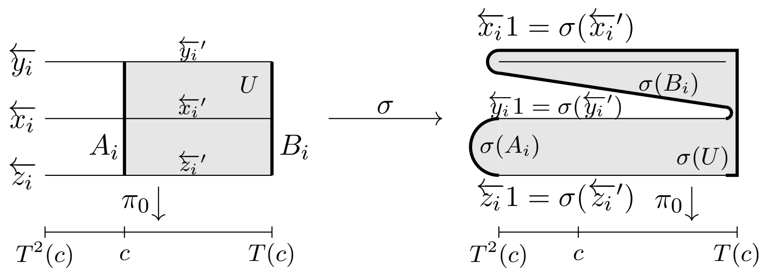

FIGURE 14. Shuffling of basic arcs from the proof of Theorem 8.1.

9. $\mathcal{E}$-EMBEDDINGS OF X' WITH TWO FULLY ACCESSIBLE ARC-COMPONENTS

In this section we study $\mathcal{E}$-embeddings of an arbitrary X' that allow at least two fully accessible (dense) arc-components.

Lemma 9.1. *Let $\nu = 10^{\kappa}1\dots$ and embed X' with respect to $L = \dots 0^{\kappa}10^{\kappa}10^{\kappa}1$. The smallest left-infinite tail with respect to $\prec_L$ is $S = S_0 = \dots 10^{\kappa}10^{\kappa}10^{\kappa} \not\subset \mathcal{U}_L$. Moreover, both $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible and dense in X'.*

Proof. It is straightforward to calculate S, infinitely many changes occur because $0^{\kappa+1}$ is not admissible, *i.e.,* symbol 0 alters L, see Definition 6.2.

To prove that $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible, it is enough to show that every basic arc in $\mathcal{U}_L \cup \mathcal{U}_S$ is at the top or the bottom of some cylinder.
Proposition 5.6 shows that $\mathcal{U}_L$ is fully accessible. Assume that $\overleftarrow{x} \subset \mathcal{U}_S$ and take $k \in \mathbb{N}$ such that $x_{k+i} = s_{k+i}$ for every $i \in \mathbb{N}$ and such that $\kappa + 1$ divides k, where $S = \dots s_2 s_1$. Then $\overleftarrow{x} = \dots 10^{\kappa}10^{\kappa}x_k \dots x_1$. Note that if $\#_1(10^{\kappa}x_k \dots x_1)$ and $\#_1(l_{k+\kappa+1} \dots l_1)$ have the same parity, then $S_{10^{\kappa}x_k\dots x_1} = \overleftarrow{x}$ and $L_{10^{\kappa}x_k\dots x_1} = \overleftarrow{x}$ in the other case.

To show that $\mathcal{U}_L$ and $\mathcal{U}_S$ are dense, fix a point $x \in X'$ with backward itinerary $\overleftarrow{x} = \dots x_2 x_1$ and fix $n \in \mathbb{N}$. Denote $\nu = 10^{\kappa}10^{\kappa_2}10^{\kappa_3}1\dots$, where $\kappa_2, \kappa_3 \geq 0$. If $\kappa_3 = 0$, then for γ large the tail $(0^{\kappa}1)^{\infty}0^{\kappa_2}101^{\gamma}x_n \dots x_1$ is admissible (recall that X' is assumed to be non-renormalizable thus if $\kappa = 1$, then $\nu = 10(1)^{\alpha}0\dots$ for even $\alpha > 1$). If $\kappa_3 > 1$ then for γ large enough the tail $(0^{\kappa}1)^{\infty}0^{\kappa_2}1^{\gamma}x_n \dots x_1$ is admissible. So there are points from both $\mathcal{U}_L$ and $\mathcal{U}_S$ arbitrary close to x. $\square$

Theorem 9.2. *For every X' there exists a planar embedding with two non-degenerate fully accessible dense arc-components.*

Proof. Let $\nu = 10^{\kappa}1\dots$ and construct $\varphi_L(X')$ with respect to $L = \dots 0^{\kappa}10^{\kappa}10^{\kappa}1$. Using Lemma 9.1 we conclude that $\mathcal{U}_S$ and $\mathcal{U}_L$ are fully accessible and dense and the claim follows. $\square$

In a special case when the orbit of c is finite and only $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible get the following corollary.

Corollary 9.3. *If orbit of the critical point is finite and only $\mathcal{U}_L$ and $\mathcal{U}_S$ are fully accessible, then there exists a planar embedding of X' with two simple dense canals.*

Proof. Take the embedding constructed in Lemma 9.1. Note that $\mathcal{U}_L$ and $\mathcal{U}_S$ do not contain endpoints for any chosen $\nu = 10^{\kappa}1\dots$ (since the kneading sequence $\nu = (10^{\kappa})^{\infty}$ does not appear as a kneading sequence in the tent map family) and are thus lines. If ν is periodic, the endpoints of X' are not accessible by Corollary 7.15. That in combination with Proposition 4.4 gives two simple dense canals. If ν is preperiodic and $T^3(c)$ is not periodic, the conclusion again follows analogously as above. We only have to argue that Type 3 folding points do not exist for a chosen L. Since L is periodic of period $\kappa+1$, it follows that $\sigma^{\kappa+1}: \varphi_L(X') \to \sigma^{\kappa+1}(\varphi_L(X'))$ is extendable to the whole plane.
Assume that the point $p \in X'$ is a Type 3 folding point. Thus $\sigma^{\kappa+1}(p)$ is also Type 3 folding point. For $\nu = 10(c_3\dots c_{n+2})^{\infty}$, the itineraries of folding points are periodic of period $n \geq \kappa$. Thus $(\kappa+1)|n$. If $\kappa+1 = n$, since $c_{n+2} = 1$ it holds that $c_3\dots c_{n+2} = 0^{\kappa-1}11$, which is even, a contradiction with Remark 7.23. From the circle of prime ends we get that there can be at most two Type 3 accessible folding points and thus $n = 2(\kappa+1)$. Since $\#_1(c_3\dots c_{n+2})$ is odd, it follows that $\dots 0P^{2k+1} \succ_L \dots 1P^{2k+1}$ and $\dots 0P^{2k} \prec_L \dots 1P^{2k}$ for all $k \in \mathbb{N}$, where $P = c_3\dots c_{n+2}$. That is a contradiction with Lemma 7.24. □

The following Example shows that for L as in 9.1 and ν of specific form there can exist more than two fully accessible arc-components. In specific we improve the upper bound on the number of fully accessible non-degenerate arc-components from three to four; compare to Example 9.

Example 15. Assume ν is of the following form $\nu = 10^{\kappa}10^{\kappa-1}110\dots$ with $\kappa > 1$. If $L = (0^{\kappa}1)^{\infty}$, then $\varphi_L(X')$ has four fully-accessible dense arc-components. Note that $S = (10^{\kappa})^{\infty}$ and note that for κ even we have $L_{1^{\kappa+1}} = (1110^{\kappa-1}10^{\kappa-1})^{\infty}11^{\kappa+1}$ and $S_{01^{\kappa+1}} = (010^{\kappa-1}1110^{\kappa-2})^{\infty}01^{\kappa+1}$. For κ odd we get $S_{1^{\kappa+1}} = (1110^{\kappa-1}10^{\kappa-1})^{\infty}11^{\kappa+1}$ and $L_{01^{\kappa+1}} = (010^{\kappa-1}1110^{\kappa-2})^{\infty}01^{\kappa+1}$. In any case, we get at least four accessible tails. To see they are fully-accessible and dense we use the same arguments as in the proof of Lemma 9.1.

The characterization of fully accessible arc-components of $\mathcal{E}$-embeddings of X' (excluding the standard embeddings, see Section 10 and Section 11) is still outstanding. Thus we pose the following question.

Question: Do there exist more than four fully accessible dense arc-components in non-standard (Section 10 and Section 11) $\mathcal{E}$-embeddings of X'? Specifically, if c is periodic, do there exist $\mathcal{E}$-embeddings of X' so that more than four dense arc-components are fully accessible?

We suspect the answer is yes, but lack the symbolic techniques to make a general construction. Note that for every $n \in \mathbb{N}$ there exists X' such that the Brucks-Diamond embedding of X' has n fully-accessible (dense) arc-components. See [10] and Section 11.

10. Bruin's embeddings $\varphi_{\mathcal{R}}(X')$

In this section we study the core X' as the subset of the plane by the Bruin's embedding constructed in [16], *i.e.,* for $L = 1^\infty$. Recall that we denote these embeddings by $\varphi_{\mathcal{R}}(X')$. If the slope $s = 2$ and thus $X' = X$, it follows from Corollary 5.12 and Remark 7.4 that $\mathcal{R}$ and $\mathcal{C}$ are both fully accessible and since there is no other folding point in Knaster continuum except the endpoint, no other point from $\varphi_{\mathcal{R}}(X')$ is accessible. Thus from the circle of prime ends we conclude that there exists exactly one simple dense canal for $\varphi_{\mathcal{R}}(X')$. Therefore, we from now onwards restrict to cases when $X \neq X'$ (*i.e.,* $s \neq 2$). Embeddings $\varphi_{\mathcal{R}}(X')$ can be viewed as global attractors of orientation reversing planar homeomorphisms, since Bruin showed in [16] that $\sigma : \varphi_{\mathcal{R}}(X') \to \varphi_{\mathcal{R}}(X')$ is extendable to the plane.

Theorem 10.1. *Say that $X \neq X'$. In embeddings $\varphi_{\mathcal{R}}(X')$ the arc-component $\mathcal{R}$ is fully accessible and no other point from $\varphi_{\mathcal{R}}(X')$ is accessible. There exists one simple dense canal for every $\varphi_{\mathcal{R}}(X')$.*

Proof. For embeddings given by Bruin in [16] it holds that $L = 1^\infty$ and thus $\mathcal{U}_L = \mathcal{R}$. We will explicitly calculate the tops/bottoms of an admissible cylinder $[a_n \dots a_1] \subset \{0,1\}^n$ for $n \in \mathbb{N}$.

