PLANAR EMBEDDINGS OF CHAINABLE CONTINUA

ANA ANUŠIĆ, HENK BRUIN, JERNEJ ČINČ

Abstract. We prove that for a chainable continuum X and every non-zigzag $x \in X$ there exists a planar embedding $\varphi : X \to \varphi(X) \subset \mathbb{R}^2$ such that $\varphi(x)$ is accessible, partially answering the question of Nadler and Quinn from 1972. Two embeddings $\varphi, \psi : X \to \mathbb{R}^2$ are called strongly equivalent if $\varphi \circ \psi^{-1} : \psi(X) \to \varphi(X)$ can be extended to a homeomorphism of $\mathbb{R}^2$. We also prove that every indecomposable chainable continuum can be embedded in the plane in uncountably many strongly non-equivalent ways.

1. Introduction

A *continuum* (compact connected metric space) is *chainable*, if it admits an ε-mapping on the interval $[0,1] \subset \mathbb{R}$ for every $\varepsilon > 0$. It is well-known that every chainable continuum can be embedded in the plane, see [7]. In this paper we develop methods to study non-equivalent planar embeddings, similar to the methods used by Lewis in [15] and Smith in [26] for the study of planar embeddings of the pseudo-arc. Following Bing's approach from [7] (see Lemma 3.1), we construct nested intersections of discs which are small tubular neighbourhoods of polygonal lines obtained from the bonding maps. Later we show that this approach produces all possible embeddings of chainable continua which can be covered with chains with *connected* links. From that we can produce non-equivalent planar embeddings of the same chainable continuum.

Definition 1.1. *Let X be a chainable continuum. Two embeddings $\varphi, \psi : X \to \mathbb{R}^2$ are called* strongly equivalent *if $\varphi \circ \psi^{-1} : \psi(X) \to \varphi(X)$ can be extended to a homeomorphism of $\mathbb{R}^2$. They are* weakly equivalent *if there is a homeomorphism h of $\mathbb{R}^2$ such that $h(\varphi(X)) = \psi(X)$.*

Notions of equivalence have an interpretation when observed as dynamical systems. Let φ, ψ be planar embeddings of X and assume X is generated by a single bonding

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map so the shift homeomorphism σ is well defined. Then $(\varphi(X), \varphi \circ \sigma \circ \varphi^{-1})$ and $(\psi(X), \psi \circ \sigma \circ \psi^{-1})$ are conjugate, and the natural (but not unique) conjugacy is given by $\psi \circ \varphi^{-1}$. The embeddings φ and ψ are strongly equivalent if the natural conjugacy can be extended, and weakly equivalent if a mere homeomorphism can be extended to the plane.

Clearly, strong equivalence implies weak equivalence, but in general not the other way around, see for instance Remark 9.19.

Question 1. *Are there uncountably many nonequivalent embeddings of every chainable indecomposable continuum?*

This question is listed as Problem 141 in a collection of Continuum Theory problems from 1983 by Lewis [16] and was also posed by Mayer in his thesis in 1983 [17], without specifying the precise definition of equivalent embeddings.

Throughout the paper, we will use "equivalent" for "strongly equivalent", and with this version of equivalent, we give a positive answer to the above question, see Theorem 9.14. If the continuum is the inverse limit space of a unimodal map and not hereditarily decomposable, then the result holds for both definitions of equivalent, see Remark 9.20.

In terms of weak equivalence, this generalizes the result in [2], where we prove that every unimodal inverse limit space with bonding map of positive topological entropy can be embedded in the plane in uncountably many non-equivalent ways. The special construction in [2] uses the symbolic techniques which enables direct computation of accessible sets and prime ends, see [3]. Here we utilize a more direct geometric approach.

The main motivation for the study of planar embeddings of tree-like continua is the infamous *plane fixed point property*, which is considered to be one of the most important open problems in continuum theory. Is it true that every non-separating plane continuum X has the fixed-point property, *i.e.,* every continuous $f : X \to X$ has a fixed point? There are examples of tree-like continua without the fixed point property, see *e.g.* Bellamy's example in [6]. It is not known whether Bellamy's example can be embedded in the plane. Although chainable continua are known to have the fixed point property (see [11]), insight in their planar embeddings may be of use to the general setting of tree-like continua.

Another motivation for this study is the following long-standing open problem:

Question 2 (Nadler and Quinn 1972, [25])**.** *Let X be a chainable continuum and $x \in X$. Can X be embedded in the plane such that x is accessible?*

Definition 1.2. *Let $X \subset \mathbb{R}^2$. We say that $x \in X$ is* accessible *(from the complement of X) if there exists an arc $A \subset \mathbb{R}^2$ such that $A \cap X = \{x\}$.*

We will introduce the notion of a *zigzag* related to the admissible permutations of graphs of bonding maps and answer the Nadler and Quinn question in the affirmative for the class of *non-zigzag* chainable continua (see Corollary 7.4). Nevertheless, it is commonly believed that there should exist a counterexample to Question 2. The most promising example is the one suggested by Piotr Minc, see Figure 15 and the description in [20]. However, there still lack techniques to prove that point $p \in X_M$ cannot be made accessible (or how to make it accessible), even with the use of thin embeddings, see Definition 8.2.

Section 2 gives basic notation, and we review the construction of natural chains in Section 3. In Section 4 we introduce the main technique of permuting branches of graphs of linear interval maps. In Section 5 we connect the techniques developed in Section 4 to chains. In Section 6 we apply the techniques developed so far to accessibility of points of chainable continua; this is the content of Theorem 6.1 which is used as a technical tool afterwards. Section 7 introduces the concept of zigzags of a graph of interval map. Moreover, it gives a partial answer to Question 2 and provides some interesting examples by applying the results from this section. In Section 8 we prove that the permutation technique yields all possible thin embeddings of chainable continua. Furthermore, we pose some related open problems at the end of this section. Finally, in Section 9 we construct uncountably many non-equivalent embeddings (in the strong sense) of every chainable continuum which contains an indecomposable subcontinuum and thus answer on Question 1 for the strong version of equivalence. We conclude the paper with some remarks and open questions emerging from the study in the final section.

2. Notation

Let $f_i : I = [0,1] \to I$ be continuous surjections for $i \in \mathbb{N}$ and let *inverse limit space*

$$X_\infty = \varprojlim\{I, f_i\} = \{(\ldots, x_2, x_1, x_0) : f_i(x_i) = x_{i-1}, i \in \mathbb{N}\} \subset I^\infty$$

be equipped with the product topology. Denote by $\pi_i : X_\infty \to I$ the *coordinate projections* for $i \in \mathbb{N}_0$.

Let X be a metric space. A *chain in* X is a set $\mathcal{C} = \{\ell_1 \ldots, \ell_n\}$ of open subsets of X called *links*, such that $\ell_i \cap \ell_j \neq \emptyset$ if and only if $|i-j| \leq 1$. We say that a chain $\mathcal{C}$ is *nice* if all links are open discs (in X) and $\overline{\ell_i} \cap \overline{\ell_j} \neq \emptyset$ if and only if $|i-j| \leq 1$. Here $\overline{A}$ denotes the *closure* of A in X.

Definition 2.1. *The* mesh of a chain $\mathcal{C}$ *is defined as mesh*$(\mathcal{C}) = \max\{\operatorname{diam} \ell_i : i = 1, \ldots, n\}$. *A space* X *is* chainable *if there exist chain covers of* X *of arbitrary small mesh.*

We say that $\mathcal{C}' = \{\ell'_1, \ldots, \ell'_m\}$ *refines* $\mathcal{C}$ and write $\mathcal{C}' \preceq \mathcal{C}$ if for every $j \in \{1, \ldots, m\}$ there exists $i \in \{1, \ldots, n\}$ such that $\ell'_j \subset \ell_i$. We say that $\mathcal{C}'$ *properly refines* $\mathcal{C}$ and write $\mathcal{C}' \prec \mathcal{C}$ if $\ell'_j \subset \ell_i$ implies $\overline{\ell'_j} \subset \ell_i$.

Let $\mathcal{C}' \preceq \mathcal{C}$ be as above. The *pattern of* $\mathcal{C}'$ *in* $\mathcal{C}$ denoted by $Pat(\mathcal{C}', \mathcal{C})$ is the ordered m-tuple $(a_1, \ldots, a_m)$ such that $\ell'_j \subset \ell_{a(j)}$ for every $j \in \{1, \ldots, m\}$ where $a(j) \in \{1, \ldots, n\}$. If $\ell'_j \subset \ell_i \cap \ell_{i+1}$ we take $a(j) = i$, but that choice is made just for completeness.

For chain $\mathcal{C} = \{\ell_1, \dots, \ell_n\}$, write $\mathcal{C}^* = \cup_{i=1}^n \ell_i$.

3. Construction of natural chains, patterns and nested intersections

First we construct *natural chains* $\mathcal{C}_n$ for every $n \in \mathbb{N}$. Take some nice chain $C_0 = \{l_1^0, \dots, l_{k(0)}^0\}$ of I and define $\mathcal{C}_0 := \pi_0^{-1}(C_0) = \{\ell_1^0, \dots, \ell_{k(0)}^0\}$, where $\ell_i^0 = \pi_0^{-1}(l_i^0)$. Note that $\mathcal{C}_0$ is a chain cover of X_∞ (but the links are not necessarily connected sets in X_∞).

Now take a nice chain $C_1 = \{l_1^1, \dots, l_{k(1)}^1\}$ of I such that for every $j \in \{1, \dots, k(1)\}$ there exists $j' \in \{1, \dots, k(0)\}$ such that $f_1(\overline{l_j^1}) \subset l_{j'}^0$ and define $\mathcal{C}_1 := \pi_1^{-1}(C_1)$. Note that $\mathcal{C}_1$ is a chain cover of X_∞. Also note that $\mathcal{C}_1 \prec \mathcal{C}_0$ and $Pat(\mathcal{C}_1, \mathcal{C}_0) = \{a_1^1, \dots, a_{k(1)}^1\}$ where $f_1(\pi_1(\ell_j^1)) \subset \pi_0(\ell_{a_j^1}^0)$ for all $j \in \{1, \dots, k(1)\}$. So the pattern $Pat(\mathcal{C}_1, \mathcal{C}_0)$ can easily be calculated by just following the graph of f_1.

Inductively we construct $\mathcal{C}_{n+1} = \{\ell_1^{n+1}, \dots, \ell_{k(n+1)}^{n+1}\} := \pi_{n+1}^{-1}(C_{n+1})$, where $C_{n+1} = \{l_1^{n+1}, \dots, l_{k(n+1)}^{n+1}\}$ is some nice chain of I such that for every $j \in \{1, \dots, k(n+1)\}$ there exists $j' \in \{1, \dots, k(n)\}$ such that $f_{n+1}(\overline{l_j^{n+1}}) \subset l_{j'}^n$. Note that $\mathcal{C}_{n+1} \prec \mathcal{C}_n$ and $Pat(\mathcal{C}_{n+1}, \mathcal{C}_n) = (a_1^{n+1}, \dots, a_{k(n+1)}^{n+1})$, where $f_{n+1}(\pi_{n+1}(\ell_j^{n+1})) \subset \pi_n(\ell_{a_j^{n+1}}^n)$ for all $j \in \{1, \dots, k(n+1)\}$.

Note that links of C_n can be chosen small enough to ensure that $\text{mesh}(\mathcal{C}_n) \to 0$ as $n \to \infty$ and note that $X_\infty = \cap_{n \in \mathbb{N}_0} \mathcal{C}_n^*$.

