An alternative to the Euclidean path length in the plane

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joint with:
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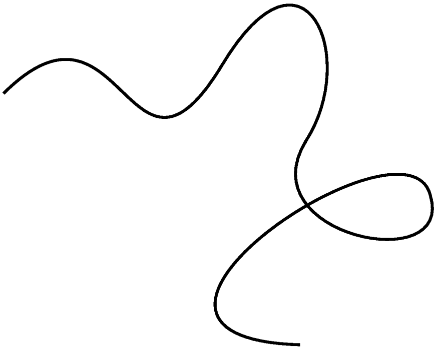

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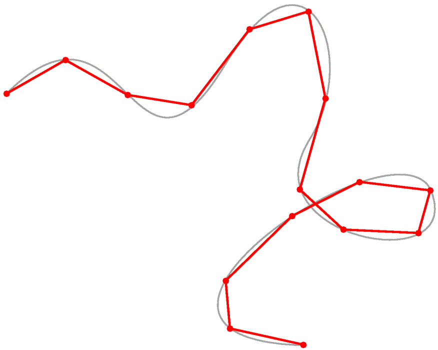

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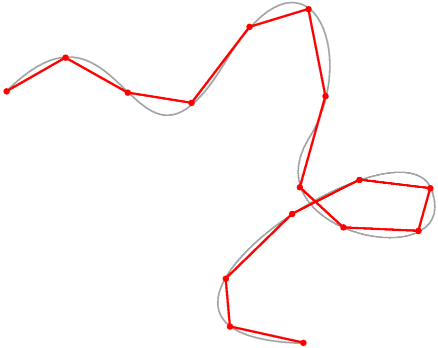

Euclidean length of γ:

$$\mathrm{length}(\gamma) = \sup\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \cdots < t_n = b\}$$

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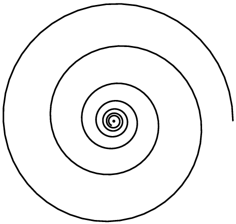

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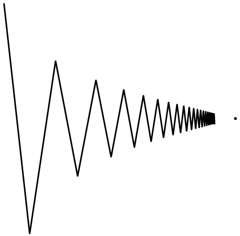

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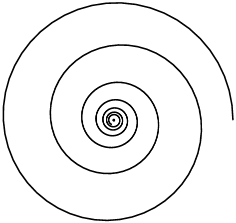

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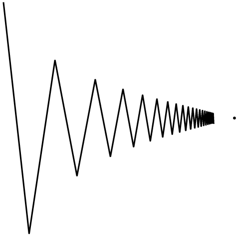

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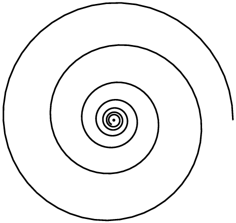

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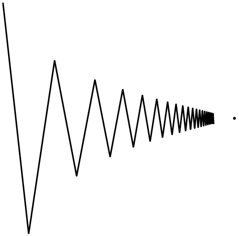

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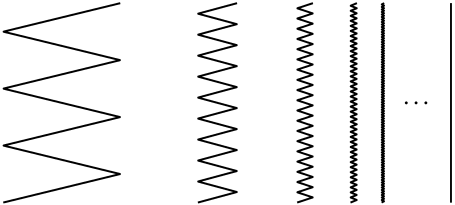

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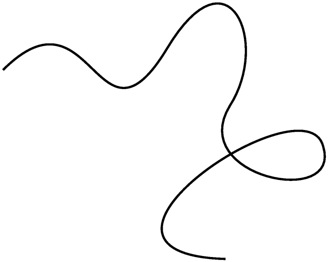

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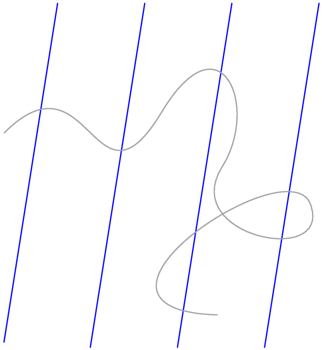

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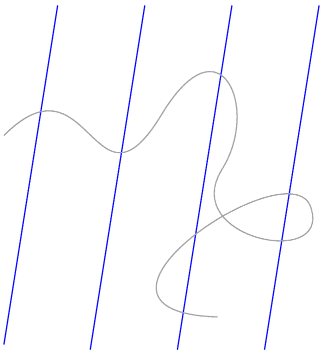

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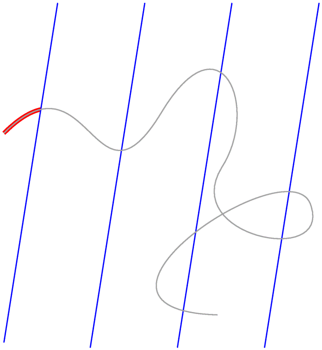

