Let's Abolish Pencil-and-Paper Arithmetic

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Abstract

This article proposes that paper-and-pencil arithmetic no longer be taught in elementary school and that it be replaced by a curriculum which emphasizes mental arithmetic much more than at present and in which calculators are used for instructional purposes in all grades including kindergarten. The article analyzes and refutes the arguments made by "back-to-basics" proponents against the use of calculators and for traditional instruction in the algorithms of pencil-and-paper arithmetic. The value of mental arithmetic in achieving all the aims - and more - of the traditional curriculum is argued. Also considered is the outline of an elementary school mathematics curriculum without pencil-and-paper arithmetic. As well, the impact of such a curriculum on secondary school and college mathematics is discussed. Finally, the barriers to achieving what the article advocates are assessed.

``One may be a mathematician of the first rank without being able to compute. It is possible to be a great computer without having the slightest idea of mathematics."

- Novalis (1772-1801)

Yes, I mean it. Achieving any level of proficiency in pencil-and-paper arithmetic (hereafter PPA) should no longer be a goal of elementary school mathematics. Which is not - of course - to say that teachers should not use PPA algorithms for whatever purposes they wish. It is only to say that children should not be expected to learn these algorithms and certainly they should not be tested on them. In the remainder of this paper I will explain the motivation for this proposal and, in particular, I will discuss what I believe should replace this portion of elementary school mathematics.

But, first, I want to consider the possible objections to this proposal in order to put my proposed alternative into context.

Back to Basics

It seems peculiarly perverse in the current climate of mathematics education, particularly in the United States and Great Britain, to be proposing the abandonment of PPA. In the

US the recently approved standards adopted by the California State Board of Education [California, 1998] which are a substantially revised version of standards issued by the California Academic Standards Commission [California, 1997] effectively mandate a PPA-only regimen in elementary school by banning calculators from all state tests in grades K-6. This back-to-basics approach[1] is the antithesis of what this paper proposes.

1. Not everyone accepts that ``back to basics" is a fair description of the California standards [Wu, 1998] but, as this term implies a return to PPA only in elementary school, then the California standards are, indeed, back-to-basics standards.

In Britain a report [DFEE, 1998a] commissioned by the Secretary of State for Education, David Blunkett, of the new Labour government discourages ``as far as possible the use of calculators" in mathematics instruction for children through the age of 11. Although the final report [DFEE, 1998b], in response to widespread criticism of the preliminary report, does ameliorate this advice, it still espouses little or no use of calculators until ``the later years" of primary school (i.e. when children are 10 or 11). In fact, the current British government itself, even more than its predecessor, espouses a back-to-basics philosophy.

Although mathematics educators in the US have widely condemned the California standards and, similarly, mathematics educators in Britain are almost all (but not all - [Gardiner, 1998]) vehemently opposed to the government's position, many others in both countries are firm supporters of back to basics. Educators other than math educators, many parents and most politicians have always deplored putting calculators in the hands of young children[2].

2. A particular irony here is that most of these same people approve wholeheartedly of computers in schools. Why is this? Is it that computers are expensive - and therefore prestigious - whereas calculators are cheap - and therefore gimmickry? Or is it that everyone thinks they understand elementary school mathematics and, therefore, can have valid opinions on it whereas most people have forgotten, if, indeed, they ever learned, most secondary school mathematics?

But perhaps those most strongly in favor of the new California standards are research mathematicians including many from prestigious institutions like Stanford and Berkeley. (In Britain the research mathematics community has been relatively quiet about [DFEE, 1998a and 1998b] but a significant cadre of that community are clearly opposed to having calculators establish any non-trivial beachhead in elementary school mathematics despite respected studies [Shuard, 1991] which show the efficacy of this approach.) The research mathematics community has good reason to be concerned about the state of pre-university American mathematics education, a topic I shall return to in the next section. Here I note only that the American research mathematical community has not understood very well the impact of technology on mathematics at the university level so it is hardly surprising that research mathematicians have generally, although there are some notable exceptions, utterly failed to understand its impact on the pre-university mathematics curriculum. Indeed, again with the caveat about exceptions, as a rule research mathematicians just don't know what they are talking about when they criticize what happens in elementary school [American, 1995-1997]. They have diagnosed the problem correctly - poor preparation for university mathematics - but they don't understand the causes of the problem or its cure[3].

3. A similar stricture is appropriate for university computer scientists. For example, Gelernter [1998] believes ``Calculators should be banned from American elementary schools".

The Value of Learning PPA

Two factors have largely been the spur of the back-to-basics, anti-calculator movement:

1. The poor showing of American (and British) students on international comparisons [Schmidt et al, 1997]. Since such comparisons at the elementary school level focus on PPA prowess, if American and British students do badly, then the reason must be their failure to learn PPA properly[4]. And what is more likely to have caused this failure than the increasing use of calculators over the past quarter century? For secondary school comparisons, PPA does not play nearly so important a role (but, nevertheless, not a trivial one). However, it is often argued that failure to become proficient in PPA in elementary school inevitably hinders a student at all subsequent levels of mathematics.