Lemma 3.1. *Let X and Y be compact metric spaces and let $\mathcal{C}_n$ and $\mathcal{D}_n$ be chains in X and Y respectively such that $\mathcal{C}_{n+1} \prec \mathcal{C}_n$, $\mathcal{D}_{n+1} \prec \mathcal{D}_n$ and $Pat(\mathcal{C}_{n+1}, \mathcal{C}_n) = Pat(\mathcal{D}_{n+1}, \mathcal{D}_n)$ for all $n \in \mathbb{N}_0$. Assume also that $mesh(\mathcal{C}_n) \to 0$, $mesh(\mathcal{D}_n) \to 0$ as $n \to \infty$. Then $X' = \cap_{n \in \mathbb{N}_0} \mathcal{C}_n^*$ and $Y' = \cap_{n \in \mathbb{N}_0} \mathcal{D}_n^*$ are non-empty and homeomorphic.*

Proof. To see that X' and Y' are non-empty, use the Cantor's intersection theorem. Denote by $\mathcal{C}_k = \{\ell_1^k, \dots, \ell_{n(k)}^k\}$ and $\mathcal{D}_k = \{L_1^k, \dots, L_{n(k)}^k\}$ for all $k \in \mathbb{N}_0$. Let $x \in X'$. Then $x = \cap_{k \in \mathbb{N}_0} \ell_{i(k)}^k$ for some $\ell_{i(k)}^k \in \mathcal{C}_k$ such that $\overline{\ell_{i(k)}^k} \subset \ell_{i(k-1)}^{k-1}$ for every $k \in \mathbb{N}$. Define $h : X' \to Y'$ as $h(x) := \cap_{k \in \mathbb{N}_0} L_{i(k)}^k$. Since the patterns agree and diameters tend to zero, this map is a well-defined bijection. We show that it is continuous. First note that $h(\ell_{i(m)}^m \cap X') = L_{i(m)}^m \cap Y'$ for every $m \in \mathbb{N}_0$ and every $i(m) \in \{1, \dots, n(m)\}$, since if $x = \cap_{k \in \mathbb{N}_0} \ell_{i(k)}^k \subset \ell_{i(m)}^m$, then there is $k' \in \mathbb{N}_0$ such that $\ell_{i(k)}^k \subset \ell_{i(m)}^m$ for all $k \geq k'$. But then $L_{i(k)}^k \subset L_{i(m)}^m$ for all $k \geq k'$, thus $h(x) = \cap_{k \in \mathbb{N}_0} L_{i(k)}^k \subset L_{i(m)}^m$. The other direction follows analogously. Now let $U \subset Y'$ be an open set and $x \in h^{-1}(U)$. Since diameters tend to zero, there is $m \in \mathbb{N}_0$ and $i(m) \in \{1, \dots, n(m)\}$ such that $h(x) \in L_{i(m)}^m \cap Y' \subset U$ and thus $x \in \ell_{i(m)}^m \cap X' \subset h^{-1}(U)$. So $h^{-1}(U) \subset X'$ is open and that concludes the proof. $\square$

In the following sections we will construct nested intersections of planar nice chains such that their patterns are the same as the patterns of refinements $\mathcal{C}_n \prec \mathcal{C}_{n-1}$ of natural chains of X_∞ and such that the diameters of links tend to zero. By the previous lemma, that gives the embeddings of X_∞ in the plane. We note that the previous lemma holds in a more general setting, *i.e.,* for graph-like continua and graph chains, see *e.g.* [19].

4. Permuting the graph

Let $C = \{l_1, \dots, l_n\}$ be a chain of I and let $f : I \to I$ be a continuous surjection which is piecewise linear with finitely many critical points $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$ (so we include the endpoints of $I = [0, 1]$ to the set of critical points). Without loss of generality we assume that $f([t_i, t_{i+1}])$ is not contained in a single link of C for every $i \in \{0, \dots, m\}$.

Define $H_j = f([t_j, t_{j+1}]) \times \{j\}$ for all $j \in \{0, \dots, m\}$ and $V_j = \{f(t_j)\} \times [j-1, j]$ for all $j \in \{1, \dots, m\}$. Note that H_{j-1} and H_j are joined at their left endpoints by V_j if t_j is a local minimum of f and they are joined at their right endpoints if it is a local maximum of f, see Figure 1. The line $H_0 \cup V_1 \cup H_1 \cup \ldots \cup V_m \cup H_m =: G_f$ is called the *flattened graph of f in $\mathbb{R}^2$*.

Definition 4.1. *A permutation $p : \{0, 1, \dots, m\} \to \{0, 1, \dots, m\}$ is called C-*admissible permutation of G_f *if for every $i \in \{0, \dots, m-1\}$ and $k \in \{0, \dots, m\}$ such that $p(i) < p(k) < p(i+1)$ or $p(i+1) < p(k) < p(i)$ it holds that:*

1. *$f(t_{i+1}) \notin f([t_k, t_{k+1}])$, or*
2. *$f(t_{i+1}) \in f([t_k, t_{k+1}])$ but $f(t_k)$ or $f(t_{k+1})$ is contained in the same link of C as $f(t_{i+1})$.*

If p is a C-admissible permutation of G_f, define the permuted graph of f with respect to *C by $p^C(G_f) = p(H_0) \cup p(V_1) \cup \ldots \cup p(V_m) \cup p(H_m)$ such that $p(H_j) = f([\tilde{t}_j, \tilde{t}_{j+1}]) \times \{p(j)\}$ and $p(V_j) = \{f(\tilde{t}_j)\} \times [p(j-1), p(j)]$, where $\tilde{t}_j$ are chosen such that $f(t_j)$ and $f(\tilde{t}_j)$ are contained in the same link of C and such that $p^C(G_f)$ has no self intersections. Denote by $E(p^C(G_f))$ the endpoint of $p(H_0)$ corresponding to $(f(\tilde{t}_0), p(0))$.*

Note that $p(V_j)$ from Definition 4.1 is a vertical line which joins the endpoints of $p(H_{j-1})$ and $p(H_j)$ at $f(\tilde{t}_j)$, see Figure 1.

Definition 4.2. *If $p(J) = m$, we say that H_J is* at the top of *$p^C(G_f)$.*

5. Chain refinements, their composition and stretching

Definition 5.1. *Let $\mathcal{C} = \{\ell_1, \dots, \ell_n\}$ be a nice planar chain. We say that an arc $A \subset \mathcal{C}^*$ is a* nerve of *$\mathcal{C}$ if $A \cap \ell_i \neq \emptyset$ and $A \cap \ell_i$ is connected for every $i \in \{1, \dots, n\}$. Let $f : I \to I$ be a piecewise linear surjection, p an admissible C-permutation of G_f and $\varepsilon > 0$. A nice planar chain $\mathcal{C} = \{\ell_1, \dots, \ell_n\}$ will be called a* tubular ε-chain with nerve *$p^C(G_f)$ if*

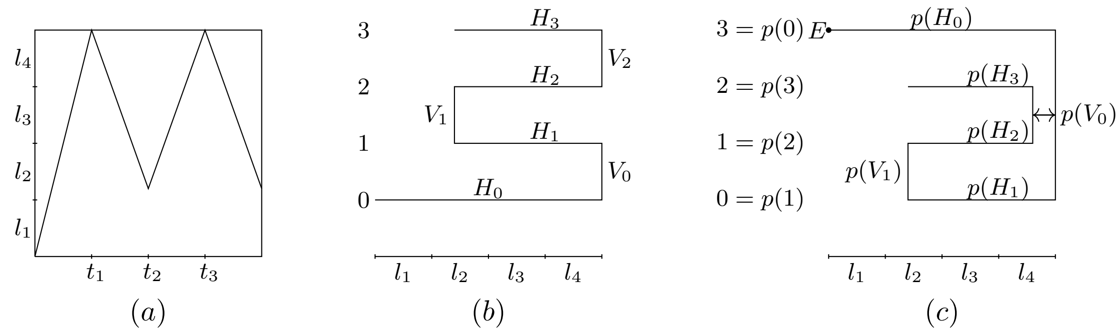

FIGURE 1. Flattened graph and its permutation. Note that H_0 is at the top of $p^C(G_f)$.

- *$p^C(G_f)$ is a nerve of $\mathcal{C}$*
- *there exists $n \in \mathbb{N}$ and arcs $A_1 \cup \ldots \cup A_n = p^C(G_f)$ such that ℓ_i is the ε-neighbourhood of A_i for every $i \in \mathbb{N}$.*

Write $\mathcal{N}_{\mathcal{C}} = p^C(G_f)$. When there is no need to specify ε and the nerve $\mathcal{N}_{\mathcal{C}}$ we just say that $\mathcal{C}$ is tubular.

Definition 5.2. *A planar chain $\mathcal{C} = \{\ell_1, \ldots, \ell_n\}$ will be called* horizontal *if there are $\delta > 0$ and a chain of open intervals $\{l_1, \ldots, l_n\}$ in $\mathbb{R}$ such that $\ell_i = l_i \times (-\delta, \delta)$ for every $i \in \{1, \ldots, n\}$.*

Remark 5.3. *Let $\mathcal{C}$ be a tubular chain. There exists a homeomorphism $\widetilde{H} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\widetilde{H}(\mathcal{C})$ is a horizontal chain and $\widetilde{H}^{-1}(\mathcal{C}')$ is tubular for every tubular $\mathcal{C}' \prec \widetilde{H}(\mathcal{C})$. Moreover, for $\mathcal{C} = \{\ell_1, \ldots, \ell_n\}$ denote by $\mathcal{N}_{\widetilde{H}(\mathcal{C})} = I \times \{0\}$. Note that $\mathcal{C} \setminus (\ell_1 \cup \ell_n \cup \mathcal{N}_{\mathcal{C}})$ has two components and thus it makes sense to call the components upper and lower. Denote by S the upper component of $\mathcal{C} \setminus (\ell_1 \cup \ell_n \cup \mathcal{N}_{\mathcal{C}})$.*

There exists a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ which has all the properties of a homeomorphism $\widetilde{H}$ above and in addition satisfies:

- *the endpoint $H(E(p^C(G_f))) = (0, 0)$ (recall Definition 4.1) and*
- *$H(S)$ is the upper component of $H(\mathcal{C}^*) \setminus (H(\ell_1) \cup H(\ell_n) \cup H(A))$.*

Applying H to a chain $\mathcal{C}$ is called stretching *of $\mathcal{C}$. See Figure 2.*

Remark 5.4. *Let $X_\infty, (C_n)_{n \in \mathbb{N}_0}, (\mathcal{C}_n)_{n \in \mathbb{N}_0}$ be as defined in Section 3. For some $i \in \mathbb{N}$ let $\mathcal{D}_i$ be a horizontal chain with the same number of links as $\mathcal{C}_i$ and such that $p^C(G_{f_{i+1}}) \subset \mathcal{D}_i^*$ for some C_i-admissible permutation p. Fix $\varepsilon' > 0$. There exists $0 < \varepsilon < \varepsilon'$ and an ε-tubular chain $\mathcal{D}_{i+1} \prec \mathcal{D}_i$ with the nerve $p^{C_i}(G_{f_{i+1}})$ such that $Pat(\mathcal{D}_{i+1}, \mathcal{D}_i) = Pat(\mathcal{C}_{i+1}, \mathcal{C}_i)$. However, the length of vertical segments of $p^{C_i}(G_{f_{i+1}})$ could force $mesh(\mathcal{D}_{i+1}) \geq \varepsilon$. This problem can be easily resolved by adding more links to $\mathcal{C}_{i+1}$,* i.e., *refining the chain C_{i+1} of I. From now onwards we assume that no such problem occurs,* i.e., *$mesh(\mathcal{D}_{i+1}) < \varepsilon$. See Figure 3.*

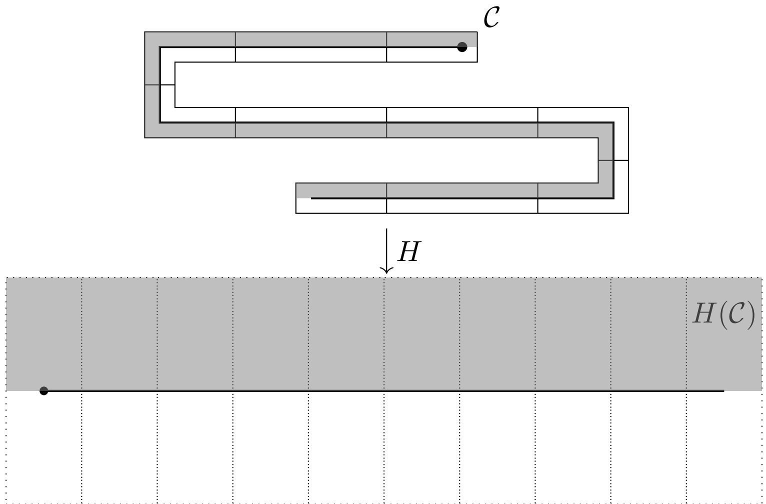

FIGURE 2. Stretching the chain $\mathcal{C}$.

