J. Math. Anal. Appl. 466 (2018) 952–960

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Linear extension operators of bounded norms

Dmitri Shakhmatov [a], Vesko Valov [b], Takamitsu Yamauchi [a,*]

[a] *Division of Mathematics, Physics and Earth Sciences, Graduate School of Science and Engineering, Ehime University, Matsuyama, 790-8577, Japan*
[b] *Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, P.O. Box 5002, North Bay, ON, P1B 8L7, Canada*

A R T I C L E   I N F O

Article history:
Received 27 December 2017
Available online 19 June 2018
Submitted by D. Repovš

Keywords:
Dugundji space
Function space
Extension operator

A B S T R A C T

Dugundji spaces were introduced by Pełczyński as compact Hausdorff spaces X such that every embedding of X into a Tychonoff cube $[0,1]^A$ admits a linear extension operator $u : C(X) \to C([0,1]^A)$ such that $\|u\| = 1$ and $u(1_X) = 1_{[0,1]^A}$, where 1_X is the constant function on X taking value 1. In this paper we show that a compact space X is Dugundji provided that there exists a linear extension operator $u : C(X) \to C([0,1]^A)$ satisfying one of the following conditions:

(a) $\|u\| < 2$ and $|u(f \cdot g)| \le \|g\| \cdot |u(|f|)|$ for all $f, g \in C(X)$;
(b) $\|u\| = 1$.

1. Introduction

All spaces in this paper are assumed to be compact and Hausdorff. By $C(X)$ we denote the Banach space of all continuous real-valued functions on a space X endowed with the sup-norm. For a real number c, the constant function on X taking value c is denoted by c_X. For spaces X and Y and an embedding $e : X \to Y$, we say that an operator $u : C(X) \to C(Y)$ is:

- an *extension operator* if $u(f) \circ e = f$ for every $f \in C(X)$,
- a *regular operator* if u is linear and satisfies $\|u\| = 1$ and $u(1_X) = 1_Y$.

A space X is called a *Dugundji space* [8] if for any embedding of X into another space Y there exists a regular extension operator $u : C(X) \to C(Y)$. This is equivalent to the existence of an embedding of X into a Tychonoff cube $\mathbb{I}^A$ and a regular extension operator $u : C(X) \to C(\mathbb{I}^A)$, where $\mathbb{I}$ is the interval $[0,1]$ and A is a set [8, Proposition 6.2].

* Corresponding author.
E-mail addresses: dmitri.shakhmatov@ehime-u.ac.jp (D. Shakhmatov), veskov@nipissingu.ca (V. Valov), yamauchi.takamitsu.ts@ehime-u.ac.jp (T. Yamauchi).

https://doi.org/10.1016/j.jmaa.2018.06.030

In this paper, we are concerned with the following question motivated by Pełczyński's problem [8, Problem 4, p. 65] and the results of Amir [1] and Cambern [3] extending the classical Banach–Stone theorem to the existence of isomorphisms of norm smaller than 2.

Question 1.1. *Is it true that a space X is a Dugundji space if there exist an embedding of X into a Tychonoff cube $\mathbb{I}^A$ and a linear extension operator $u : C(X) \to C(\mathbb{I}^A)$ satisfying $\|u\| < 2$?*

For $f, g \in C(X)$, we write $f \le g$ if $f(x) \le g(x)$ for every $x \in X$. An operator $u : C(X) \to C(Y)$ is said to be *positive* if $f \in C(X)$ and $f \ge 0_X$ implies $u(f) \ge 0_Y$. According to [8, Proposition 1.2], a bounded linear operator u is regular if and only if $u(1_X) = 1_Y$ and u is positive.

The next proposition lists three notions weaker than regularity and positivity defined above.

Proposition 1.2. *Let $u : C(X) \to C(Y)$ be a linear extension operator for some embedding of a space X into a space Y. Consider the following conditions:*

(1) $\|u\| = 1$;
(2) *if $x \in Y$, $u(1_X)(x) = 1$, $f \in C(X)$ and $f \ge 0_X$, then $u(f)(x) \ge 0$;*
(3) *if $x \in Y$, $u(1_X)(x) = 1$ and $f, g \in C(X)$, then $|u(f \cdot g)(x)| \le \|g\| \cdot |u(|f|)(x)|$.*

Then (1) ⇒ (2) ⇒ (3).

