A NOTE ON SPACES WITH LATTICES OF d-OPEN MAPPINGS

V. M. Valov

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Ščepin [9] introduced the class of k-metrizable spaces and proved [10] that compact X is k-metrizable iff X possesses a lattice of open mappings. In the present paper we consider an extension of the class of all k-metrizable compact spaces — the class of all completely regular spaces with a lattice of d-open [14] mappings. Some properties of this extended class are given.

1. Notations. We shall consider only completely regular spaces and continuous mappings.

Let τ be a cardinal number. Following Ščepin [9] we shall say that a family $\mathcal{L}$ of mappings, defined on a space X, is a τ-lattice of X if the following conditions are fulfilled:

1. If $\{h_s: s\in S\}\subset\mathcal{L}$ and for every finite set $A\subset S$ the diagonal mapping $\Delta_{s\in A} h_s$ belongs to $\mathcal{L}$, then $\Delta_{s\in S} h_s\in\mathcal{L}$.

2. For every mapping f, defined on X, there exist mappings $h\in\mathcal{L}$ and $g: h(X)\to f(X)$ such that $f=g\circ h$ and $w(h(X))\leq\tau\,.\,w(f(X))$.

An $\aleph_0$-latice is called simply a lattice.

Let f be a mapping from X to Y and let $H\subset X$. Following Arkhangel'skii [1] we shall say that a family $\mathcal{A}$ of open subsets of X is a pseudo-π-base of H with respect to f if $H\subset\overline{\cup\{U\in\mathcal{A}: f(U)\subset V\}}$ for every open subset V of Y containing $f(H)$ The pseudo-π-character of H with respect to f will be denoted by $p\pi\chi(f, H)$. In the case when f is the identical mapping on X we write $p\pi\chi(H, X)$ instead of $p\pi\chi(id, H)$. By $\chi(H, X)$, $\psi\chi(H, X)$ and $\pi\chi(H, X)$ we denote the character, the pseudocharacter and the π-character of H in X respectively. Obviously, $\pi\chi(H, X)\leq p\pi\chi(H, X)\leq\chi(H, X)$ and $p\pi\chi(x, X)=\pi\chi(x, X)$ for every $x\in X$.

A subset of a space X will be called a G_τ-set in X if it is an intersection of τ open subsets of X and a union of G_τ-subsets of X will be called a G_Σ^τ-set. For a space X the τ-modification of X is the topological space with the same underlying set and topology consisting of the G_Σ^τ-sets. The closure of $A\subset X$ with respect to the τ-modification of X will be denoted by $G_\tau(A)$.

A mapping f from X to Y is said to be d-open [14] if $f(U)$ is dense in some open subset of Y for every open set U in X.

A space X is said to be of pointwise τ-type if for every $x\in X$ there is a compact set $H(x)\subset X$ such that $x\in H(x)$ and $\chi(H(x), X)\leq\tau$

An embedding j of X in a space Y is called a regular embedding [13] if to every open subset U of $j(X)$ we can assign an open subset $e(U)$ of Y such that: (i) $e(U)\cap j(X)=U$; (ii) $e(U_1)\cap e(U_2)\neq\emptyset$ implies $U_1\cap U_2\neq\emptyset$ for each two open subsets U_1 and U_2 of $j(X)$.

If $\mathscr{A}$ is a family of subsets of X by $\bigcup\mathscr{A}$ we denote $\bigcup\{F: F\in\mathscr{A}\}$.

The cellularity and the Lindeloff number of X are denoted by $c(X)$ and $l(X)$ respectively.

2. Results. First, we give some examples of spaces with a τ-lattice of d-open (resp. open) mappings.

A) Let $Y=\Pi\{Y_\alpha: \alpha\in A\}$, where $w(Y_\alpha)\leq\tau$ for every $\alpha\in A$. Then any regularly embedded subset X of Y possesses a τ-lattice of d-open mappings. If in addition there exists an upper semi-continuous mapping r from Y to X such that $r(x)=x$ for each $x\in X$ the space X possesses a τ-lattice of open mappings.