If $\#_1(a_n \dots a_1)$ equals (does not equal) to the parity of natural number n, then $L_{a_n\dots a_1} = 1^\infty a_n \dots a_1$ ($S_{a_n\dots a_1} = 1^\infty a_n \dots a_1$), since $1^\infty a_n \dots a_1$ is always admissible by Lemma 5.2. Also, $S_{a_n\dots a_1} = 1^\infty 01^k a_n \dots a_1$ ($L_{a_n\dots a_1} = 1^\infty 01^k a_n \dots a_1$), where $k \in \mathbb{N}_0$ is the smallest nonnegative integer such that $01^k a_n \dots a_1$ is admissible.
Assume by contradiction that such k does not exists. Then $01^i a_n \dots a_1 \prec c_2c_3\dots$ for every $i \in \mathbb{N}_0$. Since the word 01^i is always admissible, it follows that $c_2c_3\dots = 01^i$ for every $i \in \mathbb{N}_0$, *i.e.,* $\nu = 101^\infty$ and the unimodal interval map which corresponds to this kneading sequence ν is renormalizable, a contradiction.

There remains a prime end P on the circle of prime ends that is either of the second, third, or fourth kind.
Assume first by contradiction that P is of the second kind, *i.e.,* it corresponds to an accessible folding point. Since $\sigma : \varphi_{\mathcal{R}}(X') \to \varphi_{\mathcal{R}}(X')$ is extendable to the plane, it follows that P needs to correspond to accessible point ρ (since $\bar\rho = \dots 11.11 \dots$ is the only σ-invariant itinerary of a point in X'). However, $A(1^\infty)$ is the top or the bottom of a cylinder, so ρ corresponds to a first kind prime end on the circle of prime ends, a contradiction.
Therefore the remaining point P on the circle of prime ends is either of the third or the fourth kind. Since $\mathcal{R}$ is dense in X' (see Proposition 1 from [13]) and $\mathcal{R}$ bounds the canal in X' it follows that $\Pi(P) = X'$ and thus $I(P) = \Pi(P) = X'$. Thus there exists one simple dense canal for every $\varphi_{\mathcal{R}}(X')$. $\square$

11. Brucks-Diamond embeddings $\varphi_{\mathcal{C}}(X')$

In this section we study the core X' as the subset of the plane by the Brucks-Diamond embedding $\varphi_{\mathcal{C}}$ constructed in [14], *i.e.,* for $L = 0^\infty 1$. If the slope $s = 2$, *i.e.,* $X = X'$ is the Knaster continuum, it follows from Corollary 5.12 and Remark 7.4 that $\mathcal{U}_L = \mathcal{C}$ is fully accessible and that no other point from $\varphi_{\mathcal{C}}(X')$ is accessible (observe the circle of prime ends). In specific there is no simple dense canal.
Thus we restrict to cases when $X \neq X'$ (*i.e.,* $s \neq 2$). Embeddings $\varphi_{\mathcal{C}}(X')$ can be viewed as global attractors of orientation preserving planar homeomorphisms as described by Barge and Martin in [6]. Therefore, $\sigma\colon \varphi_{\mathcal{C}}(X') \to \varphi_{\mathcal{C}}(X')$ can be extended to a planar homeomorphism. For $\varphi_{\mathcal{C}}(X)$ the set of accessible points is $\mathcal{C}$ and it forms an infinite canal which is dense in the core. However, if $\mathcal{C}$ is stripped off, the set of accessible points and the prime ends of $\varphi_{\mathcal{C}}(X')$ become very interesting. Recently Boyland, de Carvalho and Hall gave in [10] a complete characterization of prime ends for embeddings $\varphi_{\mathcal{C}}$ of unimodal inverse limits satisfying certain regularity conditions which hold also for tent map inverse limits with indecomposable cores. In this section we obtain an analogous characterization of accessible points as in [10] using symbolic computations. What this sections adds to the results from [10] is the characterization of types of accessible folding points, specially in the irrational height case (see the definitions below). By knowing the exact symbolic description of points in X' we can determine whether they are folding points or not, and if they are, whether they are endpoints of X'. The classification of accessible sets differentiates (as in [10]) according to the *height* of the kneading sequence which we introduce shortly in this section (for more details see [20]).

We denote by L' the left infinite itinerary which is the largest admissible sequence in the embedding X' for $L = 0^\infty 1$ (as in [14]) after $\mathcal{C}$ is removed. Therefore we need to find which basic arc of X' is the closest to the basic arc $A(0^\infty 1)$. This was calculated in [9].

Definition 11.1. *Let $q \in (0, \frac{1}{2})$. For $i \in \mathbb{N}$ define*

$$\kappa_i(q) = \begin{cases} \lfloor \frac{1}{q} \rfloor - 1, & \text{if } i = 1,\\ \lfloor \frac{i}{q} \rfloor - \lfloor \frac{i-1}{q} \rfloor - 2, & \text{if } i \geq 2. \end{cases}$$

If q is irrational, we say that the kneading sequence

$$\nu = 10^{\kappa_1(q)}110^{\kappa_2(q)}110^{\kappa_3(q)}11\dots$$

has height q *or that it is of* irrational type*. If $q = \frac{m}{n}$, where m and n are relatively prime, we define*

$$c_q = 10^{\kappa_1(q)}110^{\kappa_2(q)}11\dots110^{\kappa_m(q)}1,$$
$$w_q = 10^{\kappa_1(q)}110^{\kappa_2(q)}11\dots110^{\kappa_m(q)-1}.$$

By $\hat{a}$ we denote the reverse of a word a, so $\hat{w}_q = 0^{\kappa_m(q)-1}110^{\kappa_{m-1}(q)}11\dots110^{\kappa_1(q)}1$. We say that a kneading sequence has rational height q *if $(w_q1)^\infty \preceq \nu \preceq 10(\hat{w}_q1)^\infty$. Denote by $\mathrm{lhe}(q) := (w_q1)^\infty$, $\mathrm{rhe}(q) := 10(\hat{w}_q1)^\infty$. If $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$ we say that ν is of* rational interior type*, and* rational endpoint type *otherwise. Every kneading sequence* *that appears in the tent map family is either of rational endpoint, rational interior or irrational type, see Lemma 8 and Lemma 9 in* [9] *(for further information see also* [20]*).*

Remark 11.2. *The values of $\kappa_i(q)$ can be obtained in the following way (see Lemma 2.5 in* [20] *for details). Draw the graph Γ_ζ of the function $\zeta\colon \mathbb{R} \to \mathbb{R}$, $\zeta(x) = qx$. Then $\kappa_i(q) = N_i - 2$, where N_i is the number of intersections of the graph Γ_ζ with vertical lines $x = N$, $N \in \mathbb{N}_0$ in the segment $[i-1, i]$, see Figure 15. Note that it automatically follows that the word $\kappa_1(q)\kappa_2(q)\dots\kappa_m(q)$ is a palindrome and thus c_q is a palindrome. Furthermore, for every $i \in \mathbb{N}$ either $\kappa_i(q) = \kappa_1(q)$ or $\kappa_i(q) = \kappa_1(q) - 1$.*

Remark 11.3. *Assume $q = m/n$ is rational with m and n being relatively prime. Take $k \in \{1, \dots, n-1\}$ such that $\lceil kq \rceil - kq$ attains the smallest value; such k is unique, since m and n are relatively prime. Denote by $K = \lceil kq \rceil$ and note that for every $i \in \{1, \dots, k\}$ the line that joins $(0,0)$ with (k, K) intersects a vertical line in $[i-1, i]$ if and only if qx intersects a vertical line in $[i-1, i]$. Thus $\kappa_1(q)\dots\kappa_K(q)$ is a palindrome; it is the longest palindrome among $\kappa_1(q)\dots\kappa_i(q)$ for $i < m$. By studying the line which joins (k, K) with (n, m) we conclude that $\kappa_{K+1}(q)\dots\kappa_{m-1}(q)(\kappa_m(q)-1)$ is also a palindrome, see Figure 15. Thus for every rational q there exist palindromes Y, Z such that $c_q = Y1Z01$.*

Remark 11.4. *Note that $\{\kappa_i(q)\}_{i \geq 1}$ is a Sturmian sequence for irrational q and thus there exist infinitely many palindromic prefixes of increasing length (see* e.g. [19]*, Theorem 5) which are of even parity. This can also be concluded by studying the rational approximations of q. Namely, if $k \in \mathbb{N}$ is such that $\lceil iq \rceil - iq$ achieves its minimum in $i = k$ for all $i \in \{1, \dots, k\}$, then the word $\kappa_1(q)\dots\kappa_k(q)$ is a palindrome. Note that $10^{\kappa_1(q)}11\dots110^{\kappa_k(q)}1$ is also a palindrome and it is an even word. By choosing better rational approximations of q from above, we see that k can be taken arbitrary large, and thus the beginning of c_q consists of arbitrary long even palindromes.*

Lemma 11.5. *Let $q = \frac{m}{n}$. Then there exists $N \in \mathbb{N}$ such that $\sigma^N(\mathrm{rhe}(q)) = \mathrm{lhe}(q)$.*

Proof. Recall that $\mathrm{lhe}(q) = (w_q 1)^\infty$, $\mathrm{rhe}(q) = 10(\hat{w}_q 1)^\infty$, where $c_q = w_q 01$. By Remark 11.3, there exist palindromes Y, Z such that $c_q = Y1Z01$, so $w_q = Y1Z$. It follows that $\mathrm{lhe}(q) = (Y1Z1)^\infty$ and $\mathrm{rhe}(q) = 10(Z1Y1)^\infty$ which finishes the proof. □

Remark 11.6. *The height of a kneading sequence is the rotation number of the natural mapping on the circle of prime ends. We will only need symbolic representation of the height of a kneading sequence here; for a more detailed study of height see* [20]*.*

