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Strongly locally homogeneous generalized continua of finite cohomological dimension

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ABSTRACT

The notion of a V^n-continuum was introduced by Alexandroff [1] as a generalization of the concept of n-manifold. In this note we consider the cohomological analogue of V^n-continuum and prove that any strongly locally homogeneous generalized continuum X with cohomological dimension $\dim_G X = n$ is a generalized V^n-space with respect to the cohomological dimension $\dim_G$. In particular, every strongly locally homogeneous continuum of covering dimension n is a V^n-continuum in the sense of Alexandroff. This provides a partial answer to a question raised in [12]. An analog of the Mazurkiewicz theorem that no subset of covering dimension $\leq n-2$ cuts any region of the Euclidean n-space is also obtained for strongly locally homogeneous generalized continua X of cohomological dimension $\dim_G X = n$.

1. Introduction

By a *space* we mean a locally compact separable metric space, unless otherwise stated, and each of its connected open subsets is called a *region* or a *generalized continuum*. *Maps* are continuous mappings. The covering dimension of a space X is denoted by $\dim X$. Čech cohomology groups $H^n(X;G)$ with coefficients in an Abelian group G are considered everywhere below.

The *cohomological dimension* $\dim_G X$ of a space X is the largest number n such that there exists a closed subset $A \subset X$ with

$$H^n(X, A; G) \neq 0.$$

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Equivalently, $\dim_G X \leq n$ if and only if for any closed pair $A \subset B$ in X the homomorphism $j^n_{X,A}$: $H^n(B;G) \to H^n(A;G)$, generated by the inclusion $A \hookrightarrow B$, is surjective, see [5].

If X is a finite-dimensional space, then for every G we have $\dim_G X \leq \dim_{\mathbb{Z}} X = \dim X$ [11].

The notion of a V^n-continuum was introduced by Alexandroff [1] as a generalization of the concept of n-manifold. A continuum X is a V^n*-continuum* if for every two closed disjoint subsets P and Q of X, both having non-empty interiors $\operatorname{int} P$, $\operatorname{int} Q$, there exists an open cover ω of X such that no partition $C \subset X$ between P and Q admits an ω-map into a space Y with $\dim Y \leq n-2$ (by an ω-map on C we mean an $\omega|C$-map, where $\omega|C = \{U \cap C : U \in \omega\}$). We extend the notion of V^n-continua as follows.

Definition 1.1. A space X is a $V^n(G)$*-space* if for any two closed disjoint sets P, Q in X with $\operatorname{int} P \neq \varnothing \neq \operatorname{int} Q$ there exists an open cover ω of X such that no partition in X between P and Q admits an ω-map into a space Y of cohomological dimension $\dim_G Y \leq n-2$.

Compact $V^n(G)$-spaces were defined in [7] under the name of V^n-continua with respect to the class of all spaces with $\dim_G \leq n-2$ and studied in [12] as Alexandroff manifolds with respect to the class. Trivially, any connected space is $V^1(G)$. One can easily observe that each $V^n(G)$-space is connected, so it is a generalized continuum.

Recall that a space X is *homogeneous* if for every $x, y \in X$ there is a homeomorphism $h : X \to X$ such that $h(x) = y$. A space X is *strongly locally homogeneous* if every point $x \in X$ has a basis of open neighborhoods U such that for every $y, z \in U$ there is a homeomorphism h on X such that $h(y) = z$ and h is the identity on $X \setminus U$.

Basic examples of strongly locally homogeneous spaces are the Euclidean space $\mathbb{R}^n$ and the Menger universal n-dimensional continuum $\mathcal{M}_n$ [3, Theorem 3.2.2].

Every region of a strongly locally homogeneous space is homogeneous, strongly locally homogeneous, and it is also known to be locally connected [2].

The question whether every homogeneous ANR-continuum X of dimension $\dim X = n$ is a V^n-continuum was raised in [12, Question 1.2]. In this note we prove the following theorem which provides a partial answer to that question.

Theorem 1.1. *Every strongly locally homogeneous generalized continuum X with $\dim_G X = n$ is a $V^n(G)$-space.*

Since $\dim_G \mathbb{R}^n = \dim_G \mathcal{M}_n = n$ for any Abelian group G, and both Euclidean n-manifolds and Menger $\mathcal{M}_n$-manifolds are strongly locally homogeneous, we get the following corollary.