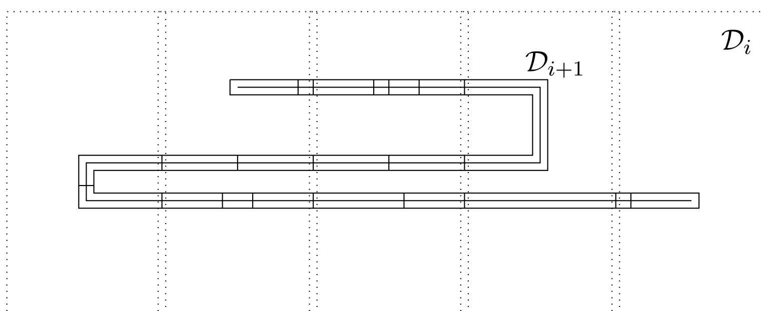

FIGURE 3. Constructing an ε-tubular chain with the nerve $p^C(G_f)$.

Definition 5.5. *Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be a stretching of some tubular chain $\mathcal{C}$. If $\mathcal{C}'$ is a nice chain in $\mathbb{R}^2$ refining $\mathcal{C}$ and there is an interval map $g : I \to I$ such that $p^C(G_g)$ is a nerve of $H(\mathcal{C}')$, then we say that $\mathcal{C}'$* follows $p^C(G_g)$ in $\mathcal{C}$.

Now we discuss compositions of chain refinements. Let $f, g : I \to I$ be piecewise linear surjections. Denote by $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$ the critical points of f and by $0 = s_0 < s_1 < \ldots < s_n < s_{n+1} = 1$ the critical points of g. Let C_1 and C_2 be nice chains of I, let $p_1 : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, m\}$ be an admissible C_1-permutation of G_f and let $p_2 : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$ be an admissible C_2-permutation of G_g.

Assume $\mathcal{C}'' \prec \mathcal{C}' \prec \mathcal{C}$ are nice chains in $\mathbb{R}^2$ such that $\mathcal{C}$ is horizontal and $p_1^{C_1}(G_f) \subset \mathcal{C}^*$ (recall that $\mathcal{C}^*$ denotes the union of links of $\mathcal{C}$), $\mathcal{C}'$ is a tubular chain with $\mathcal{N}_{\mathcal{C}'} = p_1^{C_1}(G_f)$, and $\mathcal{C}''$ follows $p_2^{C_2}(G_g)$ in $\mathcal{C}'$. Then $\mathcal{C}''$ follows $f \circ g$ in $\mathcal{C}$ with respect to a C_1-admissible permutation of $G_{f\circ g}$ which we will denote by $p_1 * p_2$. See Figures 4 and 5.

Define

$$A_{ij} = \{x \in I : x \in [s_i, s_{i+1}], g(x) \in [t_j, t_{j+1}]\},$$

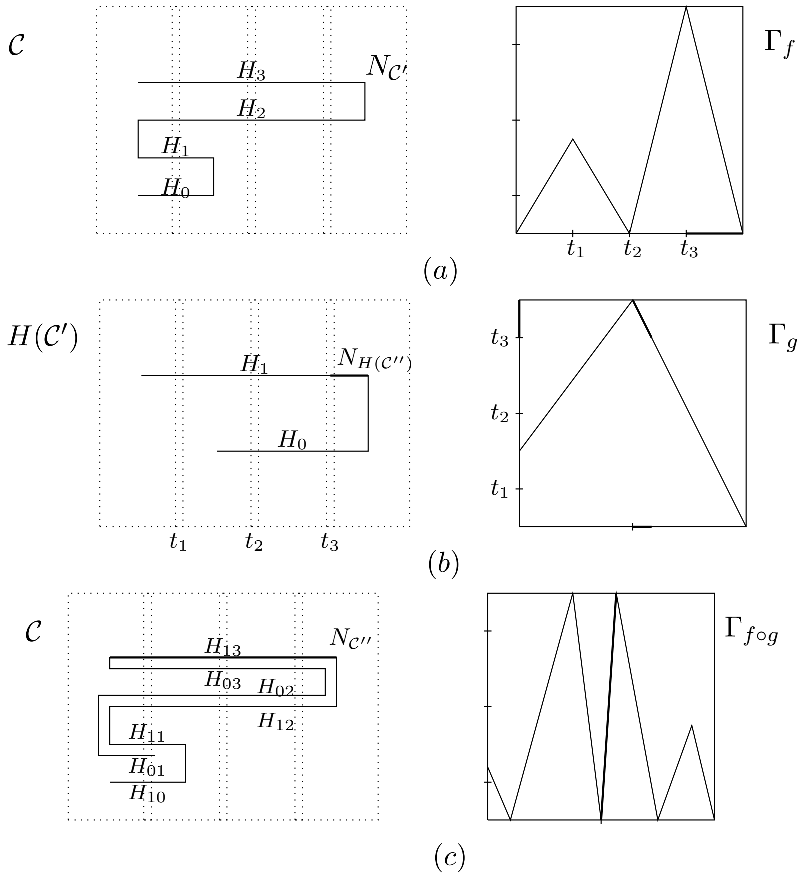

FIGURE 4. Composing refinements. In (a) the horizontal chain $\mathcal{C}$ and the nerve of $\mathcal{C}'$ are drawn. The nerve $N_{\mathcal{C}'}$ equals $G_f^{C_1}$, a flattened version of the graph Γ_f. In (b) we draw $\mathcal{C}'$ as a horizontal chain by applying H. Also, the nerve $N_{H(\mathcal{C}'')}$ is given as $G_g^{C_2}$, a flattened version of the graph Γ_g. In (c) we draw $N_{\mathcal{C}''}$ in $\mathcal{C}$. In bold we trace the arc which is the top of $(id * id)^{C_1}(G_{f\circ g}) = N_{\mathcal{C}''}$.

for $i \in \{0, 1, \dots, n\}, j \in \{0, 1, \dots, m\}$, *i.e.*, A_{ij} are maximal intervals on which $f \circ g$ is injective and possibly $A_{ij} = \emptyset$. Denote by H_{ij} the horizontal branches of $G_{f\circ g}$ corresponding to the intervals A_{ij}.

We want to see which branch H_{ij} corresponds to the top of $(p_1 * p_2)^{C_1}(G_{f\circ g})$. Denote the top of $p_1^{C_1}(G_f)$ by $p_1(H_{T_1})$, *i.e.*, $p_1(T_1) = m$. Denote by $p_2(H_{T_2})$ the top of $p_2^{C_2}(G_g)$, *i.e.*, $p_2(T_2) = n$. By the choice of orientation of H, the top of $(p_2 * p_1)^{C_1}(G_{f\circ g})$ is $H_{T_2T_1}$. See Figures 4 and 5.

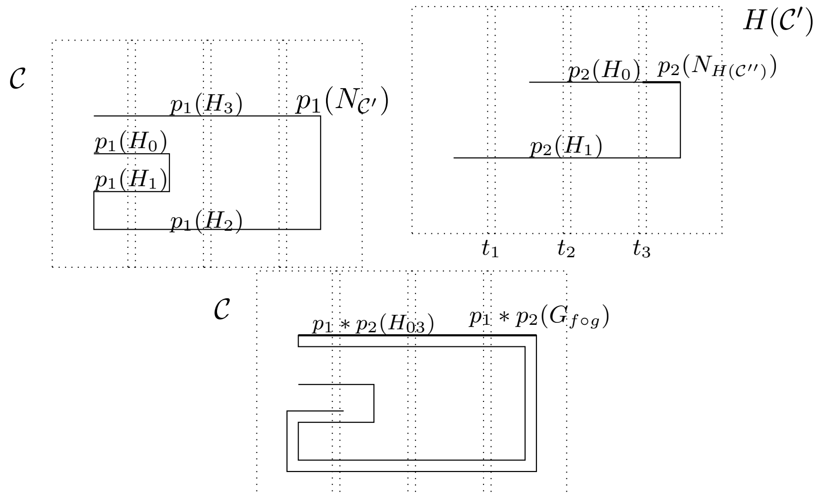

FIGURE 5. Composing permuted refinements. Here $p_1 = (0\ 2)$ and $p_2 = (0\ 1)$ are admissible. The top of $p_1(N_{\mathcal{C}'})$ is $p_1(H_3)$, so $T_1 = 3$. The top of $p_2(N_{H(\mathcal{C}'')})$ is $p_2(H_0)$, so $T_2 = 0$. Thus, the top of $(p_1 * p_2)^{C_1}(G_{f\circ g})$ is $H_{T_2T_1} = H_{03}$ (in bold).

6. Construction of the embeddings

Let $X_\infty = \varprojlim\{I, f_i\}$ where for every $i \in \mathbb{N}$ the map f_i is a continuous surjection which is piecewise linear with finitely many critical points $0 = t_0^i < t_1^i < \ldots < t_{m(i)}^i < t_{m(i)+1}^i = 1$. Denote by $I_k^i = [t_k^i, t_{k+1}^i]$ for every $i \in \mathbb{N}$ and every $k \in \{0, \ldots, m(i)\}$. We construct chains $(C_n)_{n\in\mathbb{N}_0}$ and $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ as before, such that $f_{i+1}(I_k^{i+1})$ is not contained in the same link of C_i for all $k \in \{0, \ldots, m(i+1)\}$ and all $i \in \mathbb{N}_0$. The flattened graph of f_i will be denoted by $G_{f_i} = H_0^i \cup V_1^i \cup \ldots \cup V_{m(i)}^i \cup H_{m(i)}^i$ for all $i \in \mathbb{N}_0$.

Theorem 6.1. *Assume $x = (\ldots, x_2, x_1, x_0) \in X_\infty$ is such that $x_i \in I_{k(i)}^i$ for all $i \in \mathbb{N}_0$ and assume that for every $i \in \mathbb{N}$ there exists an admissible permutation (with respect to C_{i-1}) $p_i : \{0, \ldots, m(i)\} \to \{0, \ldots, m(i)\}$ of G_{f_i} such that $p_i(k(i)) = m(i)$. Then there exists a planar embedding of X_∞ such that x is accessible.*

Proof. Fix a strictly decreasing sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ such that $\varepsilon_i \to 0$ as $i \to \infty$. Let $\mathcal{D}_0$ be a nice horizontal chain in $\mathbb{R}^2$ with the same number of links as $\mathcal{C}_0$. By Remark 5.4 we can find an ε_1-tubular chain $\mathcal{D}_1 \prec \mathcal{D}_0$ with the nerve $p_1^{C_0}(G_{f_1})$, such that $Pat(\mathcal{D}_1, \mathcal{D}_0) = Pat(\mathcal{C}_1, \mathcal{C}_0)$ and $\mathrm{mesh}(\mathcal{D}_1) < \varepsilon_1$. Note that $p_1(k(1)) = m(1)$.

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the stretching of $\mathcal{D}_1$ (see Remark 5.3). Define $F(\mathcal{D}_2) \prec F(\mathcal{D}_1)$ such that $\mathrm{mesh}(\mathcal{D}_2) < \varepsilon_2$ (F is uniformly continuous), $Pat(F(\mathcal{D}_2), F(\mathcal{D}_1)) = Pat(\mathcal{C}_2, \mathcal{C}_1)$ and the nerve of $F(\mathcal{D}_2)$ is $p_2^{C_1}(G_{f_2})$. Thus $H_{k(2)}^2$ is the top of $N_{F(\mathcal{D}_2)}$. By the arguments in the previous section, the top of $N_{\mathcal{D}_2}$ is $H_{k(2)k(1)}$.