Definition 11.7. *Given an infinite sequence $\overrightarrow{x} = x_1x_2x_3\dots$, in this section we denote its reverse by $\overleftarrow{x} = \dots x_3x_2x_1$.*

Lemma 11.8 ([9], Lemma 13)**.** *Let X' be embedded with $\varphi_{\mathcal{C}}$. Denote by L' the largest admissible basic arc in X' and by ν the kneading sequence corresponding to X'. Then,*

$$L' = \begin{cases} \overleftarrow{\mathrm{rhe}(q)}, & \textit{if } \mathrm{lhe} \prec \nu \preceq \mathrm{rhe}(q), \\ \overleftarrow{\nu}, & \textit{if } q \textit{ is irrational or } \nu = \mathrm{lhe}(q). \end{cases}$$

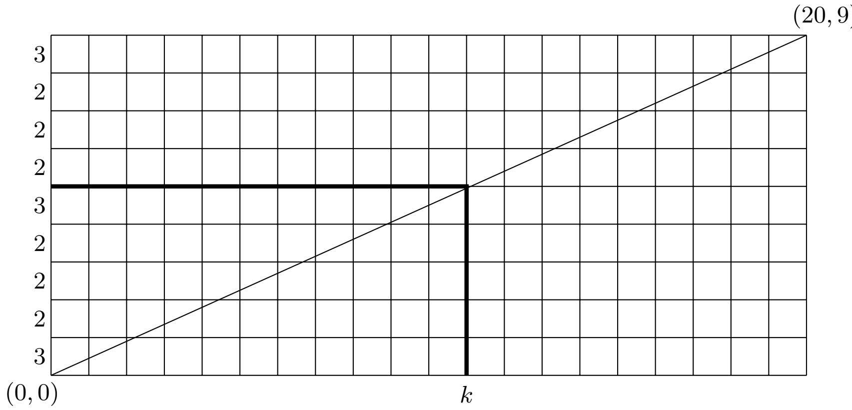

FIGURE 15. Calculating $\kappa_i(q)$ by counting the intersections of the line qx with vertical lines over integers. The picture shows the values N_i for $q = \frac{9}{20}$. It follows that $c_q = 101111111101111111101 = (101111111101)1(111111)01 = Y1Z01$. The decomposition into palindromes Y, Z follows since $\lceil \frac{9}{20}k \rceil - \frac{9}{20}k$ obtains its minimum for $k = 11 = \lfloor \frac{5}{q} \rfloor$ (bold line in the figure).

11.1. **Irrational height case.** Assume that q is irrational and note that the map T is then long-branched (since the kneading map is bounded, see [17]). Therefore, every proper subcontinuum is a point or an arc (see Proposition 3 in [13]) and consequently, every composant is an arc-component and thus either a line or a ray (every composant of X' is dense in X' so an arc can not be a composant of X'). We will show that the basic arc $A(L')$ (which is fully accessible) contains an endpoint of X'. Furthermore, we will prove that the basic arc adjacent to $A(L')$ is not an extremum of a cylinder, and thus contains a folding point which is not an endpoint. Therefore, the ray $\mathcal{U}_{L'}$ is partially accessible; only a compact arc $Q \subset \mathcal{U}_{L'}$ is fully accessible and $\mathcal{U}_{L'} \setminus Q$ is not accessible. Since σ is extendable, also $\sigma^i(Q)$ is accessible for every $i \in \mathbb{Z}$. Later in the subsection we show that no other non-degenerate arc except of $\sigma^i(Q)$ for every $i \in \mathbb{Z}$ is fully accessible. From the circle of prime ends we then see that there is still a Cantor set of points remaining to be associated to either accessible points or infinite canals of $\varphi_{\mathcal{C}}(X')$. We prove that the remaining points on the circle of prime ends correspond to accessible endpoints of $\varphi_{\mathcal{C}}(X')$ and are thus second kind prime ends. Moreover, we prove that every endpoint from $\varphi_{\mathcal{C}}(X')$ is accessible. This is an extension of Theorem 4.46 from [10]. In this subsection the usage of variables m and n should not be confused with the values in the fraction $q = \frac{m}{n}$ which will be used in the rational height case later in the paper.

Lemma 11.9. *If ν is of irrational type, then $\tau_R(L') = \infty$ and $A(L')$ is non-degenerate.*

Proof. If ν is of irrational type, then the bonding map T is long-branched, so every basic arc in X' is non-degenerate. To prove the first claim, first note that by Lemma 11.8 it holds that $L' = \overleftarrow{\nu}$. Remark 11.4 implies that there exist infinitely many even palindromes of increasing length at the beginning of ν. Thus there exists a strictly increasing sequence $(m_i)_{i\in\mathbb{N}}$ such that $l'_{m_i}\dots l'_1 = c_1\dots c_{m_i}$ and $\#_1(c_1\dots c_{m_i})$ is even for every i. Thus it follows that $\tau_R(L')=\infty$. □

The following remark follows from Remark 15 in [9] and the fact that we restrict our study only on the tent map family.

Remark 11.10. *If ν is of irrational or rational endpoint type, it holds that $\overleftarrow{t}\in\{0,1\}^\infty$ is admissible (*i.e., *every subword of $\overleftarrow{t}$ is admissible) if and only if $\overrightarrow{t}$ is admissible (*i.e., *every subword of $\overrightarrow{t}$ is admissible).*

Lemma 11.11. *Let ν be either of irrational or rational endpoint type and X' embedded with $\varphi_{\mathcal{C}}$. Then every extremum of a cylinder of $\varphi_{\mathcal{C}}(X')$ belongs to $\sigma^i(L')$ for some $i\in\mathbb{Z}$.*

Proof. Take an admissible finite word $a_n\dots a_1\in\{0,1\}^n$ and pick the smallest $k\in\{0,\dots,n-1\}$ such that $a_n\dots a_{k+1}=c_{n-k+1}\dots c_2$. If there is no such k we set $k=n$.

Assume first that $k>1$ and note that $a_k=1$.
Assume that $\#_1(a_{k-1}\dots a_1)$ is even and let us calculate $L_{a_n\dots a_1}$. If admissible, the word $L'a_{k-1}\dots a_1$ is the largest in the cylinder $[a_n\dots a_1]$. Assume that $L'a_{k-1}\dots a_1$ is not admissible. By Remark 11.10, since both L' and $a_{k-1}\dots a_1$ are admissible, there exists $i\in\{1,\dots,k-1\}$ such that $a_i\dots a_{k-1}l'_1\dots l'_j$ is not admissible for some $j\geq 1$. If $j\leq n-k+1$, then $a_i\dots a_{k-1}l'_1\dots l'_j$ is a subword of $a_1\dots a_n$ which is not admissible, a contradiction. Assume that $j>n-k+1$. In this case the word $a_i\dots a_{k-1}l'_1\dots l'_j\nsubseteq a_1\dots a_n$ is not admissible, but then $a_i\dots a_n=c_2\dots c_{2+n-i}$ which is a contradiction with k being the smallest such that $a_n\dots a_{k+1}=c_{n-k}\dots c_2$. If $\#_1(a_{k-1}\dots a_1)$ is odd we obtain that $S_{a_n\dots a_1}=L'a_{k-1}\dots a_1$ using analogous arguments as above.
Now assume that $\#_1(a_{k-1}\dots a_1)$ is odd and we calculate $L_{a_n\dots a_1}$. Say that $\#_1(a_n\dots a_k)$ is odd. Therefore, since we want to calculate the largest basic arc in the cylinder $[a_n\dots a_1]$, we need to set $L_{a_n\dots a_1}=\dots 1a_n\dots a_1$, and note that $1a_n\dots a_1$ is always admissible by Lemma 5.2. Then, knowing that $\#_1(a_n\dots a_k)$ is odd it follows from the special structure of ν in the irrational height case that the kneading sequence starts as $a_k\dots a_n11$ or $a_k\dots a_n0$ and thus the word $a_k\dots a_n10$ is admissible. It follows that $L'a_n\dots a_1$ is admissible and equals to $L_{a_n\dots a_1}$. If $\#_1(a_n\dots a_k)$ is even, it follows from the structure of ν (blocks of ones in ν are of even length) that $a_n=1$ and $a_k\dots a_n$ ends in odd number of ones. The word $a_k\dots a_n0^{\kappa_1(q)}$ is thus admissible and therefore $L_{a_n\dots a_1}=L'a_{n-1}\dots a_1$. Calculations for $S_{a_n\dots a_1}$ when $\#_1(a_{k-1}\dots a_1)$ is even follow analogously.

Now say that $k=1$. Then $L_{a_n\dots a_1}=L'$. We conclude as in the preceding paragraph that if $\#_1(a_n\dots a_1)$ is even, then $S_{a_n\dots a_1}=L'a_{n-1}\dots a_1$ and if $\#_1(a_n\dots a_1)$ is odd, then $S_{a_n\dots a_1}=L'a_n\dots a_1$.