Proof. (1) ⇒ (2). This implication follows from the argument in the proof of [8, Proposition 1.2]. We give a proof for the sake of completeness. Assume that $\|u\| = 1$, and fix $x \in Y$ and $f \in C(X)$ such that $u(1_X)(x) = 1$ and $f \ge 0_X$. Since $f \ge 0_X$, we have $\|\|f\|1_X - f\| \le \|f\|$, and hence

$$\begin{aligned} \|f\| - u(f)(x) &= \|f\|u(1_X)(x) - u(f)(x) = (\|f\|u(1_X) - u(f))(x) \\ &= u(\|f\|1_X - f)(x) \le \|u\| \cdot \|(\|f\|1_X - f)\| \le \|f\|, \end{aligned}$$

which implies $u(f)(x) \ge 0$.

(2) ⇒ (3). Assume (2). Let $x \in Y$, $u(1_X)(x) = 1$ and $f, g \in C(X)$. Since $-\|g\| \cdot |f| \le f \cdot g$, we have $(f \cdot g + \|g\| \cdot |f|)(x) \ge 0$. This, the linearity of u and (2) imply

$$u(f \cdot g)(x) + \|g\|u(|f|)(x) = u(f \cdot g + \|g\| \cdot |f|)(x) \ge 0,$$

and hence $-\|g\|u(|f|)(x) \le u(f \cdot g)(x)$. Similarly, we have $u(f \cdot g)(x) \le \|g\|u(|f|)(x)$. Therefore, $|u(f \cdot g)(x)| \le \|g\| \cdot |u(|f|)(x)|$. □

Example 1.3. *For every $\varepsilon > 0$, there exists a linear extension operator $u : C(\{0,1\}) \to C(\mathbb{I})$ satisfying $\|u\| < 1 + 2\varepsilon$ and failing condition (3) of Proposition 1.2.* Indeed, define $u : C(\{0,1\}) \to C(\mathbb{I})$ by

$$u(f)(t) = \begin{cases} f(0) + 2\varepsilon(f(0) - f(1))t & \text{if } t \in [0, \frac{1}{2}], \\ f(1) + (2 + 2\varepsilon)(f(0) - f(1))(1 - t) & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

for $f \in C(\{0,1\})$ and $t \in \mathbb{I}$. It is easy to see that u is a linear extension operator.

Let $f \in C(\{0,1\})$. Note that $u(f)(\frac{1}{2}) = (1 + \varepsilon)f(0) - \varepsilon f(1)$, so

$$|u(f)(t)| \le \max\{|f(0)|, |f(\tfrac{1}{2})|, |f(1)|\} \le (1 + 2\varepsilon)\|f\|$$

for every $t \in \mathbb{I}$. This shows that $\|u\| \le 1 + 2\varepsilon$.

To see that u does not satisfy condition (3) of Proposition 1.2, first note that $u(1_{\{0,1\}}) = 1_{\mathbb{I}}$. Define $f, g \in C(\{0,1\})$ by $f(0) = g(1) = \frac{\varepsilon}{1+\varepsilon}$ and $f(1) = g(0) = 1$. Then

$$|u(f \cdot g)(\tfrac{1}{2})| = \tfrac{\varepsilon}{1+\varepsilon} > 0 = \|g\| \cdot |u(|f|)(\tfrac{1}{2})|.$$

Hence u does not satisfy condition (3) of Proposition 1.2.

Remark 1.4. By taking ε in Example 1.3 arbitrarily close to 0, the norm $\|u\|$ of the operator u constructed in this example can be made arbitrarily close to 1 from the right. This shows that the condition from item (1) of Proposition 1.2 cannot be weakened to $\|u\| = 1 + \delta$ for any number $\delta > 0$.

2. Main result

The main result of this paper is the following theorem which provides a partial answer to Question 1.1.

Theorem 2.1. *A space X is a Dugundji space if and only if there exist an embedding of X into a Tychonoff cube $\mathbb{I}^A$ and a linear extension operator $u : C(X) \to C(\mathbb{I}^A)$ satisfying $\|u\| < 2$ and condition (3) of Proposition 1.2.*

Our proof of Theorem 2.1, given in Section 4, follows Shchepin's arguments from [13].

Theorem 2.1 would have provided a positive solution to Question 1.1 if the inequality $\|u\| < 2$ implied condition (3) of Proposition 1.2. However, Example 1.3 shows that this is not the case; indeed, it suffices to take in this example a number ε satisfying $0 < \varepsilon < 1/2$.