4. It can be doubted that American students ever were very proficient in PPA. Johnson and Rising [1967] report on a 1932 study of over 200,000 students in grades 5-12 in which only 20% of the 12th grade students could compute 2.1% of 60. Also they note that ``In 1942 Admiral Nimitz reported that 68% of 4200 freshmen in 27 United States universities and colleges were unable to pass the arithmetical reasoning portion of the examination to enter the Naval Reserve Officers' Training Corps".

2. The progressively poorer preparation of first year university students to study university mathematics. Given that this is the case - and a very wide spectrum of university mathematicians think it is - then might not the cause of this poor preparation be the use of trendy new techniques and materials - often calculators - in the teaching of school mathematics? Calculators are also often associated with a lack of rigor in teaching mathematics, a dumbing down of the curriculum.

My own belief is that, although the premises in both these arguments are correct, the conclusions are utterly wrongheaded and without basis in fact or plausibility. (I note, in particular, that, from teaching university mathematics for 30 years from 1965 to 1995, I fully subscribe to the premise in the second argument above. My students in general progressively knew less mathematics upon entering university and, as well and just as important, they knew progressively less about what mathematics is and what is important in it.)

I reason as follows:

1. There may seem to be no reason, in principle, why mixing PPA and calculators in an arithmetic curriculum should lead to poorer overall PPA performance and arithmetic understanding. It is, after all, agreed by almost everyone that the incessant drill-and-practice in arithmetic, which for many years was the hallmark of elementary school mathematics almost everywhere and still is in many places, does not instill much understanding of arithmetic, however much mechanical proficiency may be its result. Indeed, there is some evidence [Hiebert, 1986] that too much arithmetic drill-and-practice results in the message being lost to the medium. Which is to say that the message of why one does arithmetic is lost in the emphasis on computational accuracy. But it is the case that many teachers feel that the use of calculators in the classroom makes it more difficult for them to achieve PPA expertise. One might argue that, as this is the case, the reason is ineffective or prosaic use of calculators rather than imaginative use. Maybe. But my conclusion is that halfway houses are almost certain to be ineffective: *Either elementary school arithmetic should consist of PPA without calculators or calculators without PPA*.

Strangely, perhaps, the only people who are likely to agree with this proposition are the back-to-basics proponents although they, of course, will make a different choice than I would. Almost everyone who favors the use of calculators in elementary school classrooms appears to believe, perhaps out of a desire not to seem too radical in the face of the back-to-basics brigade, that this should be as a supplement to PPA or, in any case, that there should still be a major component of PPA in elementary school arithmetic. A typical opinion is: ``As one of the pro-calculator brigade, I, like many others, am also advocating that children must be fluent in ... pencil and paper methods" [Kitchen, 1998]. Support for both teaching PPA and using calculators will be found in studies that show, almost without exception, that the introduction of calculators into the classroom does not hinder the attainment of PPA skills and may even enhance them [Hembree, 1986]. But most (all?) of these studies were with teachers who were committed to trying calculators and felt comfortable using them. This is just not true for the majority of teachers who, more often than not, feel that calculators are a distraction even when they recognize that calculators are what almost everyone everywhere (in the developed world) uses to do arithmetic. Indeed, perhaps the crucial point is that children almost universally use calculators for arithmetic outside the classroom with only looming tests in PPA persuading them to practice PPA on homework or otherwise. In fact, it is the certainty that calculator use outside class does and will inevitably retard the acquisition of PPA skills which provides a telling argument against the back-to-basics regimen. *Classical PPA instruction is doomed to relative failure in a world where arithmetic is almost universally done using calculators and where even the dimmest child will see that attaining skill in PPA has almost no value in non-academic pursuits.*

So the issue is much less whether back to basics is a good idea as it is whether it can be successful no matter how much you might wish it to be. Even if it is true, as John Saxon once said [Saxon, 1990], that ``you must practice everything [in mathematics] to get it", can such practice be successful when children recognize that what they are practicing is not a useful life skill?

2. But if learning PPA is not a useful life skill, perhaps learning it is useful in later mathematics and other education and professional pursuits. First, it is clear that PPA is not useful in any professional pursuit. Although some professional mathematicians need to do considerable amounts of arithmetic, they almost always use calculators or computers when dealing with more than one or two digit numbers. And while scientists, engineers and other professionals may need to do quite a bit of arithmetic, almost without exception they do it with calculators and computers. So the argument in favor of learning PPA stands or falls insofar as this skill is necessary to learn subsequent mathematics.

