HOMOGENEOUS LOCALLY COMPACT SPACES

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Abstract. This is a survey of the recent results and unsolved problems about locally compact homogeneous metric spaces. Mostly, homogeneous finite-dimensional ANR-spaces are discussed.

1. Introduction

In this paper we survey the most recent results and unsolved problems concerning homogeneous finite-dimensional spaces. It can be considered as an continuation of our previous paper [43]. As in [43], we are mainly interested in locally compact metric homogeneous ANRs, but some problems concerning more general homogeneous spaces are also considered. Recall that a space X is *homogeneous* if for every two points $x, y \in X$ there is a homeomorphism h mapping X onto itself with $h(x) = y$. This implies that X is *locally homogeneous*, i.e. for every two points $x, y \in X$, there exists a homeomorphism h mapping a neighborhood U_x of x onto a neighborhood $U_y = h(U_x)$ of y and satisfying the condition $h(x_1) = y$.

There are many interesting problems about homogeneous spaces. Probably, the best known is the Bing-Borsuk conjecture [3] stating that every n-dimensional homogeneous metric ANR-compactum, $n \geq 3$, is an n-manifold. This conjecture is true in dimensions 1 and 2 [3]. Recently, Bryant-Ferry [11] provided a revised version of their paper containing counter-examples to that conjecture. They constructed for every $n \geq 6$ infinitely many, topologically distinct, homogeneous ANR-compacta of dimension n that are not topological manifolds. So, this conjecture is still open in dimensions 3, 4 and 5. Another open problem is whether there is a non-degenerated finite-dimensional locally homogeneous, in particular homogeneous, AR-spaces, see [3], [4]. On the other hand, finite-dimensional locally compact homogeneous

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*ANR*s share many properties with Euclidean manifolds, see for example [42], [43], [45]. So, although homogeneous finite-dimensional *ANR*-compacta may not be Euclidean manifolds, it still interesting to what extend they have common properties with Euclidean manifolds. The survey paper of Halverson-Repovš [21] contains more information for different type of homogeneity.

Recall that a metric space X is an absolute neighborhood retract (br., *ANR*) if for every embedding of X as a closed subset of a metric space M there exists a neighborhood U of X in M and a retraction $r : U \to X$, i.e. a continuous map r with $r(x) = x$ for all $x \in X$. Contractible *ANR*-spaces form the class of absolute retracts (br., *AR*).

Unless stated otherwise, all spaces are locally compact separable metric and all maps are continuous. Reduced Čech homology $\check{H}_n(X; G)$ and cohomology $\check{H}^n(X; G)$ with coefficients from an abelian group G are considered. Singular homology and cohomology groups are denoted, respectively, by $H_k(X; G)$ and $H^k(X; G)$. By a dimension we mean the covering dimension dim, the cohomological dimension with respect to a group G is denoted by $\dim_G$. Recall that $\dim_G$ is the largest integer n such that there exists a closed set $A \subset X$ with $\check{H}^n(X, A; G) \neq 0$.

2. HOMOGENEOUS SPACES AND THE BING-BORSUK CONJECTURE

It is interesting whether some of the counter examples to the Bing-Borsuk conjecture constructed by Bryant-Ferry have the stronger version of homogeneity, the so called strong local homogeneity. Recall that a space X is *strongly locally homogeneous* (br., *SLH*) if every point in X has a base of neighborhoods U such that for every $y, z \in U$ there is a homeomorphism $h : X \to X$ with $h(y) = z$ and $h(x) = x$ for all $x \notin U$. Every strongly locally homogeneous space X is homogeneous provided X is connected. Moreover, if no two-point set disconnetes a connected strongly locally homogeneous space X, then X is *n-homogeneous* for all $n \geq 1$ [2]: if A, B are two n-elements subsets of X, then there is a homeomorphism h on X such that $h(A) = B$. The question whether there is an *SLH*-counter example to the Bing-Borsuk conjecture is interesting because every Euclidean manifold has this property. So, the Bing-Borsuk conjecture can be restated:

Question 2.1. *Is it true that every n-dimensional strongly locally homogeneous ANR-compactum is an n-manifold?*

Jakobsche [19] proved that the 3-dimensional Bing-Borsuk conjecture implies the Poincare conjecture. Assuming the Poincare conjecture is not true, he constructed a 3-dimensional homogeneous compact