As in the previous section, denote the maximal intervals of monotonicity of $f_1 \circ \ldots \circ f_i$ by

$$A_{n(i)\ldots n(1)} := \{x \in I : x \in I^i_{n(i)}, f_i(x) \in I^{i-1}_{n(i-1)}, \ldots, f_1 \circ \ldots \circ f_{i-1}(x) \in I^1_{n(1)}\},$$

and denote the corresponding horizontal intervals of $G_{f_1 \circ \ldots \circ f_i}$ by $H_{n(i)\ldots n(1)}$.

Assume we have constructed a sequence of chains $\mathcal{D}_i \prec \mathcal{D}_{i-1} \prec \ldots \prec \mathcal{D}_1 \prec \mathcal{D}_0$. Take the stretching $F : \mathbb{R}^2 \to \mathbb{R}^2$ of $\mathcal{D}_i$ and define $F(\mathcal{D}_{i+1}) \prec F(\mathcal{D}_i)$ such that $\text{mesh}(\mathcal{D}_{i+1}) < \varepsilon_{i+1}$, $Pat(F(\mathcal{D}_{i+1}), F(\mathcal{D}_i)) = Pat(\mathcal{C}_{i+1}, \mathcal{C}_i)$ and such that the nerve of $F(\mathcal{D}_{i+1})$ is $p^{C_i}_{i+1}(G_{f_{i+1}})$, which is possible by Remark 5.4. Note that the top of $\mathcal{N}_{\mathcal{D}_{i+1}}$ is $H_{k(i+1)\ldots k(1)}$.

Since $Pat(F(\mathcal{D}_{i+1}), F(\mathcal{D}_i)) = Pat(\mathcal{D}_{i+1}, \mathcal{D}_i)$ for every $i \in \mathbb{N}_0$ and by the choice of the sequence (ε_i), Lemma 3.1 yields that $\cap_{n \in \mathbb{N}_0} \mathcal{D}^*_n$ is homeomorphic to X_∞. Denote by $\varphi(X_\infty) = \cap_{n \in \mathbb{N}_0} \mathcal{D}^*_n$.

To see that x is accessible, note that $H = \lim_{i \to \infty} H_{k(i)\ldots k(1)}$ is a well-defined horizontal arc in $\varphi(X_\infty)$ (possibly degenerate). Let $H = [a, b] \times \{h\}$ for some $h \in \mathbb{R}$. Note that for every $y = (y_1, y_2) \in \varphi(X_\infty)$ it holds that $y_2 \leq h$. Thus every point $p = (p_1, h) \in H$ is accessible by the vertical planar arc $\{p_1\} \times [h, h + 1]$. Since $x \in H$, the construction is complete. □

7. Zigzags

Definition 7.1. *Let $f : I \to I$ be a continuous piecewise linear surjection with finitely many critical points $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$. Denote by $I_k = [t_k, t_{k+1}]$ for every $k \in \{0, \ldots, m\}$. We say that I_k is* in a zigzag of f *if there exist critical points a and e of f such that $a < t_k < t_{k+1} < e \in I$ and either*

(1) *$f(t_k) > f(t_{k+1})$, point a is the strict minimum and point e is the strict maximum of $f|_{[a,e]}$, or*
(2) *$f(t_k) < f(t_{k+1})$, point a is the strict maximum and point e is the strict minimum of $f|_{[a,e]}$.*

We also say that $x \in I_k$ is contained in a zigzag of f *and that f* contains a zigzag*, if there exists at least one interval $J \subset I$ that in a zigzag of f. See Figure 6.*

Lemma 7.2. *Let $f : I \to I$ be a continuous piecewise linear surjection with finitely many critical points $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$. If $I_k = [t_k, t_{k+1}]$ is not in a zigzag of f for some $k \in \{0, \ldots, m\}$, then there exists an admissible permutation p of G_f (with respect to any nice chain C) such that $p(k) = m$.*

Proof. Assume I_k is not in a zigzag of f. Assume without loss of generality that $f(t_k) > f(t_{k+1})$. If $f(a) \geq f(t_{k+1})$ for all $a \in [0, t_k]$ (or if $f(e) \leq f(t_k)$ for all $e \in [t_{k+1}, 1]$) we are done, simply reflect all H_i, $i < k$ over H_k (or reflect all H_i, $i > k$ over H_k in the second case). See Figure 7.

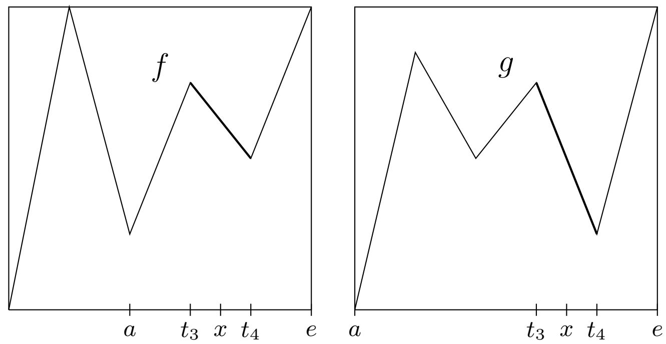

FIGURE 6. The interval $[t_3, t_4]$ is in a zigzag of f and g.

Therefore, assume that there exists $a \in [0, t_k]$ such that $f(a) < f(t_{k+1})$ and there exists $e \in [t_{k+1}, 1]$ such that $f(e) > f(t_k)$. Denote the largest such a by a_1 and the smallest such e by e_1. Since I_k is not in a zigzag, there exists $e' \in [t_{k+1}, e_1]$ such that $f(e') \leq f(a_1)$ or there exists $a' \in [a_1, t_k]$ such that $f(a') \geq f(e_1)$. Assume the first case and take e' such that it is a minimum of $f|_{[t_{k+1}, e_1]}$ (in the second case we take a' such that it is a maximum of $f|_{[a_1,t_k]}$). Reflect $f|_{[a_1,t_k]}$ over $f|_{[t_k,e']}$ (in the second case we reflect $f|_{[t_{k+1},e_1]}$ over $f|_{[a',t_{k+1}]}$). Then, H_k becomes the top of $G_{f|_{[a_1,e_1]}}$. See Figure 8.

If $f(a) \geq f(e')$ for all $a \in [0, a_1]$ (or if $f(e) \leq f(a')$ for all $e \in [e_1, 1]$ in the second case), we are done. So assume there is $a_2 \in [0, a_1]$ such that $f(a_2) < f(e')$ and take the largest such a_2. Then there exists $a'' \in [a_2, a_1]$ such that $f(a'') \geq f(e_1)$, take a'' to be a maximum of $f|_{[a_2,a_1]}$. If $f(e) \leq f(a'')$ for all $e \in [e_1, 1]$, we reflect $f|_{[a_2,a'']}$ over $f|_{[e_1,1]}$ and are done. If there is (minimal) $e_2 > e_1$ such that $f(e_2) > f(a'')$, then there exists $e'' \in [e_1, e_2]$ such that $f(e'') \leq f(a_2)$ and e'' is a minimum of $f|_{[e_1,e_2]}$. In that case we reflect $f|_{[a'',t_k]}$ over $f|_{[t_k,e']}$ and $f|_{[a_2,a'']}$ over $f|_{[t_k,e'']}$, see Figure 9. Thus we have constructed a permutation such that H_k becomes the top of $G_{f|_{[a_2,e_2]}}$. We proceed inductively. □

Theorem 7.3. *Let $X_\infty = \varprojlim\{I, f_i\}$ where $f_i : I \to I$ are continuous piecewise linear surjections with finitely many critical points. If $x = (\dots, x_2, x_1, x_0) \in X_\infty$ is such that x_i is not in a zigzag of f_i for all $i \in \mathbb{N}$, then there exists an embedding of X_∞ in the plane such that x is accessible.*

Proof. The proof follows by Lemma 7.2 and Theorem 6.1. □

Corollary 7.4. *Let $X_\infty = \varprojlim\{I, f_i\}$ where $f_i : I \to I$ are continuous piecewise linear surjections with finitely many critical points and which do not have zigzags for all $i \in \mathbb{N}$. Then, for every $x \in X_\infty$ there exists an embedding of X_∞ in the plane such that x is accessible.*

Remark 7.5. *Note that if $T : I \to I$ is a unimodal map and $x \in \varprojlim(I, T)$, then $\varprojlim(I, T)$ can be embedded in the plane such that x is accessible by the previous corollary. That is Theorem 1 of* [2]. *This easily generalizes to an inverse limit of open interval maps* (e.g. *generalized Knaster continua).*

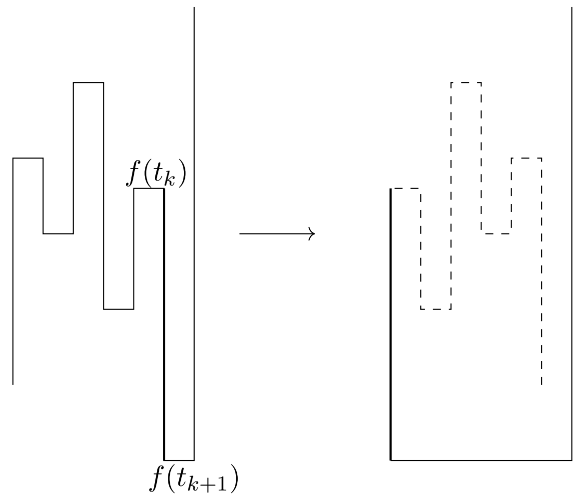

FIGURE 7. Reflections in the first part of the proof of Lemma 7.2.

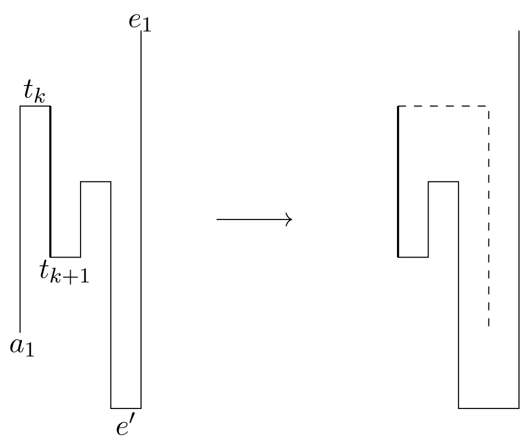

FIGURE 8. Reflections in the second part of the proof of Lemma 7.2.

The following lemma shows that given arbitrary chains (C_i), the zigzag condition from Lemma 7.2 cannot be improved.

Lemma 7.6. *Let $f : I \to I$ be a continuous piecewise linear surjection with finitely many critical points $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$. If $I_k = [t_k, t_{k+1}]$ is in a zigzag for some $k \in \{0, \ldots, m\}$, then there exists a nice chain C of I such that $p(k) \neq m$ for every admissible permutation p of G_f with respect to C.*

Proof. Take a nice chain C of I such that $\operatorname{mesh} C < \min\{|f(t_i) - f(t_j)| : i, j \in \{0, \ldots, m+1\}, f(t_i) \neq f(t_j)\}$. Assume without the loss of generality that $f(t_k) > f(t_{k+1})$ and let $t_i < t_k < t_{k+1} < t_j$ be such that t_j is a maximum and t_i is a minimum of $f|_{[t_i,t_j]}$. Assume t_i is the largest and t_j is the smallest with such properties. Let p be some permutation. If $p(i) < p(j) < p(k)$, then by the choice of C, $p(H_j)$ intersects $p(V_{i'})$ for some $i' \in \{i, \ldots, k\}$. We proceed similarly if $p(j) < p(i) < p(k)$. $\square$

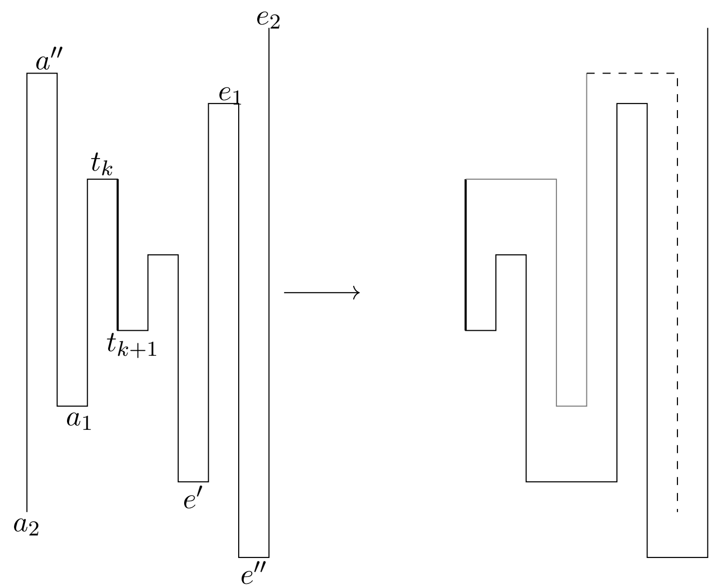

FIGURE 9. Reflections in the third part of the proof of Lemma 7.2.