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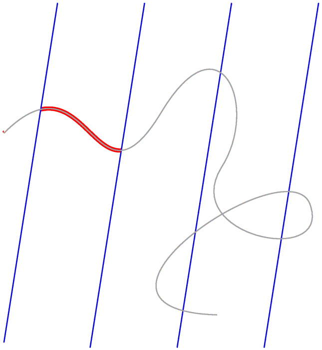

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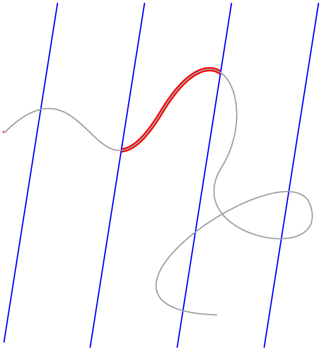

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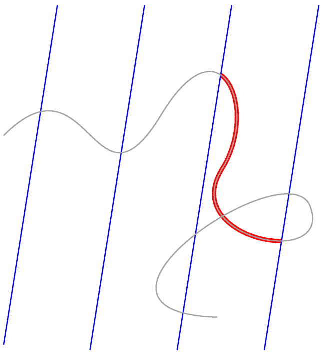

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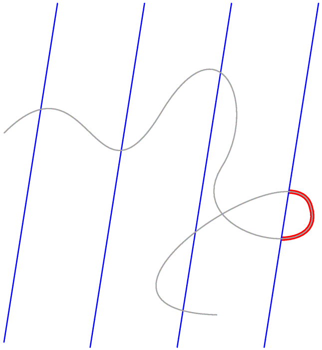

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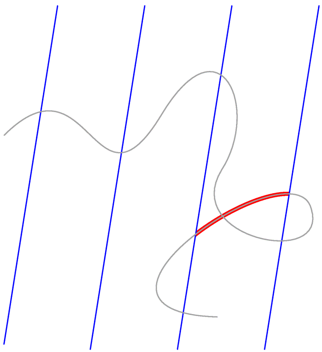

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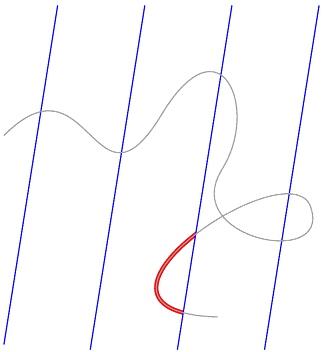

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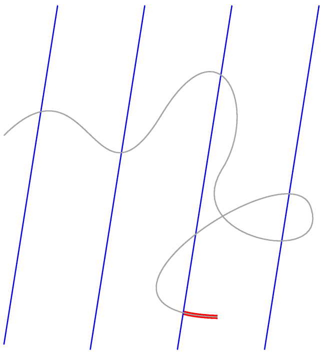

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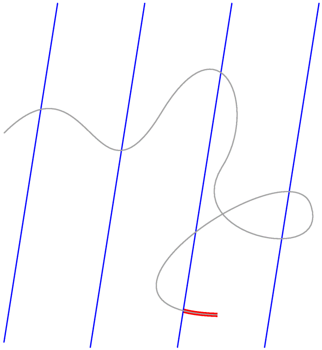

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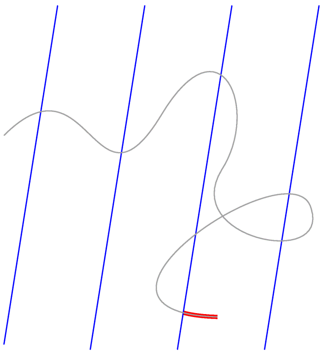

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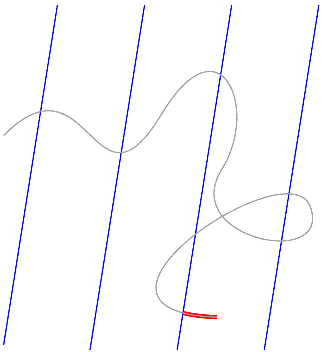

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$$\mathrm{len}(\gamma) = \int_0^1 \int_0^1 \int_0^1 L^{x,t,\mu}(\gamma)\, dx\, dt\, d\mu$$

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The function len can be defined analogously for any function γ from a locally connected continuum X to the plane.

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> ($\dagger$): *For each $\varepsilon > 0$, there is a positive integer N such that for every $\gamma \in \mathcal{F}$, there is no collection of more than N pairwise disjoint subintervals of $[0,1]$ whose images under γ have diameters $\geq \varepsilon$.*

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Conversely, if $\mathcal{F}$ *is an equicontinuous family of paths, then it satisfies the property* (†)*.*