B) Every k-metrizable space with a τ-lattice of quotient (resp. quotient and locally closed) mappings possesses a τ-lattice of d-open (resp. open) mappings. Here, a mapping f from X to Y is locally closed if for each $x\in X$ and for each neighbourhood U of x in X there is a closed neighbourhood V of x such that $f(V)$ is closed in Y.

The statement A can be proved by using Širokov's technique from [13] and the statement B follows from Ščepin's arguments ([9], the proof of Theorem 17).

The following theorem is an analogue of the lemma for strong sequences' proved by Efimov [4] for dyadic spaces and generalized by Ščepin [11] to k-adic spaces. R. Pol a. E. Pol [6] also proved an analogue of this lemma for products of spaces. Their technique adapted to the case of spaces with a lattice of open mappings, is used in the proof of Theorem 1 and Theorem 5 below.

Theorem 1. Let X possess a τ-lattice $\mathscr{L}$ of d-open (resp. open) mappings. Suppose $\mathscr{A}$ is a family of subsets of X with $p\pi\chi(F, X)<\lambda$ (resp. $\psi\chi(F, X)<\lambda$) for every $F\in\mathscr{A}$, where λ is a regular cardinal and $\lambda>\tau$. Then there exists a mapping $h\in\mathscr{L}$ and a family $\mathscr{B}\subset\mathscr{A}$ of cardinality $|\mathscr{B}|<\lambda$ such that $w(h(X))<\lambda$, $\bigcup\mathscr{B}$ is dense in $\bigcup\mathscr{A}$ and $h^{-1}(\overline{h(\bigcup\mathscr{A})})=\overline{\bigcup\mathscr{A}}$.

Remark. The conclusion of Theorem 1 remains true if the following conditions are fulfilled: (i) X possesses a τ-lattice of d-open mappings; (ii) $p\pi\chi(F, X)<\lambda$ for every $F\in A$; (iii) for each $x\in X$ there is a compact set $H(x)$ containing x such that $\chi(H(x), X)<\lambda$.

Corollary 1. Let X possess a τ-lattice $\mathscr{L}$ of d-open mappings. Then $c(X)\leq\tau$ and $\psi\chi(x, X)\leq\tau\cdot\pi\chi(x, X)$ for every $x\in X$. If in addition, $\mathscr{L}$ consists of open mappings and F is a G^τ_Σ-subset of X then $c(F)\leq\tau$.

Corollary 2. Every space with a lattice of d-open mappings is perfectly k-normal in the sense of Ščepin [12].

Theorem 2. Suppose X possesses a τ-lattice $\mathscr{L}$ of d-open mappings and let f be a mapping from M to Y, where M is dense in X. If λ is a regular cardinal larger than τ there exist a mapping $h\in\mathscr{L}$, a closed subset P of M, a closed subset Q of $h(M)$ and a mapping g from Q to Y such that the following conditions are fulfilled:

1) $w(h(X))<\lambda$ and $h(P)=Q$;

2) $g(h(x))=f(x)$ for each $x\in P$;

3) P contains the set $\{x\in M: p\pi\chi(f, x)<\lambda\}$.

Theorem 2 is an analogue of Theorem 1 from the paper [1] of Arkhangel'skii: Our proof of Theorem 2 is similar to the one of Arkhangel'skii.

Using the arguments of Tkačenko [14] (the proof of Theorem 2) the following fact can be proved. Let every canonically closed subset of X be a G_τ-set and M be a dense in X. Then each mapping from M onto a space of weight $\leq\tau$ is continuously extendable over $G_\tau(M)$. In the case when X possesses a τ-lattice of d-open mappings something more is true.

Corollary 3. Let X possess a τ-lattice of d-open mappings and let M be dense in X. Then every mapping from M to Y, where $\pi\chi(y, Y)\leq\tau$ for each $y\in Y$, is continuously extendable over $G_\tau(M)$.