If $k=0$, then $a_1\dots a_n=c_2\dots c_{n+1}$. So $S_{a_n\dots a_1}=S=\dots c_4c_3c_2$. To calculate $L_{a_n\dots a_1}$, let k' be the smallest natural number such that $a_n\dots a_{k'}=c_{n-k'+1}\dots c_1$. If k' does not exist, set $k'=n+1$. From the structure of ν (blocks of ones in ν are of even length) it follows that $\#_1(a_{k'-1}\dots a_1)$ is odd. The rest of the proof for this case follows the same as in the case for $k>1$. □

Lemma 11.12. *Assume ν is of irrational type and X' embedded with $\varphi_{\mathcal{C}}$. Then the only basic arc from $\mathcal{U}_{L'}$ which is an extremum of a cylinder is $A(L')$.*

Proof. Let $a_n \dots a_1$ be an admissible word for some $n \in \mathbb{N}$. If $n = 1$, note that $L_1 = L' \subset \mathcal{U}_{L'}$ and $L_0, S_0, S_1 \not\subset \mathcal{U}_{L'}$, since ν is not (pre)periodic.
Now assume that $n \geq 2$. Since ν is not (pre)periodic, the proof of Lemma 11.11 gives that if $L_{a_n \dots a_1}$ or $S_{a_n \dots a_1}$ are contained in $\mathcal{U}_{L'}$, then $a_1 \dots a_n = c_1 \dots c_n$ (since otherwise $L_{a_n \dots a_1}$ or $S_{a_n \dots a_1}$ would be contained in $\sigma^i(\mathcal{U}_{L'})$ for some $i \in \mathbb{Z} \setminus \{0\}$). But then, following the proof of Lemma 11.11 it holds that $L_{a_n \dots a_1} = L'$ and $S_{a_n \dots a_1} = L' a_n \dots a_1$ or $S_{a_n \dots a_1} = L' a_{n-1} \dots a_1$, depending on the parity of $\#_1(a_n \dots a_1)$. Since $L' a_n \dots a_1 \in \sigma^n(L')$ and $L' a_{n-1} \dots a_1 \in \sigma^{n-1}(L')$ the only extremum of a cylinder in $\mathcal{U}_{L'}$ is $A(L')$. □

Remark 11.13. *It follows from Lemma 11.12 that when ν has irrational height, then $\mathcal{U}_{L'}$ is partially accessible. More precisely, from Proposition 5.14 it follows that $\overleftarrow{l(L')} = \dots 110^{\kappa_3(q)} 110^{\kappa_2(q)} 110^{\kappa_1(q)-1} 11$ contains a folding point p and $A(L') \cup [a, p]$ is fully accessible, where a denotes the left endpoint of $\overleftarrow{l(L')}$. It follows from Corollary 3.3 that no other point from $\mathcal{U}_{L'}$ (which is a ray) is accessible. Since $\sigma : \varphi_{\mathcal{C}}(X') \to \varphi_{\mathcal{C}}(X')$ is extendable to the plane, also $\sigma^i(A(L') \cup [a, p])$ is accessible for every $i \in \mathbb{Z}$. In the lemmas to follow we prove that the remaining Cantor set of points on the circle of prime ends correspond to the endpoints of $\varphi_{\mathcal{C}}(X')$, and that all endpoints of $\varphi_{\mathcal{C}}(X')$ are accessible when ν is of irrational type.*

The following lemma follows directly from the fact that $(\kappa_i(q))$ is Sturmian, but we prove it here for the sake of completeness. Say that $q \in (0, \frac{1}{2})$ is irrational. Denote by $\kappa = \kappa_1(q)$, so $\kappa_i(q) \in \{\kappa, \kappa - 1\}$ for every $i \in \mathbb{N}$.

Lemma 11.14. *Let $q \in (0, \frac{1}{2})$ be irrational. There exists $J \in \mathbb{N}$ such that if $\kappa_i(q) \kappa_{i+1}(q) \dots \kappa_{i+N}(q) \kappa_{i+N+1}(q) = \kappa(\kappa - 1)^N \kappa$, then $N \in \{J, J + 1\}$.*

Proof. Let $J \in \mathbb{N}$ be such that $\kappa_2(q) = \ldots = \kappa_{J+1}(q) = \kappa - 1$ and $\kappa_{J+2}(q) = \kappa$. So there exists a sequence of J consecutive $(\kappa - 1)$s. Denote by $H_n = \lfloor \frac{n}{q} \rfloor$ for $n \in \mathbb{N}$ and note that the function $g \colon \mathbb{N} \to \mathbb{R}$ given by $g(k) = \lceil kq \rceil - kq$ achieves its minimum on $[0, H_{J+2}]$ in H_{J+2} (since $J + 2$ is minimal index $I > 1$ for which $\kappa_I = \kappa$). If we translate the graph of function $\zeta(x) = qx$ by $+\delta$ where $\delta \in (0, g(H_{J+2})]$, then the sequence of consecutive number of intersections with vertical lines over integers begins again with $(\kappa + 2)(\kappa + 1)^J (\kappa + 2)$. Since g restricted to $[0, H_{J+2})$ achieves its minimum in H_1, if $\delta \in (g(H_{J+2}), g(H_1))$, the sequence corresponding to the number of times the graph of $\zeta + \delta$ intersects vertical lines over integers begins with $(\kappa + 2)(\kappa + 1)^{J+1}(\kappa + 2)$, see Figure 16. Fix $i \geq 2$ such that $\kappa_i(q) = \kappa$. Note that then $g(H_{i-1} + 1) < g(H_1)$ since otherwise $qH_{i-1} > i - 1$ which is a contradiction. So the graph of ζ on $[H_{i-1} + 1, \infty)$ can be obtained from the graph of ζ on $[0, \infty)$ by translating it by $+\delta$ for $\delta \in (0, g(H_1))$ which finishes the proof. □

Lemma 11.15. *Let $q \in (0, \frac{1}{2})$ be irrational and $i, N \in \mathbb{N}$ such that $\kappa_{i+1}(q) \dots \kappa_{i+N}(q) = \kappa_1(q) \dots \kappa_N(q)$ and $\kappa_{i+N+1}(q) \neq \kappa_{N+1}(q)$. Then $\kappa_1(q) \dots \kappa_{N+1}(q)$ is a palindrome.*

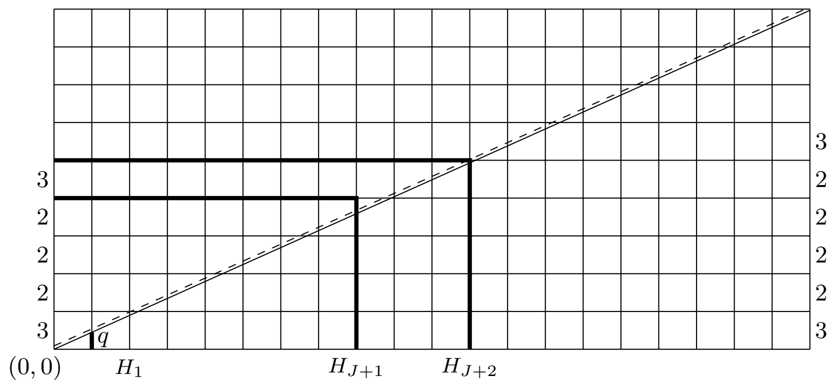

FIGURE 16. The graph of qx for $q \approx 0.4483\ldots$ with the number of intersections with vertical integer lines on the left. Dashed line represents the graph of qx translated by $\delta \in (g(H_{J+2}), g(H_1))$. On the right we count the intersections of the translated graph with vertical integer lines.

Moreover, $\kappa_{i+N+2}(q) = \kappa_1(q)$*. If* $K \in \mathbb{N}$ *is such that* $\kappa_{i+N+2}(q)\ldots\kappa_{i+N+K+1}(q) = \kappa_1(q)\ldots\kappa_K(q)$ *and* $\kappa_{i+N+K+2}(q) \neq \kappa_{K+1}(q)$*, then* $\kappa_{K+1}(q)\ldots\kappa_1(q)\kappa_{i+N+1}(q)\ldots\kappa_{i+1}(q) = \kappa_1(q)\ldots\kappa_{K+N+1}(q)$*.*

Proof. For $i \in \mathbb{N}$ denote by $H_i = \lfloor \frac{i}{q} \rfloor$ and let $f\colon \mathbb{N} \to \mathbb{R}$ be given by $f(x) = xq - \lfloor xq \rfloor$. Note that the graph of $\zeta(x) = qx$ restricted to $[H_i+1, \infty)$ is a translation of the graph of ζ on $[0,\infty)$ by some $\delta > 0$ (see *e.g.* Figure 16). The conditions $\kappa_{i+1}(q)\ldots\kappa_{i+N}(q) = \kappa_1(q)\ldots\kappa_N(q)$ and $\kappa_{i+N+1}(q) \neq \kappa_{N+1}(q)$ imply that the global minimum of f on $[H_i, H_{i+N+1}+1]$ is $H_{i+N+1}+1$. So the graph of $\zeta - f(H_{i+N+1}+1)$ on $[H_i, H_{i+N+1}+1]$ intersects vertical lines over integers the same number of times as ζ except for the point $(H_{i+N+1}+1, i+N+1)$. We conclude that $(\kappa_{i+N+1}(q)+1)\kappa_{i+N}(q)\ldots\kappa_{i+1}(q) = \kappa_1(q)\ldots\kappa_{N+1}(q)$ which concludes the first part of the proof. To see that $\kappa_{i+N+2}(q) = \kappa_1(q)$ use Lemma 11.14.
For the last part of the proof assume that $K \in \mathbb{N}$ is such that $\kappa_{i+N+2}(q)\ldots\kappa_{i+N+K+1}(q) = \kappa_1(q)\ldots\kappa_K(q)$ and $\kappa_{i+N+K+2}(q) \neq \kappa_{K+1}(q)$. That implies that the global minimum of f on $[H_i, H_{i+N+K+2}+1]$ is $H_{i+N+K+2}+1$. Again by translating the graph of ζ on $[H_i, H_{i+N+K+2}+1]$ by $-f(H_{i+N+K+2}+1)$ we conclude the second part of the proof, see Figure 17. □

Lemma 11.16. *If* ν *is of irrational type or* $\nu = lhe(q)$*, then every endpoint of* $\varphi_{\mathcal{C}}(X')$ *is accessible.*

Proof. Let $e \in X'$ be an endpoint and let $\overleftarrow{e}$ denote the left infinite symbolic description of e.
Assume that $\tau_R(\overleftarrow{e}) = \infty$ and thus there exists a strictly increasing sequence $(m_i)_{i\in\mathbb{N}}$ such that $c_1\ldots c_{m_i} = e_{m_i}\ldots e_1$ and $\#_1(e_{m_i}\ldots e_1)$ is even. Assume $(m_i)_{i\in\mathbb{N}}$ is the complete sequence for e (see Definition 7.5).
Assume that for infinitely many $i \in \mathbb{N}$ there exist admissible left infinite itineraries