The following remark is given for contrast to Theorem 7.3.

Remark 7.7. *Let $h : I \to I$ be the Henderson map from* [12] *so that $\varprojlim(I, h)$ is the pseudo-arc. Note that for every point $z \in (0, 1)$ there exists $n \in \mathbb{N}$ so that z is contained in a zigzag of a map h^n. However, since the pseudo-arc is homogeneous, every point from $\varprojlim(I, h)$ can be embedded accessibly.*

Remark 7.8. *Let $X_\infty = \varprojlim\{I, f_i\}$ and $x = (\dots, x_2, x_1, x_0) \in X_\infty$. If there exist piecewise linear continuous surjections $g_i : I \to I$ and a homeomorphism $h : X_\infty \to \varprojlim\{I, g_i\}$ such that every projection of $h(x)$ is not in a zigzag of g_i, then X_∞ can be embedded in the plane such that x is accessible. We specifically have the following two corollaries. See also Examples 7.11-7.13.*

Corollary 7.9. *Let $X_\infty = \varprojlim\{I, f_i\}$ where $f_i : I \to I$ are continuous piecewise linear surjections with finitely many critical points. If $x = (\dots, x_2, x_1, x_0) \in X_\infty$ is such that x_i is not in a zigzag of f_i for all but finitely many $i \in \mathbb{N}$, then there exists an embedding of X_∞ in the plane such that x is accessible.*

Proof. Since $\varprojlim\{I, f_i\}$ and $\varprojlim\{I, f_{i+n}\}$ are homeomorphic for every $n \in \mathbb{N}$, the proof follows using Theorem 7.3. □

Corollary 7.10. *Let f be a continuous piecewise linear surjection with finitely many critical points and $x = (\dots, x_2, x_1, x_0) \in X_\infty = \varprojlim\{I, f\}$. If there exists $k \in \mathbb{N}$ such that x_i is not in a zigzag of f^k for all but finitely many i , then there exists a planar embedding of X_∞ such that x is accessible.*

Proof. Note that $\varprojlim\{I, f^k\}$ and X_∞ are homeomorphic. □

We give applications of Corollary 7.10 in the following examples.

Example 7.11. *Let f be a piecewise linear map such that $f(0) = 0$, $f(1) = 1$ and with critical points $\frac{1}{4}, \frac{3}{4}$, where $f(\frac{1}{4}) = \frac{3}{4}$ and $f(\frac{3}{4}) = \frac{1}{4}$, see Figure 10.*

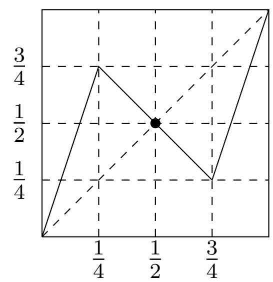

FIGURE 10. Graph of f from Example 7.11.

Note that $X = \varprojlim\{I, f\}$ are two rays compactifying on an arc and therefore, for every $x \in X$, there exists a planar embedding making x accessible. However, the point $\frac{1}{2}$ is in a zigzag of f. In Figure 11 we draw the graph of f^2. Note that the point $\frac{1}{2}$ is not contained in a zigzag of f^2 and that gives an embedding of X such that $(\dots, \frac{1}{2}, \frac{1}{2})$ is accessible.

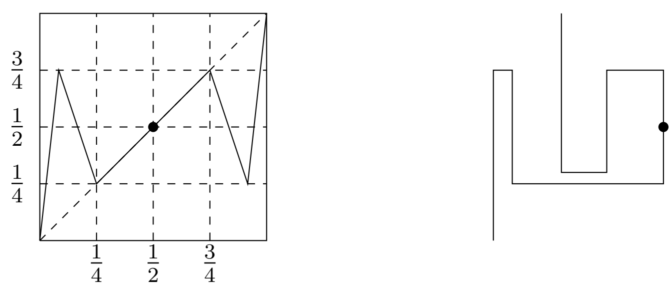

FIGURE 11. Graph of f^2 from Example 7.11.

Let $x = (\dots, x_2, x_1, x_0) \in X$ be such that $x_i \in [1/4, 3/4]$ for all but finitely many $i \in \mathbb{N}_0$. Then, the embedding in Figure 11 will make x accessible. For other points $x = (\dots, x_2, x_1, x_0) \in X$ there exists $N \in \mathbb{N}$ such that $x_i \in [0, 1/4]$ for all $i > N$ or $x_i \in [3/4, 1]$ for all $i > N$ so the standard embedding makes them accessible. In fact, the embedding in Figure 11 will make every $x \in X$ accessible.

Example 7.12. *Assume that f is a piecewise linear map with $f(0) = 0$, $f(1) = 1$ and critical points $f(\frac{3}{8}) = \frac{3}{4}$ and $f(\frac{5}{8}) = \frac{1}{4}$ (see Figure 12).*

Note that $X = \varprojlim\{I, f\}$ consists of two Knaster continua joined at their endpoints together with two rays both converging to these two Knaster continua. Note that $(\dots, \frac{1}{2}, \frac{1}{2})$ can be embedded accessibly with the use of f^2, see Figure 12. However, as opposed to the previous example, X cannot be embedded such that every point is accessible. It is proven by Minc and Transue in [21] that such an embedding of a chainable continuum exists

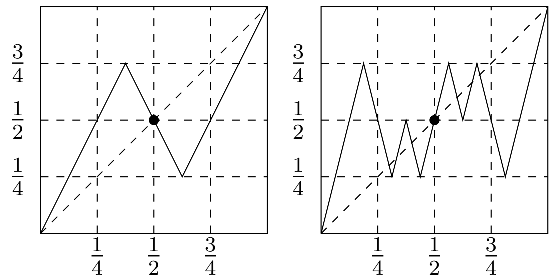

FIGURE 12. Graph of f and f^2 in Example 7.12.

if and only if it is Suslinean, i.e., *contains at most countably many mutually disjoint non-degenerate subcontinua.*

Example 7.13 (Nadler)**.** *Let $f: I \to I$ be as in Figure 13. This is Nadler's candidate from* [25] *for the negative answer to Question 2. However, in what follows we show that every point can be embedded accessibly.*

Let $n \in \mathbb{N}$. If $J \subset I$ is a maximal interval such that $f^n|_J$ is increasing, then J is not contained in a zigzag of f^n, see e.g. *Figure 13.*

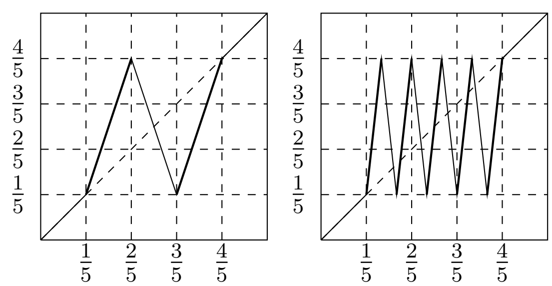

FIGURE 13. Map f and its second iterate. Bold lines are increasing branches of the restriction to $[1/5, 4/5]$. Note that they are not contained in a zigzag of f and f^2 respectively.

We will code the orbit of points in the invariant interval $[1/5, 4/5]$ in the following way. For $y \in [1/5, 4/5]$ let $i(y) = (y_n)_{n\in\mathbb{N}_0} \subset \{0,1,2\}^\infty$, where

$$y_n = \begin{cases} 0, & f^n(y) \in [1/5, 2/5], \\ 1, & f^n(y) \in [2/5, 3/5], \\ 2, & f^n(y) \in [3/5, 4/5]. \end{cases}$$

The definition is somewhat ambiguous and the problem occurs at points $2/5$ and $3/5$. Note, however, that $f^n(2/5) = 4/5$ and $f^n(3/5) = 1/5$ for all $n \in \mathbb{N}$. So every point in $[1/5, 4/5]$ will have a unique itinerary, except the preimages of $2/5$ (to which we can assign two itineraries $a_1 \dots a_n \frac{0}{1} 2222\dots$) and preimages of $3/5$, (to which we can assign two itineraries $a_1 \dots a_n \frac{1}{2} 0000 \dots$), where $\frac{0}{1}$ means "0 or 1" and $a_1, \dots, a_n \in \{0,1,2\}$.

Note that if $i(y) = 1y_2 \dots y_n 1$, where $y_i \in \{0,2\}$ for every $i \in \{2,\dots,n\}$, then y is contained in an increasing branch of f^{n+1}. This holds also if $n = 1$, i.e., *$y_2 \dots y_n = \emptyset$.*

Also, if $i(y) = 0\dots$ *or* $i(y) = 2\dots$*, then* y *is contained in an increasing branch of* f*. See Figure 14.*

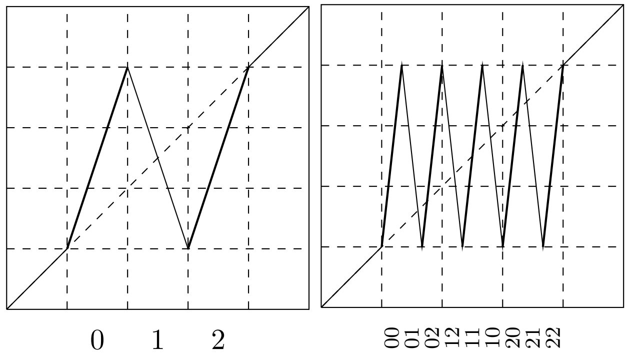

FIGURE 14. Map f and its iterate with symbolic coding of points. Note that points with itinerary $0\dots$ or $2\dots$ are contained in an increasing branch of f and points with itineraries $11\dots$ are contained in an increasing branch of f^2.