Before going further, let me first demolish some straw people often used in arguments by the anti-calculator contingent. One is epitomized by the following quote from Jaime Escalante who has been justly praised for his achievements in secondary school mathematics teaching. Criticizing the NCTM Standards [NCTM, 1989], he said that ``Whoever wrote [the Standards] must be a physical education teacher... The use of the calculator in the classroom is negative for the learning process. All students will learn is how to push buttons." This is a common reaction to calculator usage in the classroom from people who know nothing about teaching mathematics in elementary school and even from those, like Escalante, who should know better. Not only is there no evidence whatever to support this position but, I submit, the curriculum I will outline later in this paper could not possibly be criticized on these grounds.

A related straw person is that anyone who advocates calculator use in elementary school must be in favor of dumbing down the curriculum by making it easier and less work. Such a claim is just plain false. I challenge anyone who reads this paper to its conclusion to claim that I am proposing an easier or less challenging, less demanding mathematics curriculum. Another straw person is that the use of calculators in elementary school must be associated with lack of attention to other traditional portions of primary school arithmetic such as place value, fractions and proportions. There have been some trenchant criticisms [Wu, 1998; Klein, 1998] along these lines of the standards promulgated by the California Academic Standards Commission. But all these traditional - and important - topics can and should be in a calculator-based elementary school curriculum. If it is true that often when calculators are widely used, other traditional topics are short-changed, then that is the fault of those who design and teach calculator-based curricula; it is not a necessary concomitant of calculator use. Again I challenge any reader of this paper to aver that I advocate any weakening of the non-PPA elementary school curriculum.

So is a facility with PPA necessary - or even desirable - for later study of mathematics? If a student arrives in secondary school, say in a first algebra course, unable to do PPA, is that student ipso facto disadvantaged compared to students with PPA skills? I take it that it is not PPA skill itself which critics of calculator use consider important. After all, there is little secondary school or other mathematics which requires much computation *per se*. Rather it must be the ancillary benefits of developing PPA skill which are considered important. These are usually subsumed under the rubric of *numeracy* or *number sense* and include, in addition to the obvious ones of knowing the addition and multiplication tables, such things as knowing which arithmetic operation to use when, having a good sense of number size and knowing strategies to check the answers to arithmetic operations. These are all important. Is there anything about them which a calculator-based curriculum could not instill? I think the answer is, no, a proposition which will be supported by the curriculum outlined below.

What many of the critics of calculator use in elementary school deplore is the seeming loss of *technique* - if you will, the ability to understand and manipulate symbols - a loss which seems so obvious to so many university mathematicians. They generally ascribe this to increased use of calculators. This is ironic because, despite some use of calculators in some elementary school classrooms and rather more in secondary school classrooms, the actual use of calculators and computers in school mathematics is still quite limited. Thus, this failure of technique, which, like my university mathematical colleagues, I, too, recognize and deplore, must have some other source than calculator use in the classroom. In any case, let us agree that technique is crucially important. As Wu [1996] says, ``One cannot have understanding without technique". In the transition from secondary school to college, Askey [1996] deplores the ``lack of skills in algebra" which makes the learning of calculus very hard or impossible. But the issue here is not whether PPA can impart number sense and with it technique. Surely PPA may accomplish this (although it is possible to doubt how often the traditional curriculum actually accomplishes this). Rather the issue is whether technique as well as numeracy can be fully achieved in a calculator-based curriculum. I believe that the curriculum discussed below can do both as well as a PPA-based curriculum.

A related argument in favor of at least some PPA is that it provides children with their ``earliest introduction to the power of abstract mathematics" [Ernest, 1998]. Indeed, PPA may, in principle, do this although one may wonder whether children grasp much about the abstraction inherent in numbers from learning the manipulations of PPA. In any case, however, here again I would argue - and will discuss later in this paper - that this aspect of PPA need not be lost at all in a calculator-based curriculum.

Perhaps it is incumbent upon me to address, albeit briefly, the argument that PPA has always worked so why not continue to emphasize it. First, it hasn't always worked - there is lots of evidence (such as that in [Johnson and Rising, 1967] referred to in a previous footnote) that many children have remained essentially innumerate under the classical PPA curriculum. In any case, however well a PPA-based curriculum worked at one time, it is working progressively poorly in a world where almost everyone uses calculators. Anyhow, no matter how well PPA has worked and does work, that is no reason not to replace it with something better.

Finally in this section let me note the three pillars on which the argument in the remainder of this paper stands:

1. There is no research evidence - quite the contrary - that calculator use impedes children's understanding of arithmetic or acquisition of later mathematics.