Corollary 1.2. *Connected Euclidean n-manifolds and Menger $\mathcal{M}_n$-manifolds are $V^n(G)$-spaces for any Abelian group G.*

It was established in [9, Theorem 4] that every region Λ of a homogeneous and locally connected space X with $\dim_G \Lambda = n$ cannot be cut by an F_σ-set M of dimension $\dim_G M \leq n-2$. For strongly locally homogeneous spaces we get the following theorem.

Theorem 1.3. *Let X be a strongly locally homogeneous generalized continuum with $\dim_G X = n$. Then for every pair $U, V \subset X$ of disjoint, nonempty, open sets, and every set $M \subset X$ with $\dim_G M \leq n-2$, there is a continuum $K \subset X \setminus M$ joining U and V.*

It is well known that finite-dimensional homogeneous ANR-continua share many properties with Euclidean manifolds, see [13]. This fact and Theorem 1.1 suggest the following question, the positive solution of which would answer [12, Question 1.2].

Question 1.4. Let X be a homogeneous ANR-continuum with $\dim X = n$. Is it true that X is strongly locally homogeneous?

2. Proofs of Theorems 1.1 and 1.3

If ω is an open cover of X and $F \subset X$ is closed, for every closed $C \subset X$ we consider a canonical map $\kappa_{\omega_C} : (C, C \cap F) \to (N_{\omega|C}, N_{\omega|(C\cap F)})$, where $N_{\omega|C}$ and $N_{\omega|(C\cap F)}$ are the nerves of the covers $\omega|C$ and $\omega|(C \cap F)$, respectively (see, e.g., [6, §11]). The following notion was introduced in [12].

Let P and Q be disjoint, nonempty, open subsets of a continuum X, and $F \subset X$ be closed. We say that the pair (X, F) is *K^n_G-connected between P and Q* if there exists an open cover ω of X such that the following condition holds for every partition C in X between P and Q:

the map $\kappa_{\omega_C} : (C, C \cap F) \to (N_{\omega|C}, N_{\omega|(C\cap F)})$ generates a non-trivial homomorphism

$$\kappa^*_{\omega_C} : H^{n-1}(N_{\omega|C}, N_{\omega|(C\cap F)}; G) \to H^{n-1}(C, C \cap F; G).$$

If, in addition, there is $\mathrm{e} \in H^{n-1}(N_\omega, N_{\omega|(Y\cap F)})$ such that

$$\kappa^*_{\omega_C}(i^*_{\omega_C}(\mathrm{e})) \neq 0$$

for every partition C in X between P and Q, the pair (X, F) is called *strongly K^n_G-connected between P and Q*. Here, $Y = X\backslash(P \cup Q)$ and

$$i_{\omega_C} : (N_{\omega|C}, N_{\omega|(C\cap F)}) \to (N_\omega, N_{\omega|(Y\cap F)})$$

is the inclusion.

Further, (X, F) is said to be a *K^n_G-manifold* (*strong K^n_G-manifold*) if it is K^n_G-connected (resp., strongly K^n_G-connected) between any two nonempty open disjoint sets $P, Q \subset X$.

The notion of a (strong) K^n_G-manifold is justified by the following result which was actually established in [10, Theorem 1].

Proposition 2.1. *Every compactum X (not necessarily metrizable) with $\dim_G X = n$ contains a pair (Y, F) of closed sets with $\dim_G Y = n$ such that F is nowhere dense in Y and (Y, F) is a strong K^n_G-manifold.*

We need an analogous statement for locally compact spaces.

Lemma 2.2. *Every space X with $\dim_G X = n$ contains a strong K^n_G-manifold (Φ, F), where Φ is a compactum with $\dim_G \Phi = n$ and F is a closed nowhere dense subset of Φ. In particular, Φ is a $V^n(G)$-space.*

Proof. Since X is a countable union of compact sets, there exists a compactum $Y \subset X$ with $\dim_G Y = n$ (otherwise, by the countable sum theorem for $\dim_G$, $\dim_G X \leq n - 1$). Then, by Proposition 2.1, there is a pair $F \subset \Phi$ of compact sets in Y with $\dim_G \Phi = n$ such that F is nowhere dense in Φ and (Φ, F) is a strong K^n_G-manifold. Finally, according to [12, Proposition 2.2], Φ is a $V^n(G)$-space. □

Remark 2.3. According to [7, Theorem 2.6], every compactum Y with $\dim_G Y = n$ contains a compact $V^n(G)$-space Φ such that $\dim_G \Phi = n$. Lemma 2.2 is a stronger statement which will allow us to prove Theorem 1.3.