Corollary 2.2. *A space X is a Dugundji space if and only if there exist an embedding of X into a Tychonoff cube $\mathbb{I}^A$ and a linear extension operator $u : C(X) \to C(\mathbb{I}^A)$ such that $\|u\| < 2$ and $|u(f \cdot g)| \le \|g\| \cdot |u(|f|)|$ for all $f, g \in C(X)$.*

Our next corollary shows that in the definition of Dugundji space the assumption of regularity of operator u can be weakened to the condition $\|u\| = 1$; that is, the requirement $u(1_X) = 1_Y$ is superfluous.

Corollary 2.3. *A space X is a Dugundji space if and only if there exist an embedding of X into a Tychonoff cube $\mathbb{I}^A$ and a linear extension operator $u : C(X) \to C(\mathbb{I}^A)$ satisfying $\|u\| = 1$.*

Proof. Follows from the implication (1) ⇒ (3) of Proposition 1.2 and Theorem 2.1. □

Remark 2.4. *For every Dugundji space X, there exist an embedding of X into a Tychonoff cube $\mathbb{I}^A$ and a linear extension operator $u : C(X) \to C(\mathbb{I}^A)$ satisfying $\|u\| = 1$ which is not positive (and hence not regular).* Indeed, take an embedding $e_B : X \to \mathbb{I}^B$ for some set B and a regular extension operator $v : C(X) \to C(\mathbb{I}^B)$. Let $\mathbb{I}^A = \mathbb{I}^B \times \mathbb{I}$ and define $u : C(X) \to C(\mathbb{I}^A)$ by $u(f)((x,t)) = (1 - 2t)v(f)(x)$ for $f \in C(X)$, $x \in \mathbb{I}^B$ and $t \in \mathbb{I}$. Then u is a linear extension operator of norm $\|u\| = 1$ with respect to the embedding $e_A : X \to \mathbb{I}^A$ defined by $e_A(x) = (e_B(x), 0)$, $x \in X$, and u is not positive since $u(1_X)((x,1)) = -1$ for every $x \in \mathbb{I}^B$.

3. κ-Metrizable spaces and extension operators

Dugundji spaces are closely related to the wider class of κ-metrizable spaces introduced by Shchepin [14] (compactness is not required in the definition of κ-metrizability). For example, a space X is κ-metrizable if and only if its superextension λX is a Dugundji space [7]. In this section, we discuss characterizations of κ-metrizable spaces similar to our results from Section 2 and raise some natural questions.

We start with the following characterization of κ-metrizable spaces due to Shirokov [15, Theorem 2]:

Theorem 3.1. *A space X is κ-metrizable if and only if for every embedding $e_A : X \to \mathbb{I}^A$ of X into a Tychonoff cube $\mathbb{I}^A$, there exists a function $k : \mathcal{T}_{e_A(X)} \to \mathcal{T}_{\mathbb{I}^A}$, where $\mathcal{T}_{e_A(X)}$ and $\mathcal{T}_{\mathbb{I}^A}$ are the topologies of $e_A(X)$ and $\mathbb{I}^A$, respectively, such that:*

(i) *$k(U) \cap e_A(X) = U$ for each $U \in \mathcal{T}_{e_A(X)}$, and*
(ii) *if $U, V \in \mathcal{T}_{e_A(X)}$ and $U \cap V = \varnothing$, then $k(U) \cap k(V) = \varnothing$.*

Recall that a space X is said to have the *Bockstein Separation Property* (abbreviated to [B.S.P.]) if every pair of disjoint open subsets of X can be separated by open F_σ sets [8, Definition 5.9].

We shall also need the following straightforward corollary of Theorem 3.1.[1]

Corollary 3.2. *Every κ-metrizable space has [B.S.P.].*

Proof. This follows from Theorem 3.1 and the fact that $\mathbb{I}^A$ has the Bockstein Separation Property [2] (see also [9], [4]). □

An operator $u : C(X) \to C(Y)$ is said to be *monotone* if $f, g \in C(X)$ and $f \leq g$ imply $u(f) \leq u(g)$. A linear operator u is monotone if and only if it is positive.

It is worth emphasizing that the operator in item (b) of the next proposition is not assumed to be either linear or bounded.