2. There are no experiential reasons, either those of individuals or of teachers of mathematics at any level, which support not using calculators in elementary school. As I

have indicated, those experiential reasons adduced by teachers and professors just do not hold water nor would, I believe, any others which might be adduced. Most distressing about this, by the way, is that professional mathematicians whose entire ethos is based on *proof* not only do not try to prove what they say (admittedly ``proof'' on educational matters in general is very hard come by) but do not even offer any significant chain of reasoning to support their contentions. Or they unreasonably conflate calculator usage with other weaknesses in proposed curricula. For example, Klein [1998] criticizes the California Commission standards because they ``allow students to use calculators in the third grade, they under-emphasize algebra in high school and they are vague and arbitrary". A lack of emphasis on algebra and vagueness and arbitrariness are valid grounds for criticism but neither are in any way related to calculator usage in the third grade. Why is such calculator usage bad? Klein doesn't say; one reason he doesn't is because no sound argument against calculator use in the third grade can be adduced.

3. There are not even any plausible reasons to suppose that the use of calculators in elementary school should have deleterious effects on children's mathematical learning. Of course, if the use of calculators leads children to rely on them totally in elementary and secondary school, they will become mere button-pushers with disastrous effect. But not only need this not happen; it would not happen in the kind of curriculum to be discussed later in this paper.

Mental Arithmetic

You have probably noticed that there has been no mention of mental arithmetic thus far in this paper. But you must have already inferred that any proposal to do away with PPA must perforce lean heavily on mental arithmetic in any proposed replacement curriculum. In this section I want to explain both why mental arithmetic is so important in a PPA-less curriculum and, as well, how mental arithmetic can provide the benefits which PPA is supposed to provide but all too seldom does.

It is obvious - is it not? - that for reasons of efficiency, quite aside from any cognitive reasons, anyone without good PPA skills must be able to do significant amounts of mental arithmetic. It would be just too tedious to be countenanced to have to whip out a calculator to, for example, add 18 to 47. Thus, it is of some importance to specify just which calculations should be able to be done mentally in a PPA-less curriculum. But more important than such specification is to realize the benefits that will accrue if students learn to do far more mental arithmetic than is common today.

Sometimes mental arithmetic, such as learning the addition and mutiplication tables, is viewed merely as a stepping stone to learning PPA (as, for example, in [DFEE, 1998a, b]). But, as noted by Verschaffel and De Corte [1996], ``mental arithmetic is not just a stepping stone to the standard computational algorithms; to the contrary it is in independent part of the curriculum of its own right".

I have already mentioned that too often the teaching of PPA results in good (human) calculators who may understand too little of what they are doing and who thus develop too little number sense. This is because, while it is perfectly possible to become adept at PPA while still understanding what it is you are doing, it is also possible to learn PPA by rote which is what happens all too often. The result is that by the time they reach secondary school, many students are effectively lost to mathematics because they have learned essentially nothing beyond rote PPA.

Is it similarly possible to learn mental arithmetic by rote? Yes, if all that one means by mental arithmetic is memorization of the addition and multiplication tables as well as, perhaps, all one digit subtraction and division. But if, as I do, you mean something much more than this, in particular, at least all two digit addition, subtraction and multiplication, then it cannot, I believe, be argued that this can be learned by rote. Indeed, even though one can - and must - learn mental algorithms to do multi-digit arithmetic, the variety of personal[5] algorithms needed to be an effective mental calculator precludes rote application of these algorithms.

5. How multi-digit mental arithmetic is done is an individual matter with each person's approach dependent on what feels comfortable. Although teachers can steer students in certain directions, it would be unwise and, almost surely, ineffective to try to teach a single best mental algorithm for any multi-digit mental calculation. In fact, as constructivists will be quick to point out, letting children construct their own understanding of what works best for them is just what we should be aiming for in math education.

Consider then doing 2-digit by 2-digit multiplication mentally. I hope your reaction is not that we can't teach all children to do this. How do you know? (More below on the evidence about whether or not this can be done.) Just about everyone on the back-to-basics side of the argument believes that the mathematics curriculum needs to be made more demanding and that children studying mathematics need to work hard at it. And although those on the other side of the argument are often accused of wanting a ``dumbed-down" curriculum, in fact, this is a calumny because everyone - yes, everyone - in favor of a calculator-rich curriculum also believes in a demanding mathematics curriculum. So, yes, many children aged, say, 10 or 11 would find learning to multiply 46×83 mentally hard and would require many days, weeks or months to learn to do such calculations accurately. But does any reader of this paper believe that you can't teach as many children to do this as you can, say, teach to do 5-digit by 5-digit multiplication with pencil-and-paper. (Of course, it is simple madness to try to have anyone become effective at 5-digit by 5-digit multiplication; such calculations should *always* be done by calculator.)