Lemma 2.4. *Let X be a strongly locally homogeneous generalized continuum with $\dim X \geq 2$ and let $\Phi \subset X$ be a compactum containing at least two points. Suppose that U and V are nonempty open subsets of X with $U \cap V = \varnothing$. Then there exists a homeomorphism $h : X \to X$ such that $h(\Phi) \cap U \neq \varnothing \neq h(\Phi) \cap V$.*

Proof. For every $x \in X$, let $\mathcal{B}_x$ be a basis at x consisting of open sets $W \subset X$ such that for all $y, z \in W$ there is a homeomorphism h on X with $h(y) = z$ and such that h is the identity outside W.

Since X is homogeneous, we can assume that $\Phi \cap U \neq \varnothing$ and choose $x_U \in \Phi \cap U$. Let Γ be the set of all $x \in X$ such that $x \neq x_U$ and there is a homeomorphism $h_x : X \to X$ with $x \in h_x(\Phi)$ and $h_x(x_U) = x_U$. Take $x \in \Phi \backslash \{x_U\}$. Obviously, the identity homeomorphism on X witnesses that $x \in \Gamma$, so $\Gamma \setminus \{x_U\} \neq \varnothing$.

Claim 1. $\Gamma \setminus \{x_U\}$ *is open in* $X \backslash \{x_U\}$.

Indeed, let $x \in \Gamma \setminus \{x_U\}$. So, there exists a homeomorphism h_x on X such that $x \in h_x(\Phi)$ and $h_x(x_U) = x_U$. Choose $W_x \in \mathcal{B}_x$ with $x_U \notin W_x$. Then for every $y \in W_x$ there is a homeomorphism h_{xy} on X with $h_{xy}(x) = y$ and $h_{xy}(z) = z$ for all $z \notin W_x$. The composition $h_y = h_{xy} \circ h_x$ is a homeomorphism such that $y \in h_y(\Phi)$ and $h_y(x_U) = x_U$. This shows that $W_x \subset \Gamma \setminus \{x_U\}$, hence $\Gamma \setminus \{x_U\}$ is open in $X \backslash \{x_U\}$.

Claim 2. $\Gamma \setminus \{x_U\}$ *is closed in* $X \backslash \{x_U\}$.

To show Claim 2, let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $\Gamma \setminus \{x_U\}$ converging to a point $x_0 \in X \backslash \{x_U\}$. Take $W_{x_0} \in \mathcal{B}_{x_0}$ with $x_U \notin W_{x_0}$ and let $x_k \in W_{x_0}$. There exist homeomorphisms h_{x_k} and $h_{x_k x_0}$ on X such that

$$x_k \in h_{x_k}(\Phi), \quad h_{x_k}(x_U) = x_U, \quad h_{x_k x_0}(x_k) = x_0$$

and $h_{x_k x_0}$ is the identity outside W_{x_0}. Evidently, the homeomorphism $h_{x_0} = h_{x_k x_0} \circ h_{x_k}$ satisfies the conditions $x_0 \in h_{x_0}(\Phi)$ and $h_{x_0}(x_U) = x_U$. Hence, $x_0 \in \Gamma \setminus \{x_U\}$, which completes the proof of Claim 2.

Since $X \backslash \{x_U\}$ is a connected set [8], $\Gamma \setminus \{x_U\} = X \backslash \{x_U\}$ by Claims 1 and 2. So, every $x \in V$ belongs to Γ. This means that there is a homeomorphism h on X such that $h(\Phi)$ meets both sets U and V. □

Remark 2.5. Another proof of Lemma 2.4 follows from the fact that every strongly locally homogeneous space without separating points is 2-homogeneous [2]. This means that if A, B are 2-point sets in X, then there is a homeomorphism h on X with $h(A) = B$. To apply this fact, choose A, B any 2-point sets such that $A \subset \Phi$ and B meets both U and V.