Proposition 3.3. *For a space X, the following are equivalent:*

(a) *X is κ-metrizable;*
(b) *there exist an embedding of X into a Tychonoff cube $\mathbb{I}^A$ and a monotone extension operator $u : C(X) \to C(\mathbb{I}^A)$.*

Proof. The implication (a)⇒(b) follows from [16, Theorem 1.1] and the fact that every space can be embedded into a Tychonoff cube.

The implication (b)⇒(a) follows from Shapiro's characterization of κ-metrizable spaces [11, Theorem 3] and the idea of [8, Proposition 6.2]. Indeed, assume that there is an embedding $e_A : X \to \mathbb{I}^A$ for some set A and a monotone extension operator $u_A : C(X) \to C(\mathbb{I}^A)$. To apply [11, Theorem 3], let $e_B : X \to \mathbb{I}^B$ be an arbitrary embedding. Since $\mathbb{I}^A$ is an absolute extensor, there exists a continuous map $\varphi : \mathbb{I}^B \to \mathbb{I}^A$ such that $\varphi \circ e_B = e_A$. Define $u_B : C(X) \to C(\mathbb{I}^B)$ by $u_B(f) = u_A(f) \circ \varphi + |u_A(0_X) \circ \varphi|$ for $f \in C(X)$. Then u_B is a monotone extension operator such that $u_B|_{C(X)_+} : C(X)_+ \to C(\mathbb{I}^B)_+$, where $C(X)_+ = \{f \in C(X) : f \geq 0_X\}$. Thus, X is κ-metrizable by [11, Theorem 3]. □

Here is another extension condition yielding κ-metrizability.

Proposition 3.4. *If a space X admits an embedding into a Tychonoff cube $\mathbb{I}^A$ and a linear extension operator $u : C(X) \to C(\mathbb{I}^A)$ satisfying $\|u\| < 3$, then X is κ-metrizable.*

Proof. Assume that there are a set A, an embedding $e_A : X \to \mathbb{I}^A$ and a linear extension operator $u : C(X) \to C(\mathbb{I}^A)$ of norm $\|u\| < 3$. Then, by the same argument as in the proof of [17, Theorem 3.1], there exists a function $k : \mathcal{T}_{e_A(X)} \to \mathcal{T}_{\mathbb{I}^A}$ satisfying conditions (i) and (ii) from Theorem 3.1.

[1] This corollary can also be deduced from earlier results of Shchepin. Indeed, Shchepin notes in the introduction of [12] that for normal (in particular, compact) spaces, [B.S.P.] is equivalent to his notion of perfect κ-normality. Finally, it suffices to recall that κ-metrizable spaces are perfectly κ-normal by Corollary to Theorem 7 in Introduction of [14].

Let $e_B : X \to \mathbb{I}^B$ be an arbitrary embedding. Arguing as in the proof of Proposition 3.3, fix a continuous map $\varphi : \mathbb{I}^B \to \mathbb{I}^A$ such that $\varphi \circ e_B = e_A$. Define a function $k' : \mathcal{T}_{e_B(X)} \to \mathcal{T}_{\mathbb{I}^B}$ by letting $k'(U) = \varphi^{-1}(e_A(e_B^{-1}(U)))$ for each $U \in \mathcal{T}_{e_B(X)}$. Then conditions (i) and (ii) of Theorem 3.1 hold (with k replaced by k' and A replaced by B). Applying this theorem, we conclude that X is κ-metrizable. $\square$

We do not know if the converse of Proposition 3.4 holds.

Question 3.5. *Is it true that a space X is κ-metrizable if and only if there exist an embedding of X into a Tychonoff cube $\mathbb{I}^A$ and a linear extension operator $u : C(X) \to C(\mathbb{I}^A)$ satisfying $\|u\| < 3$?*

The following example demonstrates that the inequality $\|u\| < 3$ in Proposition 3.4 is sharp.

Example 3.6. *A (compact) space X having an embedding into a Tychonoff cube $\mathbb{I}^A$ and a linear extension operator $u : C(X) \to C(\mathbb{I}^A)$ satisfying $\|u\| = 3$ need not be κ-metrizable.* Indeed, let A be an uncountable set and T_A the quotient space obtained from the Cantor cube $\{0,1\}^A$ by identifying two distinct points of $\{0,1\}^A$. In [8, Example 6, p. 67], Pełczyński constructed an embedding of T_A into the Tychonoff cube $\mathbb{I}^A$ which admits a linear extension operator $u : C(T_A) \to C(\mathbb{I}^A)$ with $\|u\| = 3$. On the other hand, Pełczyński showed in [8, Example 5, p. 66] that T_A does not have [B.S.P.], so T_A is not κ-metrizable by Corollary 3.2.