ANR-space which is not a manifold. Any such an example have the additional property of n-homogeneity for all n, see [18, Theorem 8.1]. There is a strong expectation that Jacobsche's construction provides a 3-dimensional SLH compact ANR-space which is not a manifold. Therefore, we have another natural question:

Question 2.2. *Is it true that the restated Bing-Borsuk conjecture in dimension 3 imply the Poincare conjecture?*

Topological n-manifolds X have the following property: For every $x \in X$ the groups $H_k(X, X \setminus \{x\}; \mathbb{Z}) = 0$ if $k < n$ and $H_n(X, X \setminus \{x\}; \mathbb{Z}) = \mathbb{Z}$. A space with this property is said to be a *$\mathbb{Z}$-homology n-manifold.* A *generalized n-manifold* is a locally compact n-dimensional ANR-space which is a $\mathbb{Z}$-homology n-manifold. Every generalized $(n \leq 2)$-manifold is known to be a topological n-manifold [48]. On the other hand, for every $n \geq 3$ there exists a generalized n-manifold X such that X is not locally Euclidean at any point, see for example [12]. Let us mention that the Bryant-Ferry [11] counter examples to the Bing-Borsuk conjecture are generalized n-manifolds with the disjoint disks property, $n \geq 6$.

Bryant [8] suggested another modification of the Bing-Borsuk conjecture:

Conjecture 2.3. [8][*Modified Bing-Borsuk conjecture*] *Every locally compact homogeneous ANR-space of dimension* $n \geq 3$ *is a generalized n-manifold.*

A partial result concerning the Modified Bing-Borsuk conjecture is an old result of Bredon [6], reproved by Bryant [10]:

Theorem 2.4. [6],[10] *If X is a locally compact homogeneous ANR-space of dimension n such that the groups $H_k(X, X \setminus \{x\}; \mathbb{Z})$, $k \leq n$, are finitely generated, then X is a generalized n-manifold.*

Another result related to the Modified Bing-Borsuk conjecture was obtained by Bryant [7]:

Theorem 2.5. [7] *Every n-dimensional homologically arc-homogeneous ANR-compactum is a generalized manifold.*

Here, a space X is homologically arc-homogeneous [35] if for every path $\alpha : \mathbb{I} = [0, 1] \to X$ the inclusion induced map

$$H_*(X \times \{0\}, X \times \{0\} - (\alpha(0), 0)) \to H_*(X \times \mathbb{I}, (X \times \mathbb{I}) - \Gamma(\alpha))$$

is an isomorphism, where $\Gamma(\alpha)$ is the graph of α.

More information about generalized manifolds can be found in Bryant [9].

The last two theorems in this section show that $\mathbb{Z}$-homology manifolds have also common properties with Euclidean manifolds. Recall that a space X is a *Cantor n-manifold* [22], [39] if X cannot be separated by a closed subset F of dimension $\leq n-2$, (i.e., $X\backslash F$ is disconnected).

Theorem 2.6. [27] *Let X be a locally compact, locally connected $\mathbb{Z}$-homology n-manifold with* $\dim X = n > 1$ *at each point. Then X is a local Cantor manifold, i.e. every open connected subset of X is a Cantor n-manifold.*

This result was extended in [38, Corollary 4.2].

Theorem 2.7. [38] *Let X be a complete metric space which is a $\mathbb{Z}$-homology n-manifold. Then every open arcwise connected subset of X is a Mazurkiewicz arc manifold with respect to the class of all spaces of dimension $\leq n-2$.*

Note that a space X (not necessarily metrizable) is a *Mazurkiewicz arc manifold* with respect to the class of all spaces of dimension $\leq n-2$ [38] if for ever two closed disjoint sets $A, B \subset X$, both having non-empty interiors in X, and every F_σ-set $F \subset X$ with $\dim F \leq n-2$, there is an arc C in $X\backslash F$ joining A and B.

3. HOMOGENEOUS ANR-SPACES

We show in this section that finite-dimensional homogeneous ANR-spaces share many properties with Euclidean manifolds. In particular, the local cohomological and homological structure of homogeneous n-dimensional ANR-spaces is similar to the corresponding local structure of $\mathbb{R}^n$. We also discuss another two problems of Bing-Borsuk [3] and their relation to the problem whether there exists a finite-dimensional non-degenerated homogeneous AR-compactum.