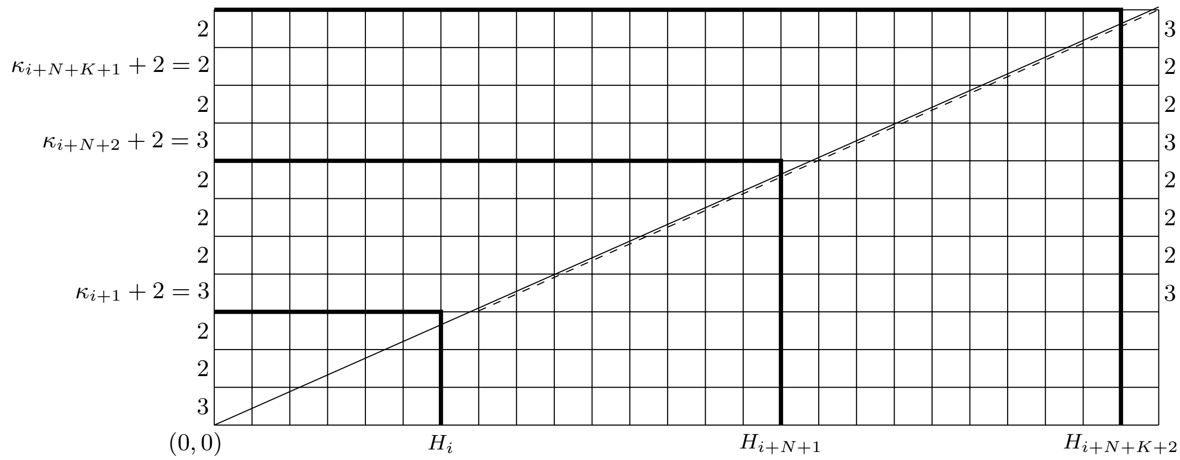

FIGURE 17. Graphic representation of the proof of Lemma 11.15 for $q \approx 0.443\ldots$. Dashed line represents the graph of $\zeta(x) = qx$ on $[H_i + 1, H_{i+N+K+2}+1]$ translated by $-f(H_{i+N+K+2}+1)$. On the right side of the grid we count intersections of the dashed line with vertical integer lines.

$\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e} \prec_L x^{I(i)}$ so that $\overleftarrow{x}^{O(i)}, \overleftarrow{x}^{I(i)} \to \overleftarrow{e}$ as $i \to \infty$, $\overleftarrow{x}^{O(i)}, \overleftarrow{x}^{I(i)}$ differ only at the index $m_i + 1$ and equal $c_1 \ldots c_{m_i}$ on the first m_i places (if we are able to construct such $\overleftarrow{x}^{O(i)}, \overleftarrow{x}^{I(i)}$ the arcs will cap the endpoint e which would thus be inaccessible - compare with the proof of Theorem 7.12). So, $\overleftarrow{x}^{O(i)}$ and $\overleftarrow{x}^{O(i)}$ are of the form:

$$\overleftarrow{x}^{I(i)} = \ldots 110^{\kappa_1(q)}110^{\kappa_2(q)}11\ldots 110^{\kappa_j(q)}1.$$

$$\overleftarrow{e} = \ldots 110^{\kappa_1(q)}110^{\kappa_2(q)}11\ldots 110^{\kappa_j(q)}1.$$

$$\overleftarrow{x}^{O(i)} = \ldots 010^{\kappa_1(q)}110^{\kappa_2(q)}11\ldots 110^{\kappa_j(q)}1.$$

Note first that $0e_{m_i}\ldots e_1$ is indeed admissible. Since $\#_1(e_{m_i}\ldots e_1)$ is even it holds that $\overleftarrow{x}^{O(i)} \prec_L \overleftarrow{e}$ for every $i \in \mathbb{N}$. Thus we need to find $\overleftarrow{x}^{I(i)} \succ_L \overleftarrow{e}$ in order to cap e. Denote by $J \in \mathbb{N}$ the smallest natural number such that

$$\overleftarrow{e} = \ldots 110^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\ldots 110^{\kappa_2(q)}110^{\kappa_1(q)}110^{\kappa_2(q)}11\ldots 110^{\kappa_j(q)}1.$$

By Lemma 11.15 it follows that $\kappa_J(q)\ldots\kappa_2(q)\kappa_1(q)$ is a palindrome and thus $10^{\kappa_J(q)}11$ $0^{\kappa_{J-1}(q)}11\ldots 110^{\kappa_2(q)}11$ equals the beginning of ν.

We want to find $\overleftarrow{x}^{I(i)} \succ \overleftarrow{e}$. Note that none of $00^{\kappa_2(q)}110^{\kappa_1(q)}, \ldots, 00^{\kappa_{J-1}(q)}11\ldots 110^{\kappa_1(q)}$ are admissible. If we set

$$\overleftarrow{x}^{I(i)} = \ldots 00^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\ldots 110^{\kappa_2(q)}110^{\kappa_1(q)}11\ldots 110^{\kappa_j(q)}1,$$

then also

$$\overleftarrow{x}^{O(i)} = \ldots 00^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\ldots 110^{\kappa_2(q)}010^{\kappa_1(q)}11\ldots 110^{\kappa_j(q)}1.$$

But since $100^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\ldots 110^{\kappa_2(q)}$ equals the beginning of ν, the word $00^{\kappa_J(q)-1}11$ $0^{\kappa_{J-1}(q)}11\ldots 110^{\kappa_2(q)}0$ is not admissible, a contradiction.

Thus we need to set

$$\overleftarrow{x}^{I(i)} = \ldots 110^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\ldots 110^{\kappa_2(q)}110^{\kappa_1(q)}11\ldots 110^{\kappa_j(q)}1.$$

By Lemma 11.14 it follows that

$$\overleftarrow{e} = \ldots 110^{\kappa_1(q)}110^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\ldots 110^{\kappa_2(q)}110^{\kappa_1(q)}110^{\kappa_2(q)}11\ldots 110^{\kappa_j(q)}1.$$

Now take the smallest $K \in \mathbb{N}$ such that

$$\begin{aligned}\overleftarrow{e} &= \ldots 110^{\kappa_{K+1}(q)-1}110^{\kappa_K(q)}11\ldots 110^{\kappa_1(q)}110^{\kappa_J(q)-1}1\\ &\quad 10^{\kappa_{J-1}(q)}11\ldots 110^{\kappa_2(q)}110^{\kappa_1(q)}110^{\kappa_2(q)}11\ldots 110^{\kappa_j(q)}1.\end{aligned}$$

By Lemma 11.15 it follows that $10^{\kappa_{K+1}(q)}11\ldots 110^{\kappa_1(q)}110^{\kappa_J(q)-1}110^{\kappa_{J-1}(q)}11\ldots 110^{\kappa_2(q)}11$ is the beginning of ν. Thus we analogously argue that

$$\begin{aligned}\overleftarrow{x}^{I(i)} &= \ldots 110^{\kappa_{K+1}(q)-1}110^{\kappa_K(q)}11\ldots 110^{\kappa_1(q)}110^{\kappa_J(q)-1}1\\ &\quad 10^{\kappa_{J-1}(q)}11\ldots 110^{\kappa_2(q)}110^{\kappa_1(q)}110^{\kappa_2(q)}11\ldots 110^{\kappa_j(q)}1,\end{aligned}$$

which agrees with $\overleftarrow{e}$. Continuing inductively we conclude that $\overleftarrow{x}^{I(i)} = \overleftarrow{e}$. Thus e is not capped. $\square$

Remark 11.17. *We can expand the definition of Type 3 folding point introduced in the preperiodic orbit case. A point p will be called a Type 3 folding point, if it is not an endpoint, it is accessible, and there is an arc $p \in V \subset \mathcal{U}_p$ such that $V \setminus \{p\}$ is not accessible, see Figure 13.*

Lemma 11.18. *If ν is of irrational type or rational endpoint type and X' is embedded with $\varphi_{\mathcal{C}}$, then there are no Type 3 folding points.*

Proof. Assume by contradiction that there is a basic arc $\overleftarrow{x} = \ldots x_2x_1$ and an accessible folding point $p \in A(\overleftarrow{x})$ of Type 3. Since p is a folding point by Proposition 2.4 there exist blocks of symbols of ν of increasing length in $\overleftarrow{x}$.

We claim that if $c_n \ldots c_{n+k} = c_m \ldots c_{m+k}$ for some $m, n \in \mathbb{N}$ and there exists $i \in \{0, \ldots, k\}$ such that $c_{n+i} = 0$, then $\#_1(c_1 \ldots c_{n+k}) = \#_1(c_1 \ldots c_{m+k})$ (then all the wiggles will accumulate on $A(\overleftarrow{x})$ from exactly one side of p as in Figure 11). Indeed, take the largest such index i. Then it follows that $c_n \ldots c_{n+i-1} = 1^i$. If i is even (odd) it holds that $\#_1(c_1 \ldots c_{n-1})$ is odd (even), which proves the claim.
Therefore, if for $\overleftarrow{x} = \ldots x_2x_1$ there exists $i \in \{0, \ldots k\}$ such that $c_{n+i} = 0$ and $x_j \ldots x_1 = c_n \ldots c_{n+k}$ it follows that $A(\overleftarrow{x})$ contains no Type 3 folding point.