We may also ask the following question:

Question 3.7. *Is there a non-Dugundji space X admitting an embedding into a Tychonoff cube $\mathbb{I}^A$ and a linear extension operator $u : C(X) \to C(\mathbb{I}^A)$ satisfying $\|u\| < 3$?*

Remark 3.8. Let X be a space of weight $> \omega_1$, where ω_1 is the first uncountable cardinal, and $\exp X$ the hyperspace consisting of all non-empty closed subsets of X with the Vietoris topology. According to [14, Theorem 3] and [10, Theorem 5], $\exp X$ is a κ-metrizable non-Dugundji space. We do not know whether $\exp X$ admits an embedding into a Tychonoff cube $\mathbb{I}^A$ having a linear extension operator $u : C(\exp X) \to C(\mathbb{I}^A)$ satisfying $\|u\| < 3$.

4. Proof of Theorem 2.1

We fix a space X, a set A, an embedding of X into the Tychonoff cube $\mathbb{I}^A$ and a linear extension operator $u : C(X) \to C(\mathbb{I}^A)$ satisfying $\|u\| < 2$ and condition (3) of Proposition 1.2. Without loss of generality we shall assume that X is a subset of $\mathbb{I}^A$.

Let $C \subset B \subset A$. The symbol π^B_C denotes the projection from $\mathbb{I}^B$ onto $\mathbb{I}^C$. We let

$$\pi_B = \pi^A_B, \quad X_B = \pi_B(X), \quad p_B = \pi_B|_X : X \to X_B \quad \text{and} \quad p^B_C = \pi^B_C|_{X_B} : X_B \to X_C.$$

Let Z be a space and f, g two maps defined on Z. We write $f \prec g$ if for every $x, y \in Z$, $g(x) = g(y)$ whenever $f(x) = f(y)$. The following well-known fact will be used (see [5, 2.7.12 (d)], [9]).

Fact 4.1. *For every $f \in C(X)$, there exists a countable subset C of A such that $\pi_C \prec u(f)$.*

For every set $B \subset A$, define

$$X(B) = \pi_B^{-1}(X_B) \cap H,$$

where

$$H = u(1_X)^{-1}(\{1\}).$$

Since $u(1_X)$ extends the function 1_X, $X \subset X(B)$ and $\pi_B(X(B)) = X_B$.

Definition 4.2. We say that a set $B \subset A$ is *u-admissible* if it satisfies the following two conditions:

(i) $\pi_B^{-1}(\pi_B(H)) = H$, and
(ii) $\pi_B|_{X(B)} \prec u(f)|_{X(B)}$ for every $f \in C(X)$ such that $p_B \prec f$.

The condition $p_B \prec f$ in item (ii) is equivalent to the existence of $g \in C(X_B)$ such that $f = g \circ p_B$.

Lemma 4.3. *There exists a countable set $B_H \subset A$ such that every set $B \subset A$ containing B_H is u-admissible provided $\pi_B|_{X(B)} \prec u(f \circ p_B)|_{X(B)}$ for any $f \in C(X_B)$.*

Proof. By Fact 4.1, there is a countable set $B_H \subset A$ such that $\pi_{B_H} \prec u(1_X)$. Then B_H is as required since $\pi_B^{-1}(\pi_B(H)) = H$ for every $B \subset A$ with $B_H \subset B$. □

Lemma 4.4. *For any countable set $B \subset A$, there exists a countable u-admissible set B^* containing B.*

Proof. By Fact 4.1, for every $f \in C(X)$ we can fix a countable set $\Gamma(f) \subset A$ with $\pi_{\Gamma(f)} \prec u(f)$. Let B_H be a countable subset of A as in Lemma 4.3. For every countable $C \subset A$, $C(X_C)$ is separable since X_C is a metric compactum. Thus, by induction we can define an increasing sequence $\{B(k)\}_{k=0}^{\infty}$ of countable subsets of A and a family $\mathcal{A}_k \subset C(X_{B(k)})$ satisfying the following three conditions for every k:

(4) $B(0) = B \cup B_H$;
(5) $\mathcal{A}_k$ is a countable dense subalgebra of $C(X_{B(k)})$ containing the constant function $1_{X_{B(k)}}$ and $\{g \circ p^{B(k)}_{B(i)} : g \in \mathcal{A}_i, i < k\}$;
(6) $B(k+1) = \bigcup\{\Gamma(g \circ p_{B(k)}) : g \in \mathcal{A}_k\} \cup B(k)$.