Proof of Theorem 1.1. Assume $\dim_G X = n \geq 2$ (the case $\dim_G X = 1$ is trivial). Then $\dim X \geq 2$. Suppose X is not a $V^n(G)$-space. Hence, there are closed disjoint sets $P, Q \subset X$, both having nonempty interiors, such that for every open cover ω of X there exists a partition C_ω in X between P and Q admitting an ω-map into a space Y_ω with $\dim_G Y_\omega \leq n-2$.

According to Lemma 2.2, X contains a compact $V^n(G)$-space Φ of dimension $\dim_G X = n \geq 2$. In particular, Φ is nondegenerate. By Lemma 2.4, there is a homeomorphism $h : X \to X$ such that $h(\Phi)$ meets both $\operatorname{int} P$ and $\operatorname{int} Q$.

Since $h(\Phi)$ is a $V^n(G)$-space, there is an open cover ω_0 of $h(\Phi)$ such that no partition in $h(\Phi)$ between the sets $h(\Phi) \cap P$ and $h(\Phi) \cap Q$ admits an ω_0-map into a space of dimension $\dim_G \leq n-2$. Extend ω_0 to an open cover ω_0^* of X and observe that, for the partition $C_{\omega_0^*}$, the set $C_{\omega_0^*} \cap h(\Phi)$ is a partition in $h(\Phi)$ between $h(\Phi) \cap P$ and $h(\Phi) \cap Q$ which admits an ω_0-map into the space $Y_{\omega_0^*}$ with $\dim_G Y_{\omega_0^*} \leq n-2$, a contradiction. □

Proof of Theorem 1.3. Suppose U and V are nonempty, disjoint, open subsets of X, and let $M \subset X$ be a set of dimension $\dim_G M \leq n-2$. Considering smaller subsets if necessary, we can assume that the closures $\overline{U}$ and $\overline{V}$ are also disjoint. By Lemma 2.2, there is a compact pair (Φ, F) in X with $\dim_G \Phi = n$ such that F is nowhere dense in Φ and (Φ, F) is a strong K_G^n-manifold. By Lemma 2.4, there is a homeomorphism h on X such that $h(\Phi) \cap U \neq \varnothing \neq h(\Phi) \cap V$. Then $h(\Phi)$ is a continuum joining U and V. So, if $M \cap h(\Phi) = \varnothing$, we are done. If $M \cap h(\Phi) \neq \varnothing$, we can apply [12, Theorem 3.1] (see Theorem 3.2 in the Appendix) to find a continuum $K \subset h(\Phi) \backslash M$ joining $U \cap h(\Phi)$ and $V \cap h(\Phi)$. □

3. Appendix

In this section we relax the assumption that spaces are locally compact separable metric.

In our paper [7] we considered the dimension $D_{\mathcal{K}}$ which unifies the covering and the cohomological dimensions.

Definition 3.1. *[4]* A sequence $\mathcal{K} = \{K_0, K_1, ..\}$ of CW-complexes is called a *stratum* for a dimension theory if for each (paracompact) space X admitting a perfect map onto a metrizable space, $K_n \in AE(X)$ implies both $K_{n+1} \in AE(X \times [0,1])$ and $K_{n+j} \in AE(X)$ for all $j \geq 0$ (here, $K_n \in AE(X)$ means that K_n is an absolute extensor for X).

Given a stratum $\mathcal{K}$, the dimension function $D_{\mathcal{K}}$ for a space X as above is defined in the following way:

1. $D_{\mathcal{K}}(X) = -1$ iff $X = \emptyset$;
2. $D_{\mathcal{K}}(X) \leq n$ if $K_n \in AE(X)$ for $n \geq 0$; if $D_{\mathcal{K}}(X) \leq n$ and $K_m \notin AE(X)$ for all $m < n$, then $D_{\mathcal{K}}(X) = n$;
3. $D_{\mathcal{K}}(X) = \infty$ if $D_{\mathcal{K}}(X) \leq n$ is not satisfied for any n.

We proved in [7, Theorem 2.6] that any compact space X (not necessarily metrizable) of dimension $D_{\mathcal{K}}X = n$ contains a continuum V^n with respect to $D_{\mathcal{K}}$. The definition of continuum V^n with respect to $D_{\mathcal{K}}$ is the same as that one of a $V^n(G)$-space; the only difference is that we consider the dimension $D_{\mathcal{K}}$ instead of $\dim_G$. So, in order to prove Theorem 1.1 in a more general version for the dimension $D_{\mathcal{K}}$, we can use Theorem 2.6 from [7] instead of Proposition 2.1.