Now assume that $\overleftarrow{x} = 1^\infty$. If $\kappa_1(q) > 1$, then $\ldots 1101^\alpha \succ_L \overleftarrow{x} \succ_L \ldots 1101^{\alpha+1}$ for every odd $\alpha \in \mathbb{N}$ and both $\ldots 1101^\alpha$ and $\ldots 1101^{\alpha+1}$ project to $[T^2(c), T(c)]$, which is again a contradiction with p being a Type 3 folding point.
If $\kappa_1(q) = 1$, then $\nu = 101^\beta 0\ldots$ for some even $\beta \in \mathbb{N}$. Then, basic arcs with symbolic description $1^\infty 01^\gamma$ for every $\gamma > \beta$ project to $[T^2(c), T(c)]$ and we get an analogous conclusion as in the preceding paragraph. $\square$

Lemma 11.19. *If ν is of irrational type, then there exist no third and fourth kind prime ends corresponding to $\varphi_{\mathcal{C}}(X')$.*

Proof. Since the embedding $\varphi_{\mathcal{C}}(X')$ is realized as an alignment of basic arcs along vertically embedded Cantor set connected with semi-circles, we can study crosscuts which are vertical segments in the plane joining two adjacent cylinders, see Figure 3. Note that every infinite canal is realized by such vertical crosscuts. Take two n-cylinders $A=[a_n\dots a_1]$ and $B=[b_n\dots b_1]$ for some $n\in\mathbb{N}$, such that $A\succ_L B$ and A and B are *adjacent* n-cylinders, *i.e.,* there is no n-cylinder D such that $A\succ_L D\succ_L B$. We will show that S_A and L_B have the same tail, *i.e.,* they both belong to $\sigma^i(L')$ for some $i\in\mathbb{Z}$. Since the accessible subsets of $\sigma^i(L')$ are arcs of finite length, it follows immediately that there cannot exist infinite canals for $\varphi_{\mathcal{C}}(X')$.

Take A and B as above and let $m\in\{0,\dots,n-1\}$ be the smallest nonnegative number such that $a_{m+1}\neq b_{m+1}$.
First assume that $\#_1(a_m\dots a_1)$ is odd. Then, $S_A=S_{0a_m\dots a_1}$ and $L_B=L_{1a_m\dots a_1}$, since $A\succ_L B$ are adjacent. Let $k\in\{1,\dots,m-1\}$ be the smallest number such that $c_2\dots c_{m-k+2}=a_{k+1}\dots a_m 1$, (compare with the proof of Lemma 11.11). Assume first that such k indeed exists. Since also $c_2\dots c^*_{m-k+2}\subset S_A$ is admissible, it follows that $\#_1(a_{k+1}\dots a_m)$ is odd. Thus, $\#_1(a_k\dots a_1)$ is even and since $a_k=1$ it holds that $\#_1(a_{k-1}\dots a_1)$ is odd. As in the proof of Lemma 11.11, we conclude that $L_{1a_m\dots a_1}=L'1a_m\dots a_1$. The same conclusion follows in the case when k does not exist. Note that $k=m$ is not possible. Furthermore, since $\#_1(a_k\dots a_m)$ is odd, it follows from the specific form of ν that $S_{0a_m\dots a_1}=L'0a_m\dots a_1$, which is always admissible. Therefore, S_A and L_B have the same left infinite tail.
Now assume that $\#_1(a_m\dots a_1)$ is even. Then $S_A=S_{1a_m\dots a_1}$ and $L_B=L_{0a_m\dots a_1}$, since $A\succ_L B$ are adjacent. Let $k\in\{1,\dots,m-1\}$ again be the smallest number such that $c_2\dots c_{m-k+1}=a_{k+1}\dots a_m 1$. By analogous arguments as in the preceding paragraph we obtain that $\#_1(a_{k-1}\dots a_1)$ is even and thus as in the proof of Lemma 11.11, we conclude that $S_{1a_m\dots a_1}=L'1a_m\dots a_1$. Furthermore, $L_{0a_m\dots a_1}=L'0a_m\dots a_1$ which is always admissible. Again, S_A and L_B have the same left infinite tail. Therefore, it holds that all the canals are finite, *i.e.,* there exist no third and fourth kind prime ends corresponding to $\varphi_{\mathcal{C}}(X')$. $\square$

The following theorem follows directly from the preceding eight lemmas.

Theorem 11.20. *If ν is of irrational type and X' is embedded with $\varphi_{\mathcal{C}}$, then there are countably infinitely many partially accessible rays of $\varphi_{\mathcal{C}}(X')$; these are the arc-components which are symbolically described by a tail which is a shift of $\overleftarrow{\nu}$. Each of them contains an endpoint of $\varphi_{\mathcal{C}}(X')$ and accessible set is a compact arc which contains that endpoint. Furthermore, there exist uncountably many accessible arc-components which are accessible in a single point which is an endpoint of $\varphi_{\mathcal{C}}(X')$. All (uncountably many) endpoints of $\varphi_{\mathcal{C}}(X')$ are accessible.*

11.2. **Rational endpoint case.** Let $q=\frac{m}{n}$. In this subsection we study $\varphi_{\mathcal{C}}(X')$ when ν is either $\mathrm{rhe}(q)$ or $\mathrm{lhe}(q)$. We provide a symbolic proof of Theorem 4.66 from [10].

When $\nu=\mathrm{lhe}(q)=(w_q 1)^\infty$ it follows that $L'=\overleftarrow{\mathrm{lhe}(q)}$. In Remark 11.3 we argued that there exist palindromes Y,Z such that $\mathrm{lhe}(q)=(Y1Z1)^\infty$, thus $\overleftarrow{\mathrm{lhe}(q)}=(1Z1Y)^\infty$.

Note that both Y and Z are even, from which we conclude that $\tau_R(L') = \infty$. Thus the right endpoint of $A(L')$ is also an endpoint of X' and since there are no other folding points on $\mathcal{U}_{L'}$ except of this endpoint, the ray $\mathcal{U}_{L'}$ is a fully accessible. Since σ is extendable to the plane it follows that $\sigma^i(\mathcal{U}_{L'})$ are fully accessible for every $i \in \{0, 1, \dots, n-1\}$ (where n is the period of $\mathrm{lhe}(q)$). Lemma 11.11 assures that the union of n rays is indeed the complete set of accessible points of $\varphi_{\mathcal{C}}(X')$ for $\nu = \mathrm{lhe}(q)$. Thus the circle of prime ends decomposes into n half-open intervals, where the endpoints represent the endpoints of X'. Summarizing, we have the following theorem:

Theorem 11.21. *If $\nu = lhe(q)$ for some $q = \frac{m}{n}$, where m and n are relatively prime, then in $\varphi_{\mathcal{C}}(X')$ there exist n fully accessible rays which are symbolically described by a tail which is a shift of $\overleftarrow{\nu}$ and no other point from $\varphi_{\mathcal{C}}(X')$ is accessible. Specifically, there exist no infinite canals in $\varphi_{\mathcal{C}}(X')$.*

When $\nu = \mathrm{rhe}(q)$ it holds by Lemma 11.8 that $L' = \overleftarrow{\mathrm{rhe}(q)} = (1Y1Z)^{\infty}01$. Since Y starts with 1 it holds that there exists a folding point $p \in \mathcal{U}_{L'}$ on a basic arc with itinerary $\overleftarrow{l(L')} = (1Y1Z)^{\infty}11$. Since $\mathrm{rhe}(q)$ is strictly preperiodic it follows that left tail of $\overleftarrow{l(L')}$ always differs from $\overleftarrow{\mathrm{rhe}(q)}$, so Lemma 11.11 implies that $\overleftarrow{l(L')}$ is not an extremum of any cylinder. Proposition 5.14 implies that p is Type 2 folding point and consequently $\mathcal{U}_{L'}$ is partially accessible. Moreover, since $\mathcal{U}_{L'}$ contains no other folding points we conclude that one component of $\mathcal{U}_{L'} \setminus \{p\}$ is fully accessible and the other component of $\mathcal{U}_{L'} \setminus \{p\}$ is not accessible. Since σ is extendable, $\sigma^i(\mathcal{U}_{L'})$ are also partially accessible. Lemma 11.11 implies that the circle of prime ends decomposes into n half-open intervals and their endpoints are representing the accessible folding points of Type 2. Thus we obtain the following theorem:

Theorem 11.22. *If $\nu = rhe(q)$ for some $q = \frac{m}{n}$, where m and n are relatively prime, then in $\varphi_{\mathcal{C}}(X')$ there exist n partially accessible lines which are symbolically described by a tail which is a shift of $\overleftarrow{\nu}$ and no other point from $\varphi_{\mathcal{C}}(X')$ is accessible. Specifically, there exist no infinite canals in $\varphi_{\mathcal{C}}(X')$.*

11.3. **Rational interior case.** Assume $q = \frac{m}{n}$, where m and n are relatively prime. We will show that in the rational interior case there exist n fully accessible arc-components which are dense lines in X'. We show that folding points which are not lying in the extrema of cylinders are not accessible, so the remaining n points on the circle of prime ends are simple dense canals. That is an analogue of Theorem 4.64 from [10] for tent inverse limits.

Lemma 11.23 (Theorem 16 in [9])**.** *Suppose that ν is of rational interior type for $q = m/n$, where m and n are relatively prime. Then a sequence $\overleftarrow{t} \in \{0,1\}^{\infty}$ which does not belong to $\mathcal{C}$ is admissible if and only if*

(a) $\sigma^i(\overleftarrow{t}) \preceq \mathrm{rhe}(q)$ *for all* $i \in \mathbb{N}$,
(b) $\sigma^i(\overleftarrow{t}) \preceq \mathrm{lhe}(q)$ *for all* $i \in \mathbb{N}$ *for which* $\sigma^i(\overrightarrow{t}) \succ \sigma^{n+1}(\nu)$.

Lemma 11.24. *Say that $q = m/n$, where m and n are relatively prime. If* $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$, *then all the extrema of cylinders of $\varphi_{\mathcal{C}}(X')$ have tails in $\sigma^i(L')$ for some $i \in \mathbb{Z}$.*

Proof. Fix an arbitrary admissible word $b_j \dots b_1$ for some $j \in \mathbb{N}$.