We say that a finite-dimensional space X is *dimensionally full-valued* if $\dim X \times Y = \dim X + \dim Y$ for any compactum Y. It is known that all polyhedra and all one-dimensional compacta are dimensionally full-valued. Pontryagin [34] constructed in 1930 a family $\{\Pi_p : p \text{ is prime}\}$ of 2-dimensional homogeneous but not ANR-compacta such that $\dim(\Pi_p \times \Pi_q) = 3$ for $p \neq q$. During the same time Borsuk raised the question whether $\dim X \times Y = \dim X + \dim Y$ for any ANR-compacta X and Y. Kodama [26] provided a partial answer of Borsuk's question by proving that every 2-dimensional ANR-compactum is dimensionally full-valued. In 1988 Dranishnikov [15] gave a negative answer to Borsuk's question by constructing a family of 4-dimensional metric ANR-compacta M_p, where p is a prime number, such that $\dim(M_p \times M_q) = 7$

for all $p \neq q$. The spaces M_p are not homogeneous. After Dranishnikov constructed his examples, the question whether homogeneous ANR-compacta are dimensionally full-valued was raised. It goes back to [8] and was also discussed in [13] and [17].

This question was answered recently (for 3-dimensional homogeneous ANR-compacta it was known earlier [45]).

Theorem 3.1. [42] *Let X be a finite-dimensional locally homogeneous ANR-space. Then the following holds:*

(i) *X is dimensionally full-valued;*
(ii) *If X is homogeneous, then every $x \in X$ has a neighborhood W_x such that $\mathrm{bd}\,\overline{U}$ is dimensionally full-valued for all $U \in \mathcal{B}_x$ with $\overline{U} \subset W_x$.*

According to [5], a finite-dimensional compactum X is dimensionally full-valued if and only if $\dim_G X = \dim X$ for any group G. It was shown in [45] that an n-dimensional ANR-compactum X is dimensionally full-valued iff there exists a point $x \in X$ with $\check{H}_n(X, X\backslash x; \mathbb{Z})$ is not trivial.

Suppose (K, A) is a pair of closed subsets of a space X with $A \subset K$. Then we denote by $j^n_{K,A} : \check{H}^n(K; G) \to \check{H}^n(A; G)$ the inclusion induced cohomology homomorphism (recall that $dim_G X \leq n$ if and only if $j^n_{X,A}$ is surjective for every closed $A \subset X$). We say that an element $\gamma \in \check{H}^n(A; G)$ is not extendable over K if γ is not contained in the image $j^n_{K,A}(\check{H}^n(A; G))$. If (K, A) is as above, K is called an (n, G)-*cohomology membrane spanned on A for an element* $\gamma \in \check{H}^n(A; G)$ if γ is not extendable over K, but it is extendable over any proper closed subset P of K containing A. The continuity of the Čech cohomology implies the following fact: If A is a closed subset of a compact space X and $\gamma \in \check{H}^n(A; G)$ is not extendable over X, then there is an n-cohomology membrane for γ spanned on A. A space X is said to be a cohomological (n, G)-*bubble* [40] if $\check{H}^n(X; G) \neq 0$ but $\check{H}^n(B; G) = 0$ for every closed proper subset $B \subset X$.

The next theorem shows that the local cohomological structure of homogenous n-dimensional ANR-spaces is similar to the local structure of $\mathbb{R}^n$ (this was established earlier in [45] for homogeneous ANR-compacta and countable principal domains G).

Theorem 3.2. [42] *Let X be a connected homogeneous ANR-space with $\dim X = n \geq 2$ and G be a countable group. Then every point $x \in X$ has a basis $\mathcal{B}_x$ of open sets $U \subset X$ satisfying the following conditions:*

1. (1) $\mathrm{int}\overline{\mathrm{U}} = \mathrm{U}$ *and the complement of* $\mathrm{bd}\, U$ *has exactly two components;*
2. (2) $\check{H}^{n-1}(\mathrm{bd}\, U; G) \neq 0$, $\check{H}^{n-1}(\overline{U}; G) = 0$ *and* $\overline{U}$ *is an* $(n-1, G)$*-cohomology membrane spanned on* $\mathrm{bd}\, U$ *for any non-zero* $\gamma \in \check{H}^{n-1}(\mathrm{bd}\, U; G)$*;*
3. (3) $\mathrm{bd}\, U$ *is a cohomological* $(n-1, G)$*-bubble;*

A similar description of the local homology structure of homogeneous ANR-compacta is given in [44].