As the previous paragraph implies, there are significant advantages in learning to do multi-digit mental arithmetic aside from calculational efficiency. Many, probably most children will find it difficult but, when they succeed, they will not just have learned a useful skill. They will surely have enhanced their number sense and, as well, they will have learned how to organize mentally a non-trivial thinking process (``thinking with your head" rather than ``in your head" [Sowder, 1992]) which should have the ancillary benefit of increasing their attention span. Indeed, developing multi-digit mental arithmetic ability requires just the kind of mind training in logical thinking that mathematicians have always believed is an advantage of studying their discipline quite aside from whatever subject matter is learned. It might also be noted that, since there are a variety of possible strategies for performing mental arithmetic calculations, mental arithmetic inherently embodies ways of checking calculations, a skill often emphasized by mathematics educators.

But can we reasonably expect to teach multi-digit mental arithmetic to essentially all children? Intuition tells me the answer is yes. But there is some experiential evidence also [Zhang, 1997] that this can be done. Additionally, there is evidence that not just two-digit addition and subtraction but three-digit can be taught even to third graders [Selter, 1995]. Although more research is needed on how much mental arithmetic can be taught in elementary school, I believe we know enough and can infer enough to increase greatly the amount of mental arithmetic now taught in elementary school.

Some may think a stress on mental arithmetic instead of PPA risks losing the idea of an algorithm in elementary school mathematics. Recently this fear was expressed by a mathematics teacher in an email group who wrote that ``basic levels of arithmetic [meaning PPA] competence [should be] required independently of a calculator (so that there is some understanding of what the calculator is doing)". There is some irony here because the algorithms used by calculators are typically quite different from PPA algorithms. But the wider point that children should learn the algorithmic nature of mathematics as early as possible is a telling one, particularly to someone like myself who has long urged an algorithmic approach at all levels of mathematics. Is it true then that classical PPA gives students some insight into the idea of an algorithm and that my proposal would lose this? Not at all. First, it may be doubted that any real flavor of algorithms is imparted by the teaching of most PPA. But be that as it may, learning mental arithmetic does not just, as indicated above, require that (personal) algorithms be developed and learned; it also provides an ideal opportunity to elucidate the notion of an algorithm.

As children learn to calculate mentally, they will of necessity develop their own mental algorithms for doing multi-digit arithmetic. Some of these (e.g. for two-digit by two-digit multiplication) will be quite complex. Asking children to write down and explain their methods provides an excellent opportunity to introduce algorithmic ideas. (Of course, this is a difficult activity. Explaining our thought process for almost any non-trivial task is difficult but doing so is salutary and forces intuitive notions to be made concrete.)

I should say a word about division. Although it is over 15 years now since the Cockcroft Report [1982] recommended that long division no longer be taught in British schools, this recommendation been, at most, spottily implemented. The California Board of Education standards [California, 1998, p43] would require students to master long division. This is mind-boggling. The only excuse can be that those who promulgated the California standards believe that long division is good for the soul. Not only does being able to do long division have no practical value whatever but, in addition, the time required to teach this algorithm to students is far, far in excess of any benefit which might accrue from learning it. *Of course*, students must learn what division is, when to apply it, what remainders are and how to do simple division problems mentally. But teaching long division is pertinent to none of these aims; it is as nonsensical as teaching the square root algorithm which was staple fare until recent times. I cannot help but believe that those who favor teaching long division in elementary school (and these include some research mathematicians [Klein, 1998]) are in the grip of some fantasy about what is important and useful in school mathematics[6].

6. It has been argued to me [Dubinsky, 1998] that unless paper-and- pencil long division is mastered, then students will have great difficulty understanding the division of polynomials. Not so. Let us by all means teach the Euclidean algorithm at an appropriate point and show students how it can be used (with a calculator!) to calculate quotients digit by digit. (This will, by the way, require students to use their mental arithmetic skills.) Using an understanding of this algorithm as a basis, it will be possible - even easy - to teach the division of polynomials.

Finally, a word about another delusion regarding mental arithmetic, namely that if you wish to stress mental arithmetic in elementary school, it follows that you must discourage the use of calculators. In Numeracy Matters [DFEE, 1998a] there is a praiseworthy emphasis on mental calculation (as there is also in [DFEE, 1998b]). But this leads to the *non sequitur* that ``the importance of mental calculation inevitably has implications for a judicious use of calculators" with ``judicious" meaning ``discouraging as far as possible the use of calculators" with children up to age 11. Why isn't it as clear to everyone as it is to me that mental arithmetic is more important when calculators are used in the curriculum than when they are not? This is not just - or even mainly - for the efficiency reason given at the beginning of this section. It is also because since, as we all know, the likelihood of egregious mistakes when using a calculator (because of a wrongly selected button) is greater than with PPA, calculator users must be able to *estimate mentally* the results of their calculations. Recently a teacher remarked on an email group that ``the older generation possesses an instinctive feel for number which soon sniffs out a crazy answer; this is a facility which the younger generation lacks". As this is true, it must be because this ``younger generation" has not learned to be good mental calculators and to use this ability to ``sniff out crazy answers".