We calculate top/bottom of the cylinder $[b_j \dots b_1]$. Assume that $b_j \dots b_1 \succ \sigma^{n+1}(\nu)$ and $\#_1(b_j \dots b_1)$ is even (odd). We first show that if $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ is admissible, then it equals $L_{b_j \dots b_1}(S_{b_j \dots b_1})$. Assume by contradiction that there exists an admissible $\dots x_2 x_1 b_j \dots b_1 \succ \overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ $(\dots x_2 x_1 b_j \dots b_1 \prec \overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1)$. Then $\dots x_2 x_1 \succ \overleftarrow{\mathrm{lhe}(q)}$ $(\dots x_2 x_1 \succ \overleftarrow{\mathrm{lhe}(q)})$. But that combined with $b_j \dots b_1 \succ \sigma^{n+1}(\nu)$ gives by (b) from Lemma 11.23 that $\dots x_2 x_1 b_j \dots b_1 \succ \overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ is not admissible, a contradiction. Similarly we show that if $b_j \dots b_1 \preceq \sigma^{n+1}(\nu)$, $\#_1(b_j \dots b_1)$ is even (odd) and $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ is admissible, then it equals $L_{b_j \dots b_1}(S_{b_j \dots b_1})$.

In the next two paragraphs we prove that the sequences of the form $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ in special case to which we restrict later in the proof satisfy conditions (a) and (b) from Lemma 11.23 and are thus admissible.

If $b_{i+1} \dots b_j$ does not equal the beginning of $\mathrm{rhe}(q)$ for any $i \in \{0, \dots, j-1\}$, then the sequences $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ satisfy (a) from Lemma 11.23. Assume there is an index $i \in \{0, \dots j-1\}$ such that $b_{i+1} \dots b_j$ is the beginning of $\mathrm{rhe}(q)$ and take the smallest such $i \in \{0, \dots, j-1\}$. Assume $\#_1(b_{i+1} \dots b_j)$ is odd (later in the proof we need only this special case). If $b_{\alpha+1} \dots b_j$ is also the beginning of $\mathrm{rhe}(q)$ for some $\alpha \in \{0, \dots, j-1\}$, where $\alpha \geq i$, then $\#_1(b_{\alpha+1} \dots b_j)$ is also odd. Note that $b_{\alpha+1} \dots b_j 10 \prec \mathrm{rhe}(q)$ for every such α. Thus $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ satisfy condition (a) from Lemma 11.23.

If for every $i \in \{1, \dots, j\}$ either $b_i \dots b_1 \preceq \sigma^{n+1}(\nu)$ or $b_{i+1} \dots b_j$ is not the beginning of $\mathrm{lhe}(q)$, then $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ satisfy (b) from Lemma 11.23. Assume there is $i < j$ such that $b_i \dots b_1 \succ \sigma^{n+1}(\nu)$ and $b_{i+1} \dots b_j$ is the beginning of $\mathrm{lhe}(q)$ and take the smallest such index i. If $\#_1(b_{i+1} \dots b_j)$ is odd (as in the paragraph above, later in the proof we need only this special case) and there is $\beta \in \{i, \dots, j-1\}$ such that $b_{\beta+1} \dots b_j$ is also the beginning of $\mathrm{lhe}(q)$, then $\#_1(b_{\beta+1} \dots b_j)$ is also odd and thus $b_{\beta+1} \dots b_j 10 \prec \mathrm{lhe}(q)$ for every such β. We conclude that $\overleftarrow{\mathrm{rhe}(q)} b_j \dots b_1$ and $\overleftarrow{\mathrm{lhe}(q)} b_j \dots b_1$ satisfy condition (b) from Lemma 11.23.

Recall that $L' = \overleftarrow{\mathrm{rhe}(q)} = (1w_q)^\infty 01$.

Fix an admissible word $a_N \dots a_1 \in \{0,1\}^N$ for some $N \in \mathbb{N}$. Let $k \in \{1, \dots, N\}$ be, if existent, the smallest index such that $a_k \dots a_1 \succ \sigma^{n+1}(\nu)$ and $a_{k+1} \dots a_N$ is the beginning of $\mathrm{lhe}(q)$. We set $k = N$ when $a_N \dots a_1 \succ \sigma^{n+1}(\nu)$ (then $a_{k+1} \dots a_N = \emptyset$ is the beginning of $\mathrm{lhe}(q)$). Let $k' \in \{0, 1, \dots, N-1\}$ be the smallest index such that

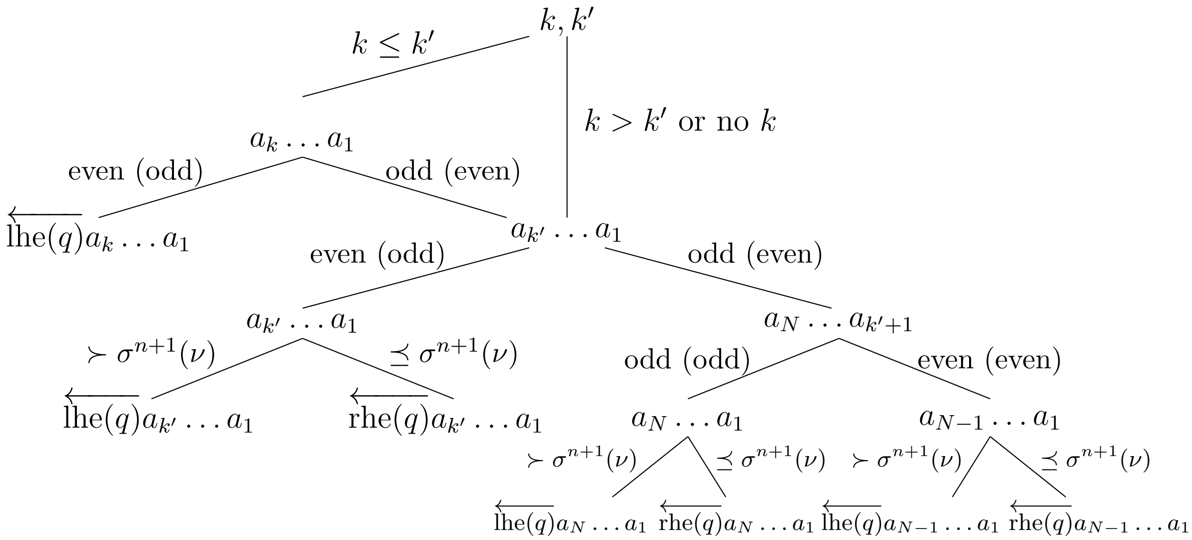

FIGURE 18. Calculating the $L_{a_N \dots a_1}$ and $S_{a_N \dots a_1}$ in the rational interior case. The graph should be read as follows: if we want to calculate $L_{a_N \dots a_1}$ we read the terms outside of the brackets and to calculate $S_{a_N \dots a_1}$ we read the terms inside the brackets. Say we want to calculate $L_{a_N \dots a_1}$ ($S_{a_N \dots a_1}$). We first calculate k and k' and compare them. Say $k > k'$ or k does not exist. We move down the right branch. Next we calculate the parity of $a_{k'} \dots a_1$. Say it is even (odd), then we move down the left branch. If $a_{k'} \dots a_1 \succ \sigma^{n+1}(\nu)$ then $L_{a_N \dots a_1} = \overleftarrow{\text{lhe}} a_{k'} \dots a_1$ ($S_{a_N \dots a_1} = \overleftarrow{\text{lhe}} a_{k'} \dots a_1$) and if $a_{k'} \dots a_1 \preceq \sigma^{n+1}(\nu)$ then $L_{a_N \dots a_1} = \overleftarrow{\text{rhe}} a_{k'} \dots a_1$ ($S_{a_N \dots a_1} = \overleftarrow{\text{rhe}} a_{k'} \dots a_1$).

$a_{k'+1} \dots a_N$ equals the beginning of rhe(q). Note that if $a_i = 1$ for some $i \in \{1, \dots, N\}$, then such k' exists. If $a_N \dots a_1 = 0^N$, then $L_{a_N \dots a_1} = \overleftarrow{\text{rhe}(q)} 0^N$ and $S_{a_N \dots a_1} = S = (1w_q)^\infty 0$.

If $a_i = 1$ for some $i \in \{1, \dots, N\}$, the diagram in Figure 18 provides an algorithm to calculate $L_{a_N \dots a_1}$ ($S_{a_N \dots a_1}$).

To see that the defined sequences are indeed $L_{a_N \dots a_1}$ ($S_{a_N \dots a_1}$) we use the first part of the proof. For example, take the case where the algorithm gives $\overleftarrow{\text{lhe}} a_N \dots a_1$. Since $a_N \dots a_1 \succ \sigma^{n+1}(\nu)$ and $\#_1(a_N \dots a_1 = a_N \dots a_{k'+1} a_{k'} \dots a_1)$ is even (odd), if $\overleftarrow{\text{lhe}} a_N \dots a_1$ is admissible then it equals $L_{a_N \dots a_1}$ ($S_{a_N \dots a_1}$). To see that it satisfies (a), note that $\#_1(a_N \dots a_{k'+1})$ is odd by assumption. To see that is satisfies (b), assume first that there exists k and $k \leq k'$. Then $\#_1(a_{k+1} \dots a_{k'})$ is even and thus $\#_1(a_N \dots a_{k+1})$ is odd. If k does not exists, we are done. If $k > k'$, then since $a_{k'+1} \dots a_N$ is the beginning of rhe(q) and $a_{k+1} \dots a_N$ is the beginning of lhe(q) it follows that $\#_1(a_{k'+1} \dots a_k)$ is even and thus $\#_1(a_{k+1} \dots a_N)$ is of the same parity as $\#_1(a_{k'+1} \dots a_N)$, which is odd. That finishes the proof in this case. Other cases follow using analogous computations.