Since $\mathcal{A}_k$ is a dense subset of $C(X_{B(k)})$, it separates the points and the closed sets of $X_{B(k)}$.

Let $B^* = \bigcup\{B(k) : k < \omega\}$ and $\mathcal{A}_{B^*}$ the subalgebra generated by $\{g \circ p^{B^*}_{B(k)} : g \in \mathcal{A}_k, k < \omega\}$, where ω is the first infinite cardinal. Since $B_H \subset B(0) \subset B^*$ by (4), it suffices to show that $\pi_{B^*}|_{X(B^*)} \prec u(f \circ p_{B^*})|_{X(B^*)}$ for any $f \in C(X_{B^*})$, see Lemma 4.3.

Claim 1. $\pi_{B^*}|_{X(B^*)} \prec u(g \circ p_{B^*})|_{X(B^*)}$ for any $g \in \mathcal{A}_{B^*}$.

Proof of Claim 1. Every $g \in \mathcal{A}_{B^*}$ is a finite sum of products $h = (g_1 \circ p^{B^*}_{B(k_1)}) \cdots (g_m \circ p^{B^*}_{B(k_m)})$ with $g_i \in \mathcal{A}_{k_i}$ for some $k_i < \omega$ and $i \le m$. Without loss of generality, we may assume that $k_i \le k_m$ for each $i \le m$. Let $h' = (g_1 \circ p^{B(k_m)}_{B(k_1)}) \cdots (g_m \circ p^{B(k_m)}_{B(k_m)})$. Then $h' \in \mathcal{A}_{k_m}$ by (5), and $h \circ p_{B^*} = h' \circ p_{B(k_m)}$.

Let $x, y \in X(B^*)$ with $\pi_{B^*}(x) = \pi_{B^*}(y)$. Since $\Gamma(h' \circ p_{B(k_m)}) \subset B(k_m + 1) \subset B^*$ by (6), we have $\pi_{\Gamma(h' \circ p_{B(k_m)})}(x) = \pi_{\Gamma(h' \circ p_{B(k_m)})}(y)$. This, $h \circ p_{B^*} = h' \circ p_{B(k_m)}$ and the choice of $\Gamma(h' \circ p_{B(k_m)})$ yield that

$$u(h \circ p_{B^*})(x) = u(h' \circ p_{B(k_m)})(x) = u(h' \circ p_{B(k_m)})(y) = u(h \circ p_{B^*})(y),$$

which shows that $\pi_{B^*}|_{X(B^*)} \prec u(h \circ p_{B^*})|_{X(B^*)}$. Finally, the linearity of u provides $\pi_{B^*}|_{X(B^*)} \prec u(g \circ p_{B^*})|_{X(B^*)}$. □

Since $1_{B(0)} \in \mathcal{A}_0$ by (5), we have $1_{X_{B^*}} \in \mathcal{A}_{B^*}$. Since each $\mathcal{A}_k$ separates the points and the closed sets of $X_{B(k)}$, so does $\mathcal{A}_{B^*}$ for X_{B^*}. Thus, by the Stone–Weierstrass Theorem, $\mathcal{A}_{B^*}$ is dense in $C(X_{B^*})$. This, Claim 1 and the continuity of u yield that $\pi_B|_{X(B)} \prec u(f \circ p_B)|_{X(B)}$ for every $f \in C(X_B)$. □

The next two lemmas are based on Shchepin's arguments from the proof of [13, Lemma 2 in Chapter 2].

Lemma 4.5. *Let B be a u-admissible subset of A. Then*

$$u((f \circ p_B) \cdot g)(x) = u(f \circ p_B)(x) \cdot u(g)(x)$$

for all $x \in X(B)$, $f \in C(X_B)$ and $g \in C(X)$.

Proof. We fix $x \in X(B)$, $f \in C(X_B)$ and $g \in C(X)$. Let $c = u(f \circ p_B)(x)$.

Claim 2. $u(|f \circ p_B - c1_X|)(x) = 0$.