Corollary 4. For every space X with a τ-lattice of d-open mappings we have $w(\overline{\{x\in X: \pi\chi(x, X)<\lambda\}})<\lambda$, where λ is a regular cardinal with $\tau<\lambda$.

Proposition 1. Let a space X of pointwise τ-type satisfy the following condition: any family $\mathscr{A}$ of G_τ-subsets of X contains a subfamily $\mathscr{B}$ of cardinality $|\mathscr{B}|\leq\tau$ such that $\bigcup\mathscr{B}$ is dense in $\bigcup\mathscr{A}$. Then every regular cardinal $\lambda\geq\tau^+$ is a caliber for X.

Corollary 5. Let a space X of pointwise τ-type possess a τ-lattice of d-open mappings and let F be a G^μ_Σ-subset of X. Then every regular cardinal $\lambda\geq(\tau\,.\,\mu)^+$ is a caliber for F.

Two disjoint subsets U and V of X are called τ-separated if there exists a mapping f from X to a space of weight $\leq\tau$ such that $f(U)\cap f(V)=\emptyset$. Such a mapping f will be called τ-separating.

Proposition 2. Let a space X of pointwise τ-type satisfy the following condition: for every G^τ_Σ-subset, F of X there is a mapping f from X to a space of weight $\leq\tau$ such that $f^{-1}(f(\bar{F}))=\bar{F}$. Suppose U and V are disjoint G_τ and G^τ_Σ subsets of X respectively. Then U and V are τ-separated.

Theorem 3. Let U and V be disjoint G_τ and G^τ_Σ subsets of X respectively.

(i) Suppose X is of pointwise τ-type and X possesses a τ-lattice of d-open mappings. Then there exists a d-open mapping f such that $w(f(X))\leq\tau$, $f(U)\cap f(V)=\emptyset$ and $f(U)$ is a G_τ-subset of $f(X)$.

(ii) Let X be compact with $c(X)\leq\lambda$ and $\chi(C, X)\leq\tau$ for each canonically closed subset C of X. Then U and V can be separated with a τ^λ-separating mapping f such that $f(U)$ is a G_τ-set in $f(X)$.

Theorem 3 follows from Proposition 2 and from results of Bandlov [2].

Corollary 6. Let X possess a τ-lattice of d-open mappings and let X can be perfectly mapped onto a space of weight $\leq\tau$. Then for every G_τ-subset P of X there exists a perfect open mapping f from X to Y such that $w(Y)\leq\tau$ and $f^{-1}(f(G_\tau(P)))=G_\tau(P)$.

Corollary 7. Let U be a G_τ-subset of a compact space X with $c(X)\leq\lambda$.

(i) Then $l(G_\tau(U))\leq\exp(\tau^\lambda)$. If in addition $G_\tau(U)=U$ or U is open we have $l(G_\tau(U))\leq\tau$.

(ii) If $\chi(C, X)\leq\tau$ for every cononically closed set C in X then any mapping from U to a space of π-character $\leq\tau^\lambda$ is continuously extendable over $G_{\tau^\lambda}(U)$.

Theorem 4. Let a space X of pointwise τ-type possess a τ-lattice of d-open mappings and let U and V be disjoint G^λ_Σ and G^μ_Σ subsets of X respectively. Then U and V are $(\tau\,.\,\mu)^\lambda$-separated.

Theorem 4 is based on the following theorem.

Theorem 5. Let a space X of pointwise τ-type possess a τ-lattice of d-open mappings and let $\mathscr{A}$ be a family of G_τ-subsets of X. Then for every cardinal λ there exists a d-open mapping h and a family $\mathscr{B}\subset\mathscr{A}$ of cardinality $|\mathscr{B}|\leq\tau^\lambda$ such that $w(h(X))\leq\tau^\lambda$, $\bigcup\mathscr{B}$ is dense in $\bigcup\mathscr{A}$ with respect to the λ-modification of X and $h^{-1}(G_\lambda(h(\bigcup\mathscr{A})))=C_\lambda(\bigcup\mathscr{A})$.