Elementary School Mathematics without PPA

Doing away with PPA in elementary school may sound radical to some but, for those of you who cannot bring yourselves to embrace this idea, how would you feel about postponing any instruction in PPA until the sixth grade? That sounds pretty radical, too, doesn't it? But, in fact, it is an old idea tried successfully in the late 1920s by the school superintendent of Manchester, New Hampshire and recorded in a paper which should be read by all mathematics educators [Benezet, 1935-36; see also Gleason]. Benezet's idea seems to have died a quick death but perhaps you will agree that a PPA-less curriculum is not so far removed from this 70-year-old idea.

In any case, what I propose for the first, say, eight years of school (K-7; an implicit assumption is that algebra should be an eighth grade subject) is the following:

1. A stress on mental arithmetic from the time when any arithmetic idea beyond counting (which, of course, is itself a mental activity) is first introduced. This means, in particular, that as each arithmetic operation is introduced, children will be expected to do mental calculations using that operation. Of course, as part of this, children will be expected to learn the addition and multiplication tables at appropriate times. In line with the paragraph above, I make no assumptions about precisely when any arithmetic operation should be introduced. It may well be that, as Benezet suggested and Gleason supported, there is no good reason to introduce arithmetic as early as is now customary. Clearly children should have many, varied, and substantial experiences with numbers from kindergarten on. This is much more important than their precise form.

2. Calculators should not only be allowed from kindergarten on but their use should be encouraged. Indeed, there is every reason why the use of calculators and the teaching of mental arithmetic should be interleaved because instruction in one will be reinforced by the use of the other and vice versa. Of course, it is obvious that the mental arithmetic portion of the curriculum must be assessed in a non-calculator environment as must also the application of mental arithmetic in problem solving. Otherwise, calculators should be generally allowed in all testing situations. The aim of calculator usage should not be the negative value of avoiding PPA but the positive one of providing exercises and problems which develop number sense and understanding of arithmetic. When decimals and fractions are introduced, calculators should be used to illustrate the relation between the two, what infinite decimals are and why they occur etc.

3. Nothing above should be meant to imply that mental arithmetic and calculators should be the only tools with which to teach arithmetic. Manipulatives and other arithmetic models (e.g. the area model for multiplication) should continue to play an important role. As well, as already noted, teachers could certainly use the algorithms of PPA to illustrate arithmetic operations as seems useful and convenient. And, in any case, pencil-and-paper as a recording and experimental medium should continue to play an important role in elementary (and secondary) school mathematics. For example, there should be no objection, in principle, to doing 2-digit by 2-digit multiplication by writing down two 1-digit by 2-digit products done mentally and then adding these mentally.

4. To reiterate a point made earlier, children should be expected to work hard and to think hard in all their mathematics instruction. Nevertheless, I believe that a mental arithmetic-calculator regimen can accomplish all the goals of the best imaginable PPA curriculum with less total instructional time because, at least, although lots of drill-and-practice will be required to create good mental calculators, the effort involved should be less than is usual now in PPA drill-and-practice.

5. To my mind one of the major advantages - and attractions - of doing away with PPA and replacing it by a mental arithmetic-calculator curriculum is the additional time which would be made available to study other mathematics in elementary school. Of course, it is

true that mathematics other than PPA is now taught in elementary school, some geometry, for example. But the dominance of PPA in the elementary school curriculum is such that not too much more than lip service is usually paid to other topics and children seldom take away from elementary school any significant mathematical skills or knowledge beyond PPA. But, of course, not just geometry but other mathematics - probability, an early introduction to aspects of algebra and others you could name - are well within the intellectual reach of elementary school students. Introduction of these topics into the elementary school mathematics curriculum would prepare students for secondary school mathematics far better than they usually are now and might, in time, mute the criticisms of university mathematicians about the mathematical knowledge of entering university students.

6. A word about abstraction. I mentioned earlier that one criticism of doing away with PPA is that PPA provides students with an early introduction to abstraction. But so similarly does mental arithmetic although I doubt very much that young children sense the idea of abstraction at all when they are first introduced to numbers. But abstraction is one of the most important ideas in mathematics and not at all beyond elementary school students, an assertion proved by the facility which many elementary school students develop for computer programming where abstraction in the form of variables may not be overt but is certainly inherent. I mentioned just above an early introduction to algebra in elementary school mathematics. Certainly this is possible, perhaps in fifth grade, more likely in sixth grade and certainly in seventh grade. What secondary school or university mathematics teacher would not agree that even if all elementary school mathematics achieved was the most elementary notion of variable, this would provide a wonderful entree to secondary school mathematics?