Proof of Claim 2. Since $x \in X(B)$, we can take $y \in X$ such that $\pi_B(x) = \pi_B(y)$. Since $p_B \prec (f \circ p_B - c1_X)$ and $p_B \prec |f \circ p_B - c1_X|$, by the u-admissibility of B, we have $\pi_B|_{X(B)} \prec u(f \circ p_B - c1_X)|_{X(B)}$ and $\pi_B|_{X(B)} \prec u(|f \circ p_B - c1_X|)|_{X(B)}$. This, $x \in X(B) \subset H$, $y \in X$ and the fact that u is a linear extension operator imply that

$$\begin{aligned} u(|f \circ p_B - c1_X|)(x) &= u(|f \circ p_B - c1_X|)(y) = |f \circ p_B - c1_X|(y) \\ &= |(f \circ p_B - c1_X)(y)| = |u(f \circ p_B - c1_X)(y)| = |u(f \circ p_B - c1_X)(x)| \\ &= |u(f \circ p_B)(x) - cu(1_X)(x)| = |u(f \circ p_B)(x) - c| = 0. \quad \square \end{aligned}$$

Since u satisfies the condition (3) in Theorem 2.1 and $x \in H$, we have

$$|u((f \circ p_B - c1_X) \cdot g)(x)| \le \|g\| \cdot |u(|f \circ p_B - c1_X|)(x)| = 0.$$

From this, $u(f \circ p_B)(x) = c$ and linearity of u, we get

$$u((f \circ p_B) \cdot g)(x) - u(f \circ p_B)(x) \cdot u(g)(x) = u((f \circ p_B - c1_X) \cdot g)(x) = 0. \quad \square$$

Lemma 4.6. *Any union of u-admissible subsets of A is also u-admissible.*

Proof. Let $\{B(\alpha) : \alpha \in \Lambda\}$ be a non-empty family of u-admissible subsets of A and $B = \bigcup\{B(\alpha) : \alpha \in \Lambda\}$. Then $\pi_B^{-1}(\pi_B(H)) = H$ because Λ is non-empty and every $B(\alpha)$ has the same property.

Let $\mathcal{A}$ be the subalgebra of $C(X_B)$ generated by $\{g \circ p^B_{B(\alpha)} : g \in C(X_{B(\alpha)}), \alpha \in \Lambda\}$.

Claim 3. $\pi_B|_{X(B)} \prec u(f \circ p_B)|_{X(B)}$ for every $f \in \mathcal{A}$.

Proof of Claim 3. Let $f \in \mathcal{A}$. Then f can be represented as a finite sum of products $(g_{\alpha_1} \circ p^B_{B(\alpha_1)}) \cdots (g_{\alpha_k} \circ p^B_{B(\alpha_k)})$ with $\alpha_i \in \Lambda$ and $g_{\alpha_i} \in C(X_{B(\alpha_i)})$ for all $i \le k$. Since each $B(\alpha_i)$ is u-admissible and $X(B) \subset \bigcap\{X(B(\alpha)) : \alpha \in \Lambda\}$, by Lemma 4.5,

$$u((g_{\alpha_1} \circ p_{B(\alpha_1)}) \cdots (g_{\alpha_k} \circ p_{B(\alpha_k)}))(x) = u(g_{\alpha_1} \circ p_{B(\alpha_1)})(x) \cdots u(g_{\alpha_k} \circ p_{B(\alpha_k)})(x)$$

for every $x \in X(B)$. Consequently, the u-admissibility of $B(\alpha)$s and the linearity of u imply $\pi_B|_{X(B)} \prec u(f \circ p_B)|_{X(B)}$. $\square$

It is easy to see that $1_{X_B} \in \mathcal{A}$ and $\mathcal{A}$ separates the points and the closed sets in X_B. Thus, by the Stone–Weierstrass Theorem, $\mathcal{A}$ is dense in $C(X_B)$. Hence, Claim 3 and the continuity of u yield that $\pi_B|_{X(B)} \prec u(f \circ p_B)|_{X(B)}$ for every $f \in C(X_B)$. Therefore, B is u-admissible. $\square$