We say that a space X has an *n-dimensional G-obstruction* at a point $x \in X$ [31] if there is $W \in \mathcal{B}_x$ such that the homomorphism $j^n_{U,W} : H^n(X, X \backslash U; G) \to H^n(X, X \backslash W; G)$ is nontrivial for every $U \in \mathcal{B}_x$ with $U \subset W$. Kuzminov [31] proved that every compactum X with $\dim_G X = n$ contains a compact set Y with $\dim_G Y = n$ such that X has an n-dimensional G-obstruction at any point of Y.

Theorems 3.1-3.2 provides more properties of homogeneous n-dimensional spaces which are typical for n-manifolds.

Corollary 3.3. *Let X be a locally homogeneous ANR-spaces with* $\dim_X = n \geq 2$ *and G be a countable group. Then*

1. (1) *$f(U)$ is open in X provided $U \subset X$ is open and $f : U \to X$ is an injective map;*
2. (2) *$\dim A = n$, where $A \subset X$ is closed, if and only if A has a non-empty interior in X;*
3. (3) *$\check{H}^n(P; G) = 0$ for any proper compact set $P \subset X$;*
4. (4) *X has an n-dimensional G-obstruction at every $x \in X$. Moreover, there is $W \in \mathcal{B}_x$ such that the homomorphism $j^n_{U,V}$ is surjective for any $U, V \in \mathcal{B}_x$ with $\overline{U} \subset V \subset \overline{V} \subset W$.*

Properties (1) and (2) were also established by Lysko [32] and Seidel [36].

We say that X is *cyclic in dimension n* if there is a group G such that $\check{H}^n(X; G) \neq 0$. If a space is not cyclic in dimension n, it is called *acyclic in dimension n*. If X is an n-dimensional ANR-compactum, the duality [22] between Čech homology and cohomology, and the universal coefficient formulas imply the following equivalence: $\check{H}_n(X; G) \neq 0$ for some group G if and only if X cyclic in dimension n.

We denote by $\mathcal{H}(n)$ the class of all homogeneous metric ANR-compacta of dimension n.

Question 3.4. [3] *Let $X \in \mathcal{H}(n)$. Is it true that:*

1. (1) *X is cyclic in dimension n?*
2. (2) *No closed subset of X, acyclic in dimension $n-1$, separates X?*

The next theorem shows that the two parts of Question 3.4 have positive or negative answers simultaneously.

Theorem 3.5. [44] *The following conditions are equivalent:*

(1) *For all* $n \geq 1$ *and* $X \in \mathcal{H}(n)$ *there exists a group* G *with* $\check{H}^n(X;G) \neq 0$ *(resp.,* $\check{H}_n(X;G) \neq 0$*);*

(2) *If* $X \in \mathcal{H}(n)$*,* $n \geq 1$*, and* $F \subset X$ *is a closed set separating* X*, then there exists a group* G *with* $\check{H}^{n-1}(F;G) \neq 0$ *(resp.,* $\check{H}_{n-1}(F;G) \neq 0$*);*

(3) *If* $X \in \mathcal{H}(n)$*,* $n \geq 1$*, and* $F \subset X$ *is a closed set separating* X *with* $\dim F \leq n-1$*, then there exists a group* G *such that* $\check{H}^{n-1}(F;G) \neq 0$ *(resp.,* $\check{H}_{n-1}(F;G) \neq 0$*).*

Note that for any finite-dimensional homogeneous continuum X (not necessarily ANR) we have the following result [24]: If $\check{H}^n(X;G) \neq 0$ for some group G, then $\check{H}^{n-1}(F;G) \neq 0$ for any closed set $F \subset X$ separating X with $\dim_G F \leq n-1$.

On the other hand, the structure of cyclic homogeneous ANR continua is described in [46] (the notion of strong V^n_G-continua is given in Section 4).