We extend symbolic coding to X*. Let* $x = (\dots, x_2, x_1, x_0) \in X$ *and denote by* $\nu(x) = (y_i)_{i\in\mathbb{Z}}$*, where* $(y_i)_{i\in\mathbb{N}_0} = i(x_0)$ *and*

$$y_i = \begin{cases} 0, & x_{-i} \in [1/5, 2/5], \\ 1, & x_{-i} \in [2/5, 3/5], \\ 2, & x_{-i} \in [3/5, 4/5], \end{cases}$$

for every $i \le 0$*. Again, the assignment is injective everywhere except at preimages of critical points* $2/5$ *or* $3/5$*.*

Now fix $x = (\dots, x_2, x_1, x_0) \in X$ *with its backward itinerary* $\overleftarrow{x} = \dots y_{-2}y_{-1}y_0$ *(assume the itinerary is unique, otherwise choose one of the two possible backward itineraries). Assume first that* $y_k \in \{0, 2\}$ *for every* $k \le 0$*. Then, for every* $k \in \mathbb{N}_0$ *it holds that* $i(x_k) = 0\dots$ *or* $i(x_k) = 2\dots$ *so* x_k *is in an increasing branch of* f *and thus not contained in a zigzag of* f*. By Theorem 7.3 it follows that there is an embedding making* x *accessible. Similarly, if there exists* $n \in \mathbb{N}$ *such that* $y_k \ne 1$ *for* $k < -n$*, we use that* X *is homeomorphic to* $\varprojlim\{I, f_j\}$ *where* $f_1 = f^n$*,* $f_j = f$ *for* $j \ge 2$*.*

Assume that $\overleftarrow{x} = \dots 1(\frac{0}{2})^{n_3}1(\frac{0}{2})^{n_2}1(\frac{0}{2})^{n_1}$ *where* $\frac{0}{2}$ *means "0 or 2" and* $n_i \ge 0$ *for* $i \in \mathbb{N}$*. We will assume that* $n_1 > 0$*; the general case follows similarly. Note that* $i(x_{n_1-1}) = (\frac{0}{2})^{n_1}\dots$ *and so it is contained in an increasing branch of* f^{n_1-1}*. Note further that* $i(x_{n_1+1+n_2}) = 1(\frac{0}{2})^{n_2}1(\frac{0}{2})^{n_1}\dots$ *and so it is contained in an increasing branch of* f^{n_2+2}*. Also* $f^{n_2+2}(x_{n_1+1+n_2}) = x_{n_1-1}$*. Further we note that* $i(x_{n_1+1+n_2+1+n_3-1}) = (\frac{0}{2})^{n_3}1(\frac{0}{2})^{n_2}1(\frac{0}{2})^{n_1}$ *and so it is contained in an increasing branch of* f^{n_3}*. Furthermore,* $f^{n_3}(x_{n_1+1+n_2+1+n_3-1}) = x_{n_1+1+n_2}$*.*

Continuing further, we see that for every even $k \geq 4$ it holds that

$$i(x_{n_1+1+n_2+1+\ldots+1+n_k}) = 1\left(\frac{0}{2}\right)^{n_k} 1\ldots 1\left(\frac{0}{2}\right)^{n_1}\ldots$$

and so it is contained in an increasing branch of f^{n_k+2}. Also, $f^{n_k+2}(x_{n_1+1+n_2+1+\ldots+1+n_k}) = x_{n_1+1+n_2+1+\ldots+1+n_{k-1}-1}$. Similarly,

$$i(x_{n_1+1+n_2+1+\ldots+1+n_k+1+n_{k+1}-1}) = \left(\frac{0}{2}\right)^{n_{k+1}} 1\ldots 1\left(\frac{0}{2}\right)^{n_1}\ldots$$

so $x_{n_1+1+n_2+1+\ldots+1+n_k+1+n_{k+1}-1}$ is in an increasing branch of $f^{n_{k+1}}$. Note also that $f^{n_{k+1}}(x_{n_1+1+n_2+1+\ldots+1+n_k+1+n_{k+1}-1}) = x_{n_1+1+n_2+1+\ldots+1+n_k}$.

So we have the following sequence

$$\ldots \xrightarrow{f^{n_5}} x_{n_1+1+\ldots+1+n_4} \xrightarrow{f^{n_4+2}} x_{n_1+1+n_2+1+n_3-1} \xrightarrow{f^{n_3}} x_{n_1+1+n_2} \xrightarrow{f^{n_2+2}} x_{n_1-1} \xrightarrow{f^{n_1-1}} x_0,$$

where the chosen points in the sequence are not contained in zigzags of the corresponding bonding maps. Let

$$f_i = \begin{cases} f^{n_1-1}, & i=1,\\ f^{n_i+2}, & i \text{ even},\\ f^{n_i}, & i>1 \text{ odd}.\end{cases}$$

Then, $\varprojlim\{I, f_i\}$ is homeomorphic to X and by Theorem 7.3 it can be embedded in the plane such that every $x \in \varprojlim\{I, f_i\}$ is accessible.

8. Thin embeddings

We have proven that if a chainable continuum X has an inverse limit representation such that $x \in X$ is not contained in zigzags of bonding maps, then there is a planar embedding of X making x accessible. Note that the converse is not true. The obvious example is the pseudo-arc which is homogeneous thus its every point can be embedded accessibly. However, the crookedness of the pseudo-arc implies the occurrence of zigzags in every representation. It is well-known that the pseudo-arc can be obtained as the inverse limit of the Henderson map from [12], but note that the zigzags in the Henderson map get smaller in diameter and in the limit no point is contained in an arc. That will not happen for *e.g.* Minc's continuum X_M, see Figure 15. In X_M every point is contained in an arc of the length at least $\frac{1}{3}$.

In the next definition we introduce the notion of *thin embedding*, used under this name in *e.g.* [10]. In [1] the notion of thin embedding was referred to as *C-embedding*.

Definition 8.1. *Let $Y \subset \mathbb{R}^2$ be a continuum. We say that Y is* thin chainable *if there exists a sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of chains in $\mathbb{R}^2$ such that $Y = \cap_{n\in\mathbb{N}}\mathcal{C}_n^*$, where $\mathcal{C}_{n+1} \prec \mathcal{C}_n$ for every $n \in \mathbb{N}$, mesh$(\mathcal{C}_n) \to 0$ as $n \to \infty$ and the links of $\mathcal{C}_n$ are connected sets in $\mathbb{R}^2$ (note that links are open in the topology of $\mathbb{R}^2$).*

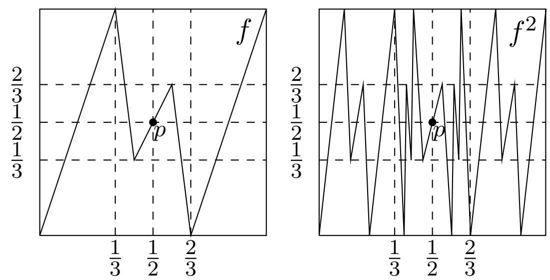

FIGURE 15. Minc's map and its second iteration.

Definition 8.2. *Let X be a chainable continuum. We say that an embedding $\varphi : X \to \mathbb{R}^2$ is a* thin embedding *if $\varphi(X)$ is thin chainable. Otherwise φ is called a* thick embedding.

Question 3. *Is there an embedding of X_M which makes p accessible? Is there a* thin embedding *of X_M which makes p accessible?*

Example 8.3 (Bing, [7])**.** *An* Elsa continuum *consists of a ray compactifying on an arc (in* [9] *this was called an arc+ray continuum). An example of a thick embedding of an Elsa continuum was constructed by Bing, see Figure 16. The terminology was introduced by Nadler in* [24].

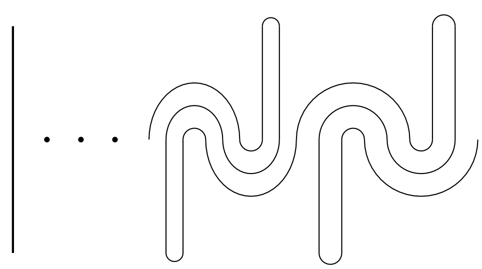

FIGURE 16. Bing's example from [7].

An example of a thick embedding of 3-Knaster continuum was given by Dębski and Tymchatyn in [10]. An arc has a unique planar embedding (up to equivalence), so its every planar embedding is a thin embedding. Therefore, it is natural to ask the following question.

Question 4 (Question 1 in [1])**.** *Which continua have a thick embedding in the plane?*

Definition 8.4. *Let X be a chainable continuum. By $\mathcal{E}_C(X)$ we denote the set of all planar embeddings of X obtained by performing admissible permutations of G_{f_i} for every representation X as $\varprojlim\{I, f_i\}$.*

Theorem 8.5. *Let X be a chainable continuum and $\varphi : X \to \mathbb{R}^2$ a thin embedding of X. Then there exists an embedding $\psi \in \mathcal{E}_C(X)$ which is weakly equivalent to φ.*

Proof. Denote by $\varphi(X) = \cap_{n\in\mathbb{N}_0}\mathcal{C}_n^*$, where the links of $\mathcal{C}_n$ are open, connected sets in $\mathbb{R}^2$ and $\mathcal{C}_{n+1} \prec \mathcal{C}_n$ for every $n \in \mathbb{N}_0$. Without loss of generality we can assume that

links of $\mathcal{C}_n$ are simply connected with a polygonal curve for a boundary. Moreover, we can assume that the intersection of every two links is simply connected. The existence of the homeomorphisms constructed in the proof follows from the generalization of the piecewise linear Schoenflies' theorem given in *e.g.* [23, Section 3]. Take a homeomorphism $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ which maps $\mathcal{C}_0$ to a horizontal chain. Then $F_0(\mathcal{C}_1) \prec F_0(\mathcal{C}_0)$ and there is a homeomorphism $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$ which is identity on $\mathbb{R}^2 \setminus F_0(\mathcal{C}_0)^*$ (recall that $\mathcal{C}^*$ denotes the union of links of $\mathcal{C}$), and which maps $F_0(\mathcal{C}_1)^*$ to a tubular neighbourhood of some permuted flattened graph with $\text{mesh}(F_1(F_0(\mathcal{C}_1))) < \text{mesh}(\mathcal{C}_1)$.
For $n \geq 1$ denote by $G_n := F_n \circ \ldots F_1 \circ F_0$ and note that $G_n(\mathcal{C}_{n+1}) \prec G_n(\mathcal{C}_n)$ and there is a homeomorphism $F_{n+1} : \mathbb{R}^2 \to \mathbb{R}^2$ which is identity on $\mathbb{R}^2 \setminus G_n(\mathcal{C}_n)^*$ and which maps $G_n(\mathcal{C}_{n+1})^*$ to a tubular neighbourhood of some flattened permuted graph with $\text{mesh}(F_{n+1}(G_n(\mathcal{C}_{n+1}))) < \text{mesh}(\mathcal{C}_{n+1})$.
Note that the sequence $(G_n)_{n\in\mathbb{N}_0}$ is uniformly Cauchy and denote by $G = \lim_{n\to\infty} G_n$. By construction $G : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism and $G \circ \varphi \in \mathcal{E}_C(X)$. ☐

Question 5 (Question 2 in [1])**.** *Is there a chainable continuum X and a thick embedding ψ of X such that the set of accessible points of $\psi(X)$ is different from the set of accessible points of $\varphi(X)$ for any thin embedding φ of X?*

9. Uncountably many non-equivalent embeddings

In this section we construct uncountably many non-equivalent embeddings of every chainable continuum which contains an indecomposable subcontinuum. Recall that we use the *strong* definition of equivalent embeddings, *i.e.*, $\varphi, \psi \colon X \to \mathbb{R}^2$ are equivalent if $\varphi \circ \psi^{-1}$ can be extended to a homeomorphism of $\mathbb{R}^2$.

The idea of the construction is to find uncountably many composants which can be embedded accessibly in more than a point. The conclusion then follows easily with the use of the following theorem.

Theorem 9.1 (Mazurkiewicz [18])**.** *Let $X \subset \mathbb{R}^2$ be an indecomposable planar continuum. There are at most countably many composants of X which are accessible in at least two points.*

Let $X = \varprojlim\{I, f_i\}$, where $f_i : I \to I$ are continuous piecewise linear surjections.

Definition 9.2. *Let $f : I \to I$ be a surjection. An interval $I' \subset I$ is called a* surjective interval *if $f(I') = I$ and $f(J) \neq I$ for every $J \subset I'$. Denote by $A_1, \ldots, A_n$, $n \geq 1$, the surjective intervals of f. For every $i \in \{1, \ldots, n\}$ define the* right accessible set *by $R(A_i) = \{x \in A_i : f(y) \neq f(x) \text{ for all } x < y \in A_i\}$, see Figure 17.*

We will first assume that the map f_i contains at least three surjective intervals for every $i \in \mathbb{N}$. We will later see that this assumption can be made without loss of generality.