Corollary 8. Let X be the same as in Theorem 5. Then for every cardinal λ the λ-modification of X has a cellularity $\leq\tau^\lambda$.

Corollary 9. Let a space X of pointwise τ-type possess a 2^τ-lattice of d-open mappings and let U be a G^τ_Σ-subset of X. Then there exists a d-open mapping h such that $w(h(X))\leq 2^\tau$ and $h^{-1}(G_{2^\tau}(U)))=G_{2^\tau}(U)$. If in addition, X can be perfectly mapped onto a space of weight $\leq 2^\tau$ then $l(G_{2^\tau}(U))\leq 2^\tau$ and every mapping from U to a space of π-character $\leq 2^\tau$ is continuously extendable over $G_{2^\tau}(U)$.

A space X satisfies the condition (*) if for every $x\in X$ there is a G_δ-subset $H(x)$ of X containing x such that $G_{\aleph_0}(P)$ coinsides with the sequential closure of P for each G_δ-subset P of $H(x)$. A space X is weakly normal [8] if every countable discrete in X set A contains a C^*-embedded in X subset M with $|M|=|A|$. Pasynkov [8] showed that every realcompact space is weakly normal.

Let us note that every upper semi-continuous retract of a product of metric spaces satisfies the condition (*).

Theorem 6. Let X possess a lattice of perfect open mappings and let X satisfy the condition (*). Suppose Y is a perfect image of X and U is a G_δ-subset of Y. Then the following conditions are equivalent:

(i) U is weakly normal;

(ii) U is realcompact;

(iii) U is normal;

(iv) U is Lindeloff.

Corollary 10. Every weakly normal G_δ-subset of a locally compact topological group is paracompact.

Question 1. Let X be weakly normal G_δ-subset of an almost metrizable topological group in the sense of Pasynkov [7]. Is X paracompact?

If X is open in an almost metrizable topological group the positive answer of this question follows from the results in Čoban [3] and Klebanov [5].

Question 1 is connected with the following question.

Question 2. Suppose M is a metric space and U is a (weakly) normal G_δ-subset of $M \times D^\tau$, where $\tau > \aleph_0$ and D is the discrete two-points set. Is U paracompact?

Question 3. Let X be a k-metrizable compact space and let U be a G_δ-subset of X. Does Theorem 6 hold for U?

Institute of Mathematics
Bulgarian Academy of Sciences
1090 Sofia, Bulgaria

REFERENCES

[1] Arkhangel'skii, A. Tr. Mosk. Mat. ob. **47**, 1984, 3. [2] Bandlov, I. Dokl. Acad. Nauk SSSR **253**, 1980, 1036. [3] Čoban, M. Ibid. **228**, 1976, 52. [4] Efimov, B. Tr. Mosk. Mat. ob. **14**, 1965, 211. [5] Klebanov, B. Abstracts of Leningrad Intern. Top. Conf. Leningrad, 1982, 81. [6] Pol, R., E. Pol. Fund. Math. **93** 1976, 57. [7] Pasynkov, B. Dokl. Acad. Nauk SSSR **161**, 1965, 281. [8] Id. Vestnik Mosk. Univ. Ser. Mat. i. Mech. **6**, 1979, 54. [9] Ščepin, E. Usp. Mat. Nauk **31**, 1976, 5, 191. [10] Id. Ibid. **36**, 1981, 3, 3. [11] Id. Izv. Acad. Nauk SSSR **43**, 1979, 3, 442. [12] Id. Dokl. Acad. Nauk SSSR **226**, 1976, 527. [13] Šhirokov, L. Ibid. **263**, 1982, 1073. [14] Tkačenko, M. Top. and Appl. **15**, 1983, 93.