The above is not - and is not intended to be - a complete elementary school mathematics curriculum. It is rather an attempt to convince you that a PPA-less elementary school mathematics curriculum could be developed - actually, I believe, easily developed - which would prepare students for secondary school mathematics far better than they are today not just in the US but anywhere (and far better than any proposed curriculum in the US would achieve). Moreover, I believe it would - anyhow could - do this without the boredom which so many students now feel in elementary school mathematics and which must lose a shocking proportion of the mathematical talent inherent in American students when they first enter school.

Secondary School and Later Mathematics

What would be the impact of a PPA-less elementary school mathematics curriculum on secondary school and university mathematics? As the intent is to have students better prepared for secondary school mathematics than they are now, the impact would be entirely positive. In particular, better prepared students would allow the secondary school curriculum to introduce useful new subject matter - statistics, discrete mathematics etc. - as is now being tried in a number of places, without abandoning traditional subjects. But how would the mathematics *style* proposed above affect what is taught in secondary school mathematics and how it is taught?

University mathematicians who oppose the use of calculators in elementary school do so mainly because as quoted earlier, of the belief, which I share, that you ``cannot have understanding without technique". I have argued that, at the level of arithmetic, that technique and the understanding which follows from it can be achieved through mental arithmetic. What is the relevant analogy to secondary school (and later) mathematics?

It will hardly surprise anyone who has read this far that, just as I favor full use of technology in elementary school, I similarly favor its full use in secondary school and universities - graphing calculators, symbolic calculator-computers and computers. But how then will students achieve technique - algebraic technique is crucial here - if they are allowed full use of calculators? As Edward Effros has said [1989]: ``I can state categorically that if a student cannot factor $x^2 - 9$ *instantly* it is extremely unlikely that [the student] will pass calculus". I agree. The path to algebraic technique in a world of calculators is essentially the same as the path to arithmetic technique, namely *mental algebra*. Surely it is not unreasonable to expect students of algebra to do a fair amount of polynomial algebra mentally. Moreover, even as normally taught now, tasks like completing the square are essentially mental tasks which use pencil and paper as the recording medium. Going beyond Effros I would expect students to be able to factor a variety of three term quadratics mentally. And, of course, as with mental arithmetic, there should be calculator-free assessment of mental algebra.

More generally, just as attaining number sense is the essential goal of teaching arithmetic, attaining *symbol sense* should be the essential goal of algebra. Symbol sense is harder to define than number sense but it includes such things as being able to predict the *form* of the result of a symbolic calculation (what is the degree of the product of two polynomials?), the ability to select the most appropriate of several equivalent forms (e.g. polynomial, completed square or factored form of a quadratic), having judgment about the reasonableness of a result (if there are six times as many students as professors, do we express this as 6P=S or 6S=P?), developing competency in *engineering* symbolic expressions (given some points in the plane, find an algebraic or trigonometric function which passes through or close to all of them) etc. (See [Arcavi, 1994] for a good discussion of symbol sense). Students who have this symbol sense, no matter how much they use graphing and symbolic calculators, will be well-prepared to study calculus.

Although university mathematicians have been slower to use technology in their teaching than scientists and engineers, they have started to come around in recent years. Still too many feel about symbolic mathematical systems (aka computer algebra systems), such as Mathematica and Maple, the way they feel about calculators for elementary school children. One result is that despite so-called calculus reform, the aim of most college calculus courses still seems to be to create a student-machine in which functions are fed to its maw and derivatives and integrals emerge at the other end. And this in spite of the demonstrable failure of calculus courses to produce students with more than a mechanical grasp of the subject. Whether or not you agree with my long held belief that discrete mathematics should play a role co-equal to calculus in the university curriculum [Ralston, 1981], there can be no excuse any more for calculus courses which do not make full use of technology and which teach students to do mentally much of what they now do mechanically.

Could My Proposal Work?

It could - almost - be implemented tomorrow. A detailed curriculum could be developed, textbooks could be written, lessons planned etc. Let's even be optimistic that politicians, parents, mathematicians - all those antediluvian groups - could be convinced of the rightness of abolishing PPA. But are elementary school teachers themselves ready for such a curriculum. I fear - but would happily be proven wrong - that even most excellent elementary school teachers would have serious difficulties teaching such a curriculum. This is mainly because their preparation to teach elementary school mathematics has been woefully lacking and has prepared them to teach little aside from classical PPA. But also it is because many (most?) teachers attracted to teaching in elementary school are less interested in mathematics than in any other subject; indeed, too often elementary school teachers suffer from mathophobia.

How many elementary school teachers, for example, are prepared to teach two-digit mental arithmetic? How many of them have the knowledge to teach the other aspects of mathematics which would enter the curriculum if PPA were replaced by a more demanding but less time-consuming mental arithmetic-calculator curriculum? Some of these problems could be alleviated with the current cadre of elementary school teachers with appropriate in-service training. For future teachers much better - and intensive - pre-service mathematics education would be necessary.