Remark 9.3. *Assume that f has $n \geq 3$ surjective intervals. Then $A_1 \cap A_n = \emptyset$ and $f([l, r]) = I$ for every $l \in A_1$ and $r \in A_n$. Also $f([l, r]) = I$ for every $l \in A_i$ and $r \in A_j$ where $j - i \geq 2$.*

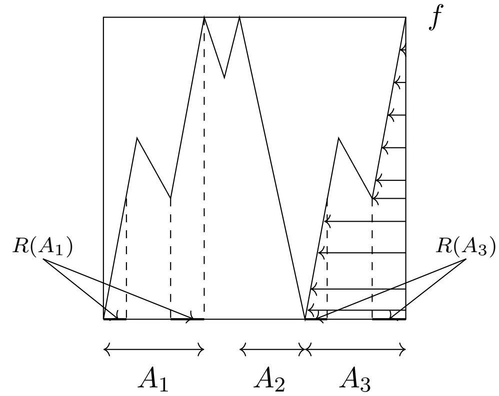

FIGURE 17. Map f has three surjective intervals. The right accessible sets in the surjective branches A_1 and A_3 of f are denoted in the picture by $R(A_1)$ and $R(A_3)$ respectively. Note that $R(A_2) = A_2$.

Lemma 9.4. *Let $J \subset I$ be a closed interval and $f : I \to I$ a map with surjective intervals $A_1, \dots A_n$, $n \geq 1$. For every $i \in \{1, \dots, n\}$ there exists an interval $J^i \subset A_i$ such that $f(J^i) = J$, $f(\partial J^i) = \partial J$ and $J^i \cap R(A_i) \neq \emptyset$.*

Proof. Denote the interval $J = [a, b]$ and fix $i \in \{1, \dots, n\}$. Let $a_i, b_i \in R(A_i)$ be such that $f(a_i) = a$ and $f(b_i) = b$. Assume first that $b_i < a_i$, see Figure 18. Find the smallest $\tilde{a}_i > b_i$ such that $f(\tilde{a}_i) = a$. Then $J^i := [b_i, \tilde{a}_i]$ has the desired properties. If $a_i < b_i$, then take $J^i = [a_i, \tilde{b}_i]$, where $\tilde{b}_i > a_i$ is the smallest such that $f(\tilde{b}_i) = b$. □

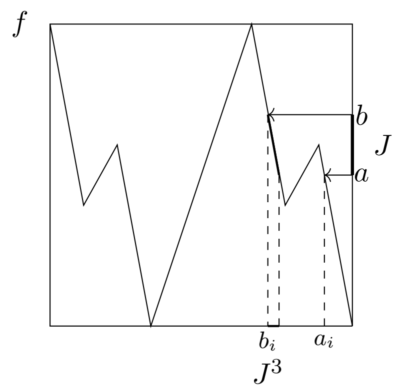

FIGURE 18. Construction of interval J^i from the proof of Lemma 9.4.

The following definition is a slight generalization of the notion of the "top" of a permutation $p(G_f)$ of the graph Γ_f.

Definition 9.5. *Let $f : I \to I$ be a piecewise linear surjection and for a chain C of I, let p be a admissible C-permutation of G_f. For $x \in I$ denote by $p(f(x))$ the point in $p(G_f)$ which corresponds to the point $f(x)$. We say that x is* topmost *in $p(G_f)$ if* *there exists a vertical ray* $\{f(x)\} \times [h, \infty)$, *where* $h \in \mathbb{R}$, *which intersects* $p(G_f)$ *only in* $p(f(x))$.

Remark 9.6. *If* $A_1, \dots, A_n$ *are surjective intervals of* $f : I \to I$, *then every point in* $R(A_n)$ *is topmost. Also, for every* $i = 1, \dots, n$ *there exists a permutation of* G_f *such that every point in* $R(A_i)$ *is topmost.*

Lemma 9.7. *Let* $f : I \to I$ *be a map with surjective intervals* $A_1, \dots A_n$, $n \geq 1$. *For* $[a, b] = J \subset I$ *and* $i \in \{1, \dots, n\}$ *denote by* J^i *the interval from Lemma 9.4. There exists an admissible permutation* p_i *of* G_f *such that both endpoints of* J^i *are topmost in* $p_i(G_f)$.

Proof. Denote by $A_i = [l_i, r_i]$. Assume first that $f(l_i) = 0$ and $f(r_i) = 1$, thus $a_i < b_i$ (recall the notation $a_i, \tilde{a}_i$ and $b_i, \tilde{b}_i$ from the proof of Lemma 9.4). Find the smallest critical point m of f such that $m \geq \tilde{b}_i$ and note that $f(x) > f(a)$ for all $x \in A_i$, $x > m$. So we can reflect $f|_{[m,r_i]}$ over $f|_{[a_i,m]}$ and $f|_{[r_i,1]}$ over $f|_{[0,l_i]}$. This makes a_i and $\tilde{b}_i$ topmost, see Figure 19. In the case when $f(l_i) = 1, f(r_i) = 0$, thus $a_i > b_i$, we have that $f(x) < f(b)$ for all $x \in A_i$, $x > m$ so we can again reflect $f|_{[m,r_i]}$ over $f|_{[a_i,m]}$ making $\tilde{a}_i$ and b_i topmost. $\square$

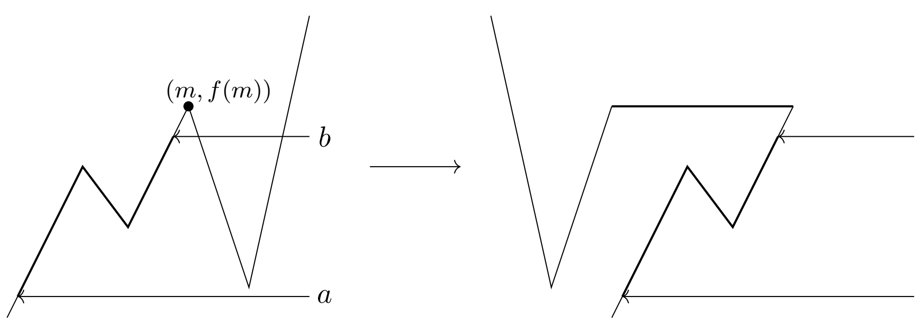

FIGURE 19. Making endpoints of J^i topmost.

Lemma 9.8. *Let* $X = \varprojlim\{I, f_i\}$, *where* $f_i : I \to I$ *are continuous piecewise linear surjections and assume that* X *is indecomposable. If* f_i *contains at least three surjective intervals for every* $i \in \mathbb{N}$, *then there exist uncountably many non-equivalent planar embeddings of* X.

Proof. For every $i \in \mathbb{N}$ denote by $k_i \geq 3$ the number of surjective branches of f_i and fix $L_i, R_i \in \{1, \dots, k_i\}$ such that $|L_i - R_i| \geq 2$. Let $J \subset I$ and $(n_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}}\{L_i, R_i\}$. Then

$$J^{(n_i)} := J \xleftarrow{f_1} J^{n_1} \xleftarrow{f_2} J^{n_1 n_2} \xleftarrow{f_3} J^{n_1 n_2 n_3} \xleftarrow{f_4} \dots$$

is a well-defined subcontinuum of X. Here we used the notation $J^{nm} = (J^n)^m$. Moreover, Lemma 9.7 and Theorem 6.1 imply that X can be embedded in the plane such that both points in $\partial J \leftarrow \partial J^{n_1} \leftarrow \partial J^{n_1 n_2} \leftarrow \partial J^{n_1 n_2 n_3} \leftarrow \dots$ are accessible.

Remark 9.3 implies that for every $f : I \to I$ with surjective intervals $A_1, \dots, A_n$, every $|i - j| \geq 2$ and every $J \subset I$ it holds that $f([J^i, J^j]) = I$, where $[J^i, J^j]$ denotes the convex hull of J^i and J^j. So if $(n_i), (m_i) \in \prod_{i\in\mathbb{N}}\{L_i, R_i\}$ differ at infinitely many places, then there is no proper subcontinuum of X which contains $J^{(n_i)}$ and $J^{(m_i)}$, *i.e.,* they are contained in different composants of X. Now Theorem 9.1 implies that there are uncountably many non-equivalent planar embeddings of X. $\square$

Next we prove that the assumption of at least three surjective intervals can be made without loss of generality for every indecomposable chainable continuum. For $X = \varprojlim\{I, f_i\}$, where $f_i : I \to I$ are continuous piecewise linear surjections, we show that there is $X' = \varprojlim\{I, g_i\}$ homeomorphic to X such that g_i has at least three surjective intervals for every $i \in \mathbb{N}$. We will build on the following remark.

Remark 9.9. *Assume $f, g : I \to I$ have two surjective intervals. Note that then $f \circ g$ has at least three surjective intervals. So if f_i has two surjective intervals for every $i \in \mathbb{N}$, then X can be embedded in the plane in uncountably many non-equivalent ways.*

Definition 9.10. *Let $\varepsilon > 0$ and let $f : I \to I$ be a continuous surjection. We say that f is P_ε if for every two segments $A, B \subset I$ such that $A \cup B = I$ it holds that $d_H(f(A), I) < \varepsilon$ or $d_H(f(B), I) < \varepsilon$, where d_H denotes the Hausdorff distance.*

Remark 9.11. *Let $f : I \to I$ and $\varepsilon > 0$. Note that f is P_ε if and only if there exist $0 \le x_1 < x_2 < x_3 \le 1$ such that one of the following holds*

(a) $|f(x_1) - 0|, |f(x_3) - 0| < \varepsilon$, $|f(x_2) - 1| < \varepsilon$, *or*
(b) $|f(x_1) - 1|, |f(x_3) - 1| < \varepsilon$, $|f(x_2) - 0| < \varepsilon$.

For $n < m$ denote by $f_n^m = f_n \circ f_{n+1} \circ \ldots \circ f_{m-1}$.

Theorem 9.12 (Kuykendall [14])**.** *X is indecomposable if and only if for every $\varepsilon > 0$ and every $n \in \mathbb{N}$ there exists $m > n$ such that f_n^m is P_ε.*

Furthermore, we will need the following strong theorem.

Theorem 9.13 (Mioduszewski, [22])**.** *Continua $\varprojlim\{I, f_i\}$ and $\varprojlim\{I, g_i\}$ are homeomorphic if and only if for every sequence of positive integers $\varepsilon_i \to 0$ there exists an infinite diagram as in Figure 20, where (n_i) and (m_i) are sequences of strictly increas-*

$$\begin{tikzcd}
I & & I \arrow[ll, "f^{n_2}_{n_1}"'] \arrow[dl] & & I \arrow[ll, "f^{n_3}_{n_2}"'] \arrow[dl] & & I \arrow[ll, "f^{n_4}_{n_3}"'] \arrow[dl] & & \cdots \arrow[ll, "f^{n_5}_{n_4}"'] \\
& I \arrow[ul] & & I \arrow[ll, "g^{m_2}_{m_1}"] \arrow[ul] & & I \arrow[ll, "g^{m_3}_{m_2}"] \arrow[ul] & & I \arrow[ll, "g^{m_4}_{m_3}"] \arrow[ul] & & \cdots \arrow[ll, "g^{m_5}_{m_4}"]
\end{tikzcd}$$

FIGURE 20. Infinite (ε_i)-commutative diagram from Mioduszewski's theorem.

ing integers, $f_{n_i}^{n_{i+1}} = f_{n_i+1} \circ \ldots \circ f_{n_{i+1}}$, $g_{m_i}^{m_{i+1}} = g_{m_i+1} \circ \ldots \circ g_{m_{i+1}}$ for every $i \in \mathbb{N}$ and every subdiagram as in Figure 21 is ε_i-commutative.

$$
\begin{tikzcd}
I & & I \arrow[ll, "f_{n_i}^{n_{i+1}}"'] \arrow[dl] & \cdots \arrow[l, "f_{n_{i+1}}^{n_{i+2}}"'] & I \arrow[l, "f_{n_k}^{n_{k+1}}"'] \arrow[dl] & \\
& I \arrow[ul] & \cdots \arrow[l, "g_{m_i}^{m_{i+1}}"] & I \arrow[l, "g_{m_{k-1}}^{m_k}"] & & I \arrow[ll, "g_{m_k}^{m_{k+1}}"] \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[dl] & & \cdots \arrow[ll, "f_{n_{i+1}}^{n_{i+2}}"'] & I \arrow[l, "f_{n_k}^{n_{k+1}}"'] & & I \arrow[ll, "f_{n_{k+1}}^{n_{k+2}}"'] \arrow[dl] \\
I & & I \arrow[ll, "g_{m_i}^{m_{i+1}}"] \arrow[ul] & \cdots \arrow[l, "g_{m_{i+1}}^{m_{i+2}}"] & & I \arrow[ll, "g_{m_k}^{m_{k+1}}"] \arrow[ul] &
\end{tikzcd}
$$

FIGURE 21. Subdiagrams which are ε_i-commutative for every $i \in \mathbb{N}$.