Lemma 4.7. *If $B \subset A$ is u-admissible, then the map p_B is open.*

Proof. We follow Shchepin's arguments from the proof of [13, Lemma 1 in Chapter 2]. Let $x_0 \in X$ and let $U(x_0)$ be a closed neighborhood of x_0 in X. We are going to show that $p_B(U(x_0))$ is a neighborhood of $y_0 = p_B(x_0)$ in X_B. To this end, take $\eta > 1$ with $\|u\|\eta < 2$ and let $f : X \to [0, \eta]$ be a function such that $f(x_0) = \eta$ and $f(x) = 0$ for all $x \notin U(x_0)$. Because the projection π_B is open and $\pi_B^{-1}(\pi_B(H)) = H$, so is the restriction $\pi_B|_H$. This implies that the function $\varphi : X_B \to [-2\eta, 2\eta]$ defined by

$$\varphi(y) = \sup\{u(f)(x) : x \in \pi_B^{-1}(\{y\}) \cap H\}$$

is lower semi-continuous; that is, $\varphi^{-1}((a, \infty))$ is open in X_B for all $a \in \mathbb{R}$. Since $\varphi(y_0) \geq \eta > 1$, it suffices to show that $\varphi(y) \leq 1$ for all $y \in X_B \setminus p_B(U(x_0))$.

To this end, suppose $y \in X_B \setminus p_B(U(x_0))$ and let x be an arbitrary point in $\pi_B^{-1}(\{y\}) \cap H$. Choose a function $\theta : X_B \to [0, 1]$ such that $\theta(y) = 1$ and $\theta(p_B(U(x_0))) = \{0\}$. Then $\|\theta \circ p_B + f\| \leq \eta$, so $u(\theta \circ p_B + f)(x) \leq \|u\|\eta < 2$. Because $y \in X_B$, there exists $\bar{x} \in X$ with $\pi_B(\bar{x}) = y = \pi_B(x)$. Using the u-admissibility of B, we obtain that

$$u(\theta \circ p_B)(x) = u(\theta \circ p_B)(\bar{x}) = \theta(\pi_B(\bar{x})) = \theta(y) = 1.$$

Since $u(\theta \circ p_B)(x) + u(f)(x) = u(\theta \circ p_B + f)(x) < 2$, we have $u(f)(x) < 1$. The last inequality is true for every $x \in \pi_B^{-1}(\{y\}) \cap H$, so $\varphi(y) \leq 1$. □

Proof of Theorem 2.1. We are going to use Haydon's result [6, Theorems 1 and 2] saying that a space X is a Dugundji space if X is the limit space of a well-ordered continuous inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ consisting of compacta X_α and open bonding maps p_α^β with X_0 being metrizable and each $p_\alpha^{\alpha+1}$ having a metrizable kernel. This means that there exists a metric compactum M_α such that $X_{\alpha+1}$ is embedded in $X_\alpha \times M_\alpha$ and $p_\alpha^{\alpha+1}$ coincides with the restriction $\varphi_\alpha|X_{\alpha+1}$, where φ_α is the projection from $X_\alpha \times M_\alpha$ onto X_α. Recall that S is continuous if for every limit ordinal α the space X_α is the limit of the inverse system $S_\alpha = \{X_\beta, p_\beta^\eta, \beta < \eta < \alpha\}$.

We can suppose that the set A fixed in the beginning of this section is the set of all ordinals $\alpha < \tau$ for some cardinal τ. According to Lemma 4.4, for every α there exists a countable u-admissible set $B(\alpha) \subset A$ with $\alpha \in B(\alpha)$. Let $\alpha < \tau$ be arbitrary. By Lemma 4.6, $A(\alpha) = \bigcup\{B(\eta) : \eta < \alpha\}$ is u-admissible and the set $A(\alpha + 1) \setminus A(\alpha) \subset B(\alpha)$ is countable.

For brevity, we let $X_\alpha = X_{A(\alpha)}$, $p_\alpha = p_{A(\alpha)}$ and $p_\alpha^\beta = p_{A(\alpha)}^{A(\beta)}$ provided $\alpha < \beta < \tau$. Because all projections p_α are open by Lemma 4.7, so are the bonding maps p_α^β. Since A is the union of all $A(\alpha)$ and each X_α is closed in $\mathbb{I}^{A(\alpha)}$, we obtain a continuous inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ with open bonding maps, whose limit space is X. Moreover, X_0 is metrizable and each $p_{\alpha+1}^\alpha$ has a metrizable kernel, as $A(\alpha+1) \setminus A(\alpha)$ is countable. Therefore, X is a Dugundji space by Haydon's theorem. □

Acknowledgments

The first author was supported by JSPS KAKENHI Grant no. 26400091. The second author was partially supported by NSERC Grant 261914-13. The paper was started during his visit to Ehime University in April 2016. He appreciates the hospitality of his colleagues at Ehime University. The third author was supported by JSPS KAKENHI Grant no. 26800040.

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