There is one claim from the proof of [7, Theorem 2.6], see Claim 2.7, which was left without proof in that paper. Below, we provide a detailed proof of the claim for completeness.

Lemma 3.1. [7, Claim 2.7] *Let $F : Z \to K$ be a map, where Z is a nonempty compact space and K is a nonempty metrizable ANR. Then for every open cover γ of K there exists an open cover ν of Z satisfying the following condition: for any closed set $B \subset Z$ and any ν-map $\varphi : B \to Y$, where Y is a paracompact space, there is a map $g : \varphi(B) \to K$ such that $g \circ \varphi$ is γ-close to the restriction $F|B$.*

Proof. We embed K as a closed subset of a normed space L and find a retraction $r : W \to K$, where W is open in L. For every $x \in K$, choose $U_x \in \gamma$ containing x, and a convex open subset $\widetilde{U}_x$ of L such that $x \in \widetilde{U}_x$ and $r(\widetilde{U}_x) \subset U_x$. Next, take an open cover γ' of K which is a star-refinement of the cover $\widetilde{\gamma} = \{\widetilde{U}_x \cap K : x \in K\}$.

Let us show that the cover $\nu = F^{-1}(\gamma')$ is as required. Suppose $B \subset Z$ is closed and $\varphi : B \to Y$ is a ν-map. We can assume that $\varphi(B) = Y$, so there is a finite open cover β of Y such that $\varphi^{-1}(\beta)$ refines ν. For every $V \in \beta$ fix $\Lambda_V \in \nu$ and a point $a_V \in F(\Lambda_V)$ with $\varphi^{-1}(V) \subset \Lambda_V$. Let N_β be the nerve of β and $\kappa_\beta : Y \to N_\beta$ the canonical map assigning to each $y \in Y$ the point $\sum_{V\in\beta} f_V(y)V$, where $y \in \cap_{V\in\beta} V$ and $\{f_V : V \in \beta\}$ is a partition of unity subordinated to β (here the elements $V \in \beta$ are considered also as vertices of N_β).

We define a map $h : N_\beta \to K$ by

$$h(\sum_{V\in\beta} t_V V) = r(\sum_{V\in\beta} t_V a_V), \quad \text{where} \quad \sum_{V\in\beta} t_V = 1, t_V \geq 0,$$

and let $g = h \circ \kappa_\beta$. To check that $g \circ \varphi$ is γ-close to $F|B$, we fix $z \in B$. Then

$$\kappa_\beta(\varphi(z)) = \sum_{V\in\beta} f_V(\varphi(z))V \quad \text{and} \quad h(\kappa_\beta(\varphi(z))) = r(\sum_{V\in\beta} f_V(\varphi(z))a_V),$$

where $\varphi(z) \in \cap_{V\in\beta} V$.

Since γ' is a star-refinement of $\widetilde{\gamma}$, there is $U_x \in \gamma$ such that $\widetilde{U}_x \cap K$ contains all points a_V. So, $\sum_{V\in\beta} f_V(\varphi(z))a_V \in \widetilde{U}_x$ because $\widetilde{U}_x$ is convex. Consequently, $h(\kappa_\beta(\varphi(z))) \in U_x$. On the other hand, $F(z) \in \cap_{V\in\beta} F(\Lambda_V) \subset U_x$. Therefore, $F|B$ and $g\circ\varphi$ are γ-close. □

The proof of Theorem 1.3 is based on Theorem 3.1 from [12]. Here is the formulation of that theorem which was actually established in [12]:

Theorem 3.2. *[12, Theorem 3.1] Let (X, F) be a strong K^n_G-manifold and M be a Lindelöf, normally placed subset of X with $\dim_G M \le n-2$. Then for every disjoint, nonempty, open subsets U and V of X there exists a continuum $K \subset X\backslash M$ such that $U\cap K \neq \varnothing \neq V\cap K$.*

Recall that a set $M \subset X$ is normally placed if every two closed disjoint sets in M have disjoint open in X neighborhoods. For example, every F_σ-subset of a normal space is normally placed. The same is true if X is a metric space and $M \subset X$ is arbitrary.

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