Note that if $\#_1(a_N \dots a_{k'+1})$ is even, then since $a_{k'+1} \dots a_N$ is the beginning of $\mathrm{rhe}(q)$ it follows that $a_N = 1$ and thus $\#_1(a_{N-1} \dots a_{k'+1})$ is odd (this is needed in the proof of the two cases in the right branch of Figure 18). $\square$

Lemma 11.25. *Say that $q = m/n$, where m and n are relatively prime. If $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$, then every admissible itinerary in $\sigma^i(\mathcal{U}_{L'})$ is realized as an extremum of a cylinder of $\varphi_{\mathcal{C}}(X')$.*

Proof. Assume that $\overleftarrow{x} = \dots x_2 x_1$ is an admissible tail and that there exists $K \in \mathbb{N}_0$ such that $\dots x_{K+2} x_{K+1} = \overleftarrow{\mathrm{lhe}(q)}$ and take K the smallest index with that property. Denote by $\mathrm{lhe}(q) = (w_q 1)^\infty = (y_1 \dots y_n)^\infty$ and note that $\mathrm{rhe}(q) = 10(\hat{w}_q 1)^\infty$ and thus $\sigma^{n+1}(\mathrm{rhe}(q)) = (1\hat{w}_q)^\infty = (y_n \dots y_1)^\infty$. Since $\mathrm{rhe}(q) \succ \nu$ and they agree on the first $n+1$ places (which equal c_q and which is a word of even parity, for details see *e.g.* [10]), it follows that $\sigma^{n+1}(\mathrm{rhe}(q)) \succ \sigma^{n+1}(\nu)$. Let $J \in \mathbb{N}$ be the smallest natural number such that $(y_n \dots y_1)^J \succ \sigma^{n+1}(\nu)$. We study the cylinder $Y = [y_n \dots y_1 (y_n \dots y_1)^J x_K \dots x_1]$. Note that $x_i \dots x_K (y_1 \dots y_n)^{J+1}$ does not agree with the beginning of $\mathrm{lhe}(q)$ for any $i \in \{1, \dots, K\}$. Also $y_i \dots y_n (y_1 \dots y_n)^j$ does not agree with the beginning of $\mathrm{lhe}(q)$ for any $i \in \{2, \dots, n\}$ and any $j \in \mathbb{N}$. Denote by $a_N \dots a_1 = y_n \dots y_1 (y_n \dots y_1)^J x_K \dots x_1$. Let $k \in \{1, \dots, N\}$ be, if existent, the smallest index such that $a_k \dots a_1 \succ \sigma^{n+1}(\nu)$ and $a_{k+1} \dots a_N$ is the beginning of $\mathrm{lhe}(q)$ (compare with the definition of k in the proof of Lemma 11.24). By the choice of J it follows that k indeed exists and $k \in \{K + Mn : M \in \{0, \dots, J\}\}$. So, if for any $i \in \{0, \dots, K-1\}$ the word $x_{i+1} \dots x_K (y_1 \dots y_n)^{J+1}$ does not equal the beginning of $\mathrm{rhe}(q)$, then Lemma 11.24 implies that $\overleftarrow{x} = L_Y$ or $\overleftarrow{x} = S_Y$, depending on the parity of $\#(x_K \dots x_1)$.
If there is $\alpha \in \{0, \dots, K-1\}$ such that the word $x_{\alpha+1} \dots x_K (y_1 \dots y_n)^{J+1}$ equals the beginning of $\mathrm{rhe}(q)$, then $x_\alpha \dots x_1 \preceq \sigma^{n+1}(\nu)$ (otherwise Y does not satisfy (b) from Lemma 11.23 and is thus not admissible). Lemma 11.24 implies that $\overleftarrow{\mathrm{rhe}(q)} x_\alpha \dots x_1$ equals L_Y or S_Y, depending on the parity of $\#(x_\alpha \dots x_1)$. Since the tails of $\mathrm{rhe}(q)$ and $\mathrm{lhe}(q)$ are shifts of one another and $J \geq 1$ it follows that $\overleftarrow{x} = \overleftarrow{\mathrm{rhe}(q)} x_\alpha \dots x_1$, which concludes the proof. $\square$

Theorem 11.26. *Say that $q = m/n$, where m and n are relatively prime. If $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$, then in $\varphi_{\mathcal{C}}(X')$ there exist n fully accessible arc-components which are dense lines in X' and n simple dense canals. Moreover, a point from $\varphi_{\mathcal{C}}(X')$ is accessible if and only if it belongs to one of these n lines.*

Proof. Lemma 11.24 shows that all the extrema of cylinders have tails in $\sigma^i(L')$ for some $i \in \mathbb{Z}$ and Lemma 11.25 shows that every admissible itinerary in $\sigma^i(\mathcal{U}_{L'})$ is realized as an extremum of a cylinder. Since L' is preperiodic of preperiod n, we obtain n fully accessible lines in $\varphi_{\mathcal{C}}(X')$. Thus the circle of prime ends can be decomposed into n open intervals and their n endpoints. We claim that the endpoints correspond to simple dense canals.
Assume by contradiction that a folding point $x \in \varphi_{\mathcal{C}}(X')$ is accessible. Then its every shift $\sigma^j(x)$ needs to be accessible for some natural number j which divides n (denoted from now onwards by $j|n$). We conclude that the tail corresponding to the point x must

be periodic of period $j|n$, *i.e.*, $\sigma^j(x) = x$. Note that there are no periodic kneading sequences ν of period $j|n$ for $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$ since $\mathrm{lhe}(q)$, $\mathrm{rhe}(q)$ and ν agree on the first $n-1$ places. Thus the basic arc $\overleftarrow{x}$ has $\tau_L(\overleftarrow{x}), \tau_R(\overleftarrow{x})$ finite. Specially, the basic arc $\overleftarrow{x}$ contains no endpoint of X' and x is the only accessible point in $\overleftarrow{x}$ and it thus needs to be Type 3 folding point. Write $\overleftarrow{x} = \dots x_3x_2x_1$. Since x is a folding point and not an endpoint, there exist arbitrarily large $M, k_i \in \mathbb{N}$ such that $x_M \dots x_1 = c_{k_i+1} \dots c_{k_i+M}$ and $x_{M+1} \neq c_{k_i}$. Now we proceed similarly as in Proposition 7.17. Fix a cylinder around $\overleftarrow{x}$ and assume that all long basic arcs in that cylinder lie below (above) $\overleftarrow{x}$. Here long basic arcs $\overleftarrow{y}$ are such that $\pi_0(x) \in \mathrm{Int}\,(\pi_0(\overleftarrow{y}))$. Specially, for M large enough and when $c_{k_i}c_{k_i+1} \dots c_{k_i+M} \neq c_2 \dots c_{M+2}$, the basic arcs $1^\infty c_{k_i}c_{k_i+1} \dots c_{k_i+M}$ are long (if $M > \tau_L(x), \tau_R(x)$ then $\pi_0(1^\infty c_{k_i}c_{k_i+1} \dots c_{k_i+M}) = [T^{\tau_L(x)}, T^{\tau_R(x)}]$). Basic arcs in the chosen cylinder which do not project to $[T^{\tau_L(x)}, T^{\tau_R(x)}]$ are of the form $\dots \frac{0}{1} c_1 c_2 \dots c_{k_i} c_{k_i+1} \dots c_{k_i+M}$. Since $c_{k_i} \neq x_{M+1}$, it follows that those arcs are on the same side of $\overleftarrow{x}$ as long arcs $1^\infty c_{k_i}c_{k_i+1} \dots c_{k_i+M}$. Since we assumed that all long basic arcs lie on the same side of $\overleftarrow{x}$ it follows that $\overleftarrow{x}$ is an extremum of a cylinder, a contradiction. The remaining case is when $c_{k_i} \dots c_{k_i+M} = c_2 \dots c_{M+2}$ for all (but finitely many) $i \in \mathbb{N}$. That is, whenever $x_M \dots x_1$ appears in the kneading sequence, then $x_M \dots x_1 = c_3 \dots c_{M+2}$ and $x_{M+1} \neq c_2 = 0$. However, $\overleftarrow{x}$ is periodic of period $j|n$ and x is a folding point, from which we conclude that $T^3(c)$ is periodic of period $j|n$ and $\overleftarrow{x} = (c_3 \dots c_{n+2})^\infty$. Note that the only kneading sequence $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$ for which $T^3(c)$ is periodic of period $j|n$ is $10(\hat{w}_q 0)^\infty$ which is actually periodic of period n. But there are no periodic kneading sequences ν of period n such that $\mathrm{lhe}(q) \prec \nu \prec \mathrm{rhe}(q)$, a contradiction. Thus no folding point $x \in \varphi_{\mathcal{C}}(X')$ is accessible.

We need to show that the n accessible lines are indeed dense in X'. It follows from Lemma 11.24 that the symbolic code of these n lines is eventually $\mathcal{U}^i = \sigma^i(\overleftarrow{\mathrm{lhe}(q)})$ for $i \in \{0, \dots, n-1\}$. Let $a \in X'$ be a point with backward itinerary $\overleftarrow{a} = \dots a_2a_1$. Note that for every β, every $i \in \{0, \dots, n-1\}$ and large enough γ the left infinite sequences $\sigma^i(\overleftarrow{\mathrm{lhe}(q)})1^\gamma a_\beta \dots a_1$ are admissible since they satisfy conditions (a) and (b) from Lemma 11.23. Thus, sending $\beta \to \infty$ we get a sequence of basic arcs from $\mathcal{U}^i$ converging to $\overleftarrow{a}$ such that their π_0 projections contain $\pi_0(a)$.

Therefore n prime ends $P_1, \dots, P_n$ on the circle of prime ends are either of the third or the fourth kind. Since the shores of the canal are lines which are dense in both directions it follows that $\Pi(P_i) = I(P_i) = X'$ for every $i \in \{1, \dots, n\}$. Therefore, there are n simple dense canals for every $\varphi_{\mathcal{C}}(X')$. □

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