Theorem 3.6. [46] *Let* X *be a homogeneous metric* ANR*-continuum such that* $\dim_G X = n$ *and* $\check{H}^n(X;G) \neq 0$ *for some group* G*. Then*

(1) X *is a cohomological* (n,G)*-bubble;*

(2) X *is a strong* V^n_G*-continuum;*

(3) $\check{H}^{n-1}(A;G) \neq 0$ *for every closed set* $A \subset X$ *separating* X*.*

Items (1) and (3) were also established by Yokoi [40] for the case G is a principal ideal domain.

Clearly, the cyclicity of finite-dimensional homogeneous ANR-compacta provides a negative answer to the next question.

Question 3.7. [3], [4] *Does there exists a non-degenerate finite-dimensional homogeneous AR-compactum*?

According to Fadell [16] there is no non-degenerate strongly homogeneous space. Here, a compactum X is *strongly homogeneous* if X is connected and for every $x_1 \in X$ there is an open set U containing x_1 and a continuous map $L_U : U \to C(X,X)$ with the following properties ($C(X,X)$ is the space of continuous maps from X into X with the compact-open topology): (i) For every $x \in U$, $L_U(x) : X \to X$ is a homeomorphism such that $L_U(x)(x_1) = x$; (ii) $L_U(x_1)$ is the identity homeomorphism on X. This kind of homogeneity seems to be quite strong because Fadell's result implies that the Hilbert cube Q is not strongly homogeneous.

On the other hand it is interesting if we consider strongly locally homogeneous spaces in Question 3.7.

Question 3.8. *Does there exists a non-degenerate finite-dimensional strongly locally homogeneous AR-compactum*?

Another questions in that direction was listed in [47].

Question 3.9. [47] *Is the Hilbert cube Q the only homogeneous non-degenerate compact AR*?

4. SEPARATION OF HOMOGENEOUS SPACES

We already observed that the existence of finite-dimensional AR-compacta is equivalent to the question whether homogeneous n-dimensional ANR-compacta can be separated by closed sets A acyclic in dimension $n-1$ with $\dim A \leq n-1$. In this section we discuss the question of separating homogeneous n-dimensional spaces by sets of a smaller dimension. Cantor manifolds defined in Section 2 (just before Theorem 2.6) were introduced by Urysohn [39] in 1925 as a generalization of Euclidean manifolds. One of the first result concerning separation of homogeneous spaces was established by Krupski [28], [29].

Theorem 4.1. *Every region in a homogeneous n-dimensional space cannot be separated by a subset of dimension $\leq n-2$.*

The notion of Cantor manifolds was generalized in different ways. Inspired by the classical result of Mazurkiewicz that any region in $\mathbb{R}^n$ cannot be cut by subsets of dimension $\leq n-2$ (a subset cuts if its compliment is not continuum-wise connected), Hadjiivanov-Todorov [20] introduced the class of *Mazurkiewicz manifolds*. This notion was generalized in [25] as follows: A normal space (not necessarily metrizable) X is a *Mazurkiewicz manifold with respect to* $\mathcal{C}$, where $\mathcal{C}$ is a class of spaces, if for every two closed, disjoint subsets $X_0, X_1 \subset X$, both having non-empty interiors in X, and every F_σ-subset $F \subset X$ with $F \in \mathcal{C}$, there exists a continuum K in $X \setminus F$ joining X_0 and X_1. Obviously, every Mazurkiewicz manifold with respect to the class at most $(n-2)$-dimensional spaces is a Cantor n-manifold.

A new dimension $\mathcal{D}_{\mathcal{K}}$, unifying both the covering and the cohomological dimension, was introduced in [25]. By $\mathcal{D}^n_{\mathcal{K}}$ we denote all spaces X with $\mathcal{D}_{\mathcal{K}}(X) \leq n$. Concerning that dimension, we have the following result:

Theorem 4.2. [30] *Let X be a homogeneous locally connected space. Then every region $U \subset X$ with $D_{\mathcal{K}}(U) = n$ is a Mazurkiewicz manifold with respect to the class $\mathcal{D}^{n-2}_{\mathcal{K}}$.*

Alexandroff [1] introduced another property which is possessed by compact closed n-manifolds, to so-called continua V^n. Here is the general notion of Alexandroff manifold, see [38]: A connected space X is an *Alexandroff manifold with respect to a given class* $\mathcal{C}$ of spaces if for every two disjoint closed subsets X_0, X_1 of X, both having non-empty interiors, there exists an open cover ω of X such that no partition P between X_0 and X_1 admits an ω-map onto a space $Y \in \mathcal{C}$. The Alexandroff *continua* V^n are compact Alexandroff manifolds with respect to the class of all spaces Y with $\dim Y \leq n-2$. Recall that a partition between two disjoint sets X_0, X_1 in X is a closed set $F \subset X$ such that $X \setminus F$ is the union of two open disjoint sets U_0, U_1 in X with $X_0 \subset U_0$ and $X_1 \subset U_1$. An ω-map $f : P \to Y$ is such a map that $f^{-1}(\gamma)$ refines ω for some open cover γ of Y.