Theorem 9.14. *Every indecomposable chainable continuum X can be embedded in the plane in uncountably many non-equivalent ways in the strong sense.*

Proof. Let $X = \varprojlim\{I, f_i\}$, where $f_i : I \to I$ are continuous piecewise linear surjections. If all but finitely many f_i have at least three surjective intervals, we are done by Lemma 9.8. If for all but finitely many i the map f_i has two surjective intervals, we are done by Remark 9.9.

Now fix a sequence (ε_i) such that $\varepsilon_i > 0$ for every $i \in \mathbb{N}$ and $\varepsilon_i \to 0$ as $i \to \infty$. Fix $n_1 = 1$ and find $n_2 > n_1$ such that $f_{n_1}^{n_2}$ is P_{ε_1}. Such n_2 exists by Theorem 9.12. For every $i \in \mathbb{N}$ find $n_{i+1} > n_i$ such that $f_{n_i}^{n_{i+1}}$ is P_{ε_i}. The space X is homeomorphic to $\varprojlim\{I, f_{n_i}^{n_{i+1}}\}$. Every $f_{n_i}^{n_{i+1}}$ is piecewise linear and there exist $x_1^i < x_2^i < x_3^i$ as in Remark 9.11. Take them to be critical points and assume without loss of generality that they satisfy condition (a) of Remark 9.11. Define a piecewise linear surjection $g_i : I \to I$ with the same set of critical points as $f_{n_i}^{n_{i+1}}$ such that $g_i(c) = f_{n_i}^{n_{i+1}}(c)$ for all critical points $c \notin \{x_1, x_2, x_3\}$ and $g_i(x_1) = g_i(x_3) = 0$, $g_i(x_2) = 1$. Then g_i is ε_i-close to $f_{n_i}^{n_{i+1}}$. By Theorem 9.13, $\varprojlim\{I, f_{n_i}^{n_{i+1}}\}$ is homeomorphic to $\varprojlim\{I, g_i\}$. Since every g_i has at least two surjective intervals, this finishes the proof by Remark 9.9. $\square$

Remark 9.15. *Specifically, Theorem 9.14 proves that the pseudo-arc has uncountably many non-equivalent embeddings in the strong sense. Lewis* [15]*, has already proven this with respect to the weaker version of equivalence, by carefully constructing embeddings with different prime end structures. Some planar embeddings of the pseudo-arc were constructed earlier by Brechner in* [8].

In the next theorem we expand the techniques from this section to construct uncountably many non-equivalent embeddings of every continuum that contains an indecomposable subcontinuum. First we give a generalization of Lemma 9.7.

Lemma 9.16. *Let $f : I \to I$ be a surjective map and let $K \subset I$ be a closed interval. Denote by $A_1, \ldots, A_n$ the surjective intervals of $f|_K : K \to f(K)$, and for $i \in \{1, \ldots, n\}$ denote by J^i the intervals from Lemma 9.4 applied to the map $f|_K$.*
Assume $n \geq 4$. Then there exist $\alpha, \beta \in \{1, \ldots, n\}$ such that $|\alpha - \beta| \geq 2$ and such that there exist admissible permutations p_α, p_β of G_f such that both endpoints of J^α are topmost in $p_\alpha(G_{f|_K})$ and such that both endpoints of J^β are topmost in $p_\beta(G_{f|_K})$.

Proof. Denote $K = [k_l, k_r]$ and $f(K) = [K_l, K_r]$. Let $x > k_r$ be the smallest local extremum of f such that $f(x) > K_r$ or $f(x) < K_l$. A surjective interval $A_i = [l_i, r_i]$ will be called increasing (decreasing) if $f(l_i) = K_l$ ($f(r_i) = K_l$).

Case 1. Assume $f(x) > K_r$, see Figure 22. If $A_i = [l_i, r_i]$ is increasing, since $f(x) > K_r$, there exists an admissible permutation which reflects $f|_{[m,x]}$ over $f|_{[a_i,m]}$ and leaves $f|_{[x,1]}$ fixed. Here m is chosen as in the proof of Lemma 9.7. Since there are at least four surjective intervals, at least two are increasing. This finishes the proof.

Case 2. If $f(x) < K_l$ we proceed as in the first case but for decreasing A_i. □

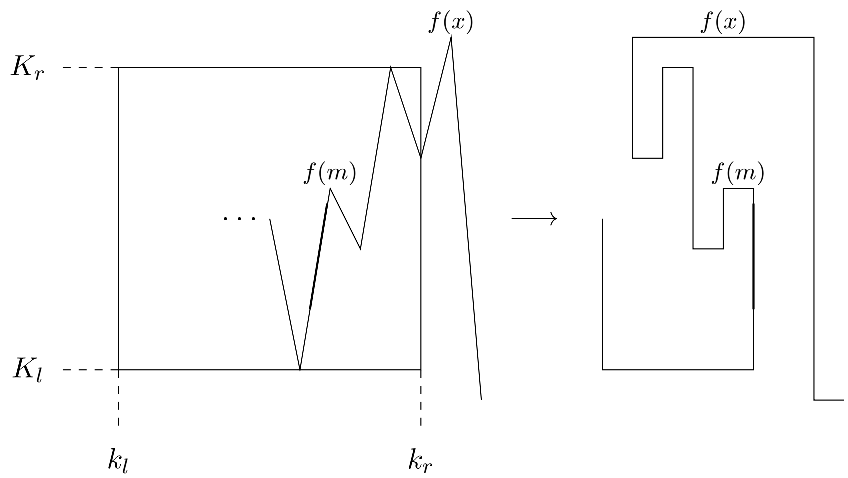

FIGURE 22. Permuting in the proof of Lemma 9.16.

Theorem 9.17. *Let X be a chainable continuum that contains an indecomposable subcontinuum Y. Then X can be embedded in the plane in uncountably many non-equivalent ways in the strong sense.*

Proof. Denote by

$$Y := Y_0 \xleftarrow{f_1} Y_1 \xleftarrow{f_2} Y_2 \xleftarrow{f_3} Y_3 \xleftarrow{f_4} \dots$$

If $\varphi, \psi : X \to \mathbb{R}^2$ are equivalent planar embeddings of X, then $\varphi|_Y, \psi|_Y$ are equivalent planar embeddings of Y. We will construct uncountably many non-equivalent planar embeddings of Y which extend to planar embeddings of X. That will complete the proof.

According to Theorem 9.12 and Theorem 9.13 we can assume that $f_i|_{Y_i} : Y_i \to Y_{i-1}$ has at least four surjective intervals for every $i \in \mathbb{N}$. For a closed interval $J \subset Y_{j-1}$ denote by α_j, β_j the integers from Lemma 9.16 applied to $f_j : Y_j \to Y_{j-1}$, and denote the appropriate subintervals of Y_j by J^{α_j}, J^{β_j}. For every sequence $(n_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}}\{\alpha_i, \beta_i\}$ we obtain a subcontinuum of Y:

$$J^{(n_i)} := J \xleftarrow{f_1} J^{n_1} \xleftarrow{f_2} J^{n_1 n_2} \xleftarrow{f_3} J^{n_1 n_2 n_3} \xleftarrow{f_4} \dots$$

We used the notation as in the proof of Lemma 9.8. Lemma 9.16 implies that for every (n_i) there exists an embedding of Y such that both points of $\partial J \leftarrow \partial J^{n_1} \leftarrow \partial J^{n_1 n_2} \leftarrow \partial J^{n_1 n_2 n_3} \leftarrow \ldots$ are accessible and which can be extended to an embedding of X. This completes the proof. $\square$

We have proven that every chainable continuum which contains indecomposable subcontinuum has uncountably many non-equivalent embeddings. Thus we pose the following question.

Question 6. *Which hereditarily decomposable chainable continua have uncountably many non-equivalent planar embeddings, in the strong and the weak sense?*

Remark 9.18. *Mayer has constructed in* [17] *uncountably many non-equivalent planar embeddings (in both senses) of the* $\sin\frac{1}{x}$ *continuum by varying the rate of convergence of the ray. This approach readily generalizes to any Elsa continuum. We do not know whether the approach can be generalized to all chainable continua which contain a dense ray. Specifically, it would be interesting to see if* $\varprojlim\{I, f_{feig}\}$ *(where* f_{feig} *denotes the logistic map at the Feigenbaum parameter of the logistic family of interval maps) can be embedded in uncountably many non-equivalent ways. However, this approach would not generalize to the remaining hereditarily decomposable continua since there exist hereditarily decomposable continua which do not contain a dense ray, see* e.g. [13].

Remark 9.19. *In Figure 23 we give examples of planar continua which have exactly* $n \in \mathbb{N}$ *or countably many non-equivalent planar embeddings. However, all the examples we know are not chainable except for the arc.*

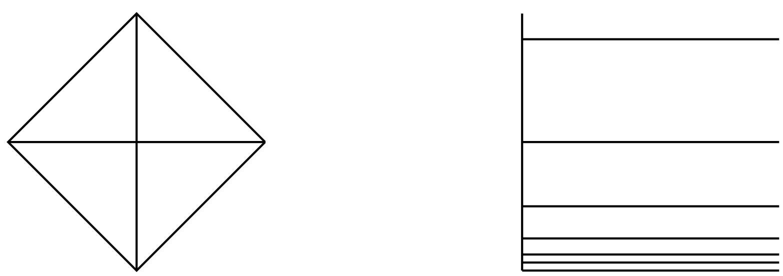

FIGURE 23. Left: Planar projection (Schlegel diagram) of the sides of the pyramid with $n \geq 4$ faces has exactly n non-equivalent embeddings in the strong sense, determined by the choice of the unbounded face. Actually any planar representation of a polyhedron with n faces would do in the previous example. We are indebted to Imre Péter Tóth for these examples. Continua with exactly $n = 2, 3$ non-equivalent planar embeddings in the strong sense are *e.g.* letters H, X respectively. In the weak sense, there is only one planar embedding of all these examples.
Right: the harmonic comb has countably many non-equivalent embeddings in both the strong and the weak sense; any finite number of non-limit teeth can be flipped over to the left to produce a non-equivalent embedding.

Remark 9.20. *For inverse limit spaces X with a single* **unimodal** *bonding map that are not hereditarily decomposable, Theorems 9.14 and 9.17 hold with weak notion of equivalence as well, see* [2]*. This is because every self-homeomorphism of X is known to be pseudo-isotopic (two self-homeomorphisms f, g of X are pseudo-isotopic if $f(C) = g(C)$ for every composant C of X) to a power of the shift homeomorphism (see* [4]*), and so any composant can only be mapped to one in a countable collections of composants. Hence, if uncountably many composants can be made accessible in at least two points, then there are uncountably many non-equivalent embeddings, also w.r.t. the weak equivalence. In general there are no such rigidity results on the group of self-homeomorphisms of chainable continua. For example, there are uncountably many self-homeomorphisms on the pseudo-arc up to pseudo-isotopy, since it is homogeneous and indecomposable. Thus we ask the following two questions.*

Question 7. *For which indecomposable chainable continua is the group of all self-homeomorphisms at most countable up to a pseudo-isotopy?*

For such continua we can conclude that there exist uncountably many non-equivalent planar embeddings in a weak sense. In view of Remark 9.19 we also ask:

Question 8. *Is there a non-arc chainable continuum for which there exist at most countably many non-equivalent planar embeddings in the weak sense?*

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