But so long as mathematics is low on the interest scale of current and prospective elementary school teachers, it is unlikely that much progress will be made this way. People with a talent for mathematics and an interest in it are just unlikely to be attracted to elementary school teaching since most of the elementary school curriculum involves, as it should, language-based subjects. This is true even of elementary school science which is almost entirely descriptive.

There is really only one solution. This is to have much, if not all elementary school mathematics taught by specialist teachers much as subjects like music and art are. Some schools have headed tentatively in this direction by appointing a mathematics specialist whose task is to give advice and help to all the teachers. Another partial solution is team teaching in which the mathematics teaching is done by those teachers who are most comfortable with mathematics. Both these ideas are all to the good but I believe that, inevitably, they cannot be fully effective in implementing the curriculum described in this paper. Even elementary school mathematics specialists seldom have the training and breadth of mathematical knowledge and perspective to teach or advise on a mental arithmetic-calculator curriculum.

I conclude that specialist mathematics teachers who do *all* the mathematics teaching in elementary school, from at least the third grade on if not earlier, are the only solution if the curriculum is to be radically changed. This is clearly a long term solution. *But I* *predict that no great progress will be made in the mathematical education of American elementary (and, therefore, also secondary) school students until this solution is implemented* . Until it is, the hand-wringing at the poor performance of American students in international comparisons and the disappointment of university teachers with the mathematical preparation of their students will be a continuing phenomenon of the American educational scene.

Do We Need Research?

Of course, we do. We always need research. But the immediate question is: Does any proposal in this paper *require* that research be done before that proposal should be tried out? Or, put another way, a guiding principle of educational innovation should be, as in medicine: Above all, do no harm. So is it possible that students ``deprived'' of the opportunity to learn PPA could be harmed thereby?

I submit that the answer is, no. Since no one argues any longer that knowledge of PPA is a useful skill in life (or, for that matter, in mathematics), the question is only whether such ``deprivation'' could leave students without the understanding or technique necessary to study further mathematics. We may not know how much mental arithmetic the average child may reasonably be expected to learn - and we should surely be doing research to find out - but I do not see how you could argue that an elementary school mathematics curriculum built around mental arithmetic and calculators, which was demanding for students and which introduced them to considerably more of mathematics outside arithmetic than is usual today, could possibly do them any *harm* . Indeed, a curriculum emphasizing number sense in this context could hardly help but prepare students for secondary school mathematics at least as well as they are now.

Yes, much of the argument in this paper has been *ex cathedra* but I make no apology for that. My arguments are, in fact, far better based on experience and research than the arguments of those who wish to continue to subject children to a PPA regimen. So, yes, more research by all means but there is no reason whatsoever why the ideas in this paper should not be tried before further research is done.

It is also worth noting that nothing in this paper is at variance with any modern theoretical, experimental or practical research on the teaching or learning of mathematics (see, for example, Kamii [1985]). Indeed, I would expect the many proponents of a constructivist approach to mathematics to applaud a curriculum which emphasizes having children construct their own understandings of arithmetic through mental calculation.

Final Remarks

The reform of mathematics education is a much debated topic currently, particularly in the United States. What has been called the ``math wars'' [Becker and Jacob, 1998; Jackson, 1997; Ross, 1998] pits mathematicians, mathematics educators, other educators, parents and politicians against each other in a bewildering array of contexts in which curriculum, pedagogy, teacher training, textbooks and technology are all subjects of controversy. One irony of all this, in my view, is that, while the crucial decisions that need to be made concern elementary school mathematics, the chief mathematical protagonists are professional mathematicians, who rarely understand the issues with respect to elementary school mathematics, and the National Council of Teachers of Mathematics which, although it is a group whose membership and interests cover both elementary and secondary school mathematics, is dominated by university mathematics educators and secondary school teachers. The result is that elementary school mathematics too often gets short shrift or misunderstanding or worse in the debates about mathematics education.

Some of the debate about mathematics education concerns such matters as how much focus there should be on problem solving in the school mathematics curriculum, whether mathematical pedagogy can and should be influenced by recent work on learning theory and on a constructivist approach to mathematics, and on whether new curriculum proposals give sufficient weight to traditional topics like algebra and trigonometry. I have not discussed these matters here because I claim that the proposals made in this paper are equally apt whichever side you take on any of these questions.

My belief is that a mental arithmetic-calculator curriculum taught by specialist mathematics in elementary school would not just be a better curriculum than is now taught (virtually) everywhere but that, more strongly, unless this approach to elementary school mathematics is adopted, major improvement in the mathematical performance of American students at the elementary school, secondary school or university levels will remain a chimera.

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