A cohomological version of V^n-continua was considered in [37]. A compactum X is a V^n_G*-continuum* [37], where G is a given group, if for every open disjoint subsets U_1, U_2 of X there is an open cover ω of $X_0 = X \setminus (U_1 \cup U_2)$ such that any partition P in X between U_1 and U_2 does not admit an ω-map g onto a space Y with $g^* : \check{H}^{n-1}(Y; G) \to \check{H}^{n-1}(P; G)$ being a trivial homomorphism. If, in addition, there is also an element $\gamma \in \check{H}^{n-1}(X_0; G)$ such that for any partition P between U_1 and U_2 and any ω-map g of P into a space Y we have $0 \neq i_P^*(\gamma) \in g^*(\check{H}^{n-1}(Y; G))$, where i_P is the embedding $P \hookrightarrow X_0$, X is called a *strong* V^n_G*-continuum* [46]. Because $\check{H}^{n-1}(Y; G) = 0$ for every space Y with $\dim Y \leq n-2$, every V^n_G-continuum is a V^n-continuum in the sense of Alexandroff.

The following question, raised in [38], is one of the remaining open problems concerning separation of homogeneous ANR-spaces.

Question 4.3. *Let* X *be a homogeneous* ANR*-continuum with* $\dim X = n$ *and* G *be a group.*

1. *Is* X *a* V^n*-continuum?*
2. *Is* X *a* V^n_G*-continuum?*

According to Theorem 3.6, Question 4.3 has a positive answer provide $\check{H}^n(X; G) \neq 0$. For strongly locally homogeneous spaces (not necessarily ANRs) the answer of Question 4.3(1) is also positive.

Theorem 4.4. [23] *Every strongly locally homogeneous connected space with* $\dim_G X = n$ *is an Alexandroff manifold with respect to the class of spaces* Y *with* $\dim_G Y \leq n-2$.

Finally, let's mention another recent result extending Theorem 4.1, as well as, the result from [33] that no region in 2-dimensional strongly locally homogeneous space cannot be separated by an arc (we say that a closed set $C \subset X$ is irreducibly separating X if there are two disjoint open sets G_1, G_2 in X such that $\overline{G}_1 \cap \overline{G}_2 = C$ and $X = \overline{G}_1 \cup \overline{G}_2$). We say that a point $x \in X$ has a *special base* $\mathcal{B}_x$ if for any neighborhoods U, V of x in X with $\overline{U} \subset V$ there is $W \in \mathcal{B}_x$ such that bdW separates $\overline{V} \backslash U$ between $\mathrm{bd\overline{V}}$ and $\mathrm{bd\overline{U}}$.

Theorem 4.5. [41] *Let* Γ *be a region in a finite-dimensional homogeneous space* X *with* $\dim_G X = n \geq 2$*, where* G *is a countable Abelian group. Then* Γ *cannot be irreducibly separated by any closed set* $C \subset X$ *with the following property:*

- (i) $\dim_G C \leq n-1$ *and* $H^{n-1}(C; G) = 0$*;*
- (ii) *There is a point* $b \in C \cap \Gamma$ *having a special local base* $\mathcal{B}_C^b$ *in* C *with* $H^{n-2}(\mathrm{bd_C U; G}) = 0$ *for every* $U \in \mathcal{B}_C^b$*.*

If X *is strongly locally homogeneous, the finite-dimensionality of* X *can be omitted and condition* (*ii*) *can be weakened to the following one:*

- (iii) *There is* $b \in C \cap \Gamma$ *having an ordinary base* $\mathcal{B}_C^b$ *in* C *with* $H^{n-2}(\mathrm{bd_C U}) = 0$*,* $U \in \mathcal{B}_C^b$*.*

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