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# Ranks and the approximate $n$ -th root property of $C^*$ -algebras

**Abstract:** This paper is a survey of some results on ranks of  $C^*$ -algebras obtained by the author and his co-authors. The main topics under consideration are: real and bounded ranks, weakly (strongly) infinite real and bounded ranks and the  $n$ -th root approximation property of  $C^*$ -algebras. A number of theorems included in the survey are accompanied by proofs.

**Keywords:** bounded and real ranks of  $C^*$ -algebras,  $n$ -th root approximation property, universal elements of  $C^*$ -algebras, weakly infinite real rank

**MSC:** Primary 46L05, 46L85; Secondary 54C40, 54F45

## 1 Introduction and preliminaries

The goal of this paper is to provide an overview of some results about ranks and approximate root properties of  $C^*$ -algebras obtained by the author and his coauthors, see [7], [8], [9]. We did not aim at the full coverage of all achievement in this very broad field. Instead, we included here the formulation and proofs of certain theorems, thus hoping to convey the key ideas to the reader.

The concept of the real rank plays an important role in a variety of problems related to general classification problems of  $C^*$ -algebras. Despite of this widely recognized fact we still see continuing attempts of defining "right dimension" (such as stable [18], analytic [14], tracial [13], exponential [17], completely positive [19] ranks) for unital  $C^*$ -algebras. Definition of the real rank [3] (as well as of its prototype – topological stable rank) of unital  $C^*$ -algebras is based on a well known characterization of the covering dimension of compact spaces. But still it is not known if the real rank of any product of unital  $C^*$ -algebras each of real rank  $\leq n$  is also  $\leq n$ . At the same time the corresponding topological fact is true. The main advantage of the bounded rank, introduced in [8], is that the analog of the above question has a positive answer. This fact is used in the construction of a separable unital universal  $C^*$ -algebra in the class of all separable unital  $C^*$ -algebras of bounded rank  $\leq n$  (see Theorem 14). A motivation for such a result lies, once again, in dimension theory. It is well known that the Menger cube  $\mu^n$  contains a topological copy of any at most  $n$ -dimensional compact metric space. The definition and the main properties

of the bounded rank, as well as, the relation between the real and bounded rank, are discussed in Section 2.

Section 3 is devoted to the definition and some properties of infinite real and bounded ranks defined in [9]. It is clear that some  $C^*$ -algebras of infinite real rank have infinite rank in a very strong sense. Distinguishing these types of infinite ranks is the main reason to introduce the weakly (strongly) infinite real and bounded ranks. As expected, every  $C^*$ -algebra of finite real rank has weakly infinite rank. On the other hand there exists non-commutative  $C^*$ -algebras of weakly infinite but not finite real rank (Corollary 8). The commutative case is completely settle by establishing (Theorem 19) that the function algebra  $C(X)$ , where  $X$  is compact, has weakly infinite real rank if and only if  $X$  is a weakly infinite-dimensional.

In the last Section 4 we consider the so called approximate  $n$ -root property of  $C^*$ -algebras. This property was introduced in [7] by modifying a similar property of the function spaces  $C(X)$  with  $X$  a compactum. The commutative and non-commutative cases are considered. One of the main results is that the class of all unital separable  $C^*$ -algebras with the approximate  $n$ -th root property has universal elements which are non-commutative.

All  $C^*$ -algebras below are assumed to be unital. When we refer to a unital  $C^*$ -subalgebra of a unital  $C^*$ -algebra we implicitly assume that the inclusion is a unital  $*$ -homomorphism. The set of all self-adjoint elements of a  $C^*$ -algebra  $X$  is denoted by  $X_{sa}$ . The product in the category of (unital)  $C^*$ -algebras, i.e. the  $\ell^\infty$ -direct sum, is denoted by  $\prod\{X_t : t \in T\}$ . For a given set  $Y$  and a cardinal number  $\tau$  the symbol  $\exp_\tau Y$  denotes the partially ordered (by inclusion) set of all subsets of  $Y$  of cardinality not exceeding  $\tau$ .

Recall that a direct system  $\mathcal{S} = \{X_\alpha, i_\alpha^\beta, A\}$  of unital  $C^*$ -algebras consists of a partially ordered directed indexing set  $A$ , unital  $C^*$ -algebras  $X_\alpha$ ,  $\alpha \in A$ , and unital  $*$ -homomorphisms  $i_\alpha^\beta : X_\alpha \rightarrow X_\beta$ , defined for each pair of indexes  $\alpha, \beta \in A$  with  $\alpha \leq \beta$ , and satisfying the condition  $i_\alpha^\gamma = i_\beta^\gamma \circ i_\alpha^\beta$  for each triple of indexes  $\alpha, \beta, \gamma \in A$  with  $\alpha \leq \beta \leq \gamma$ . The (inductive) limit of the above direct system is a unital  $C^*$ -algebra which is denoted by  $\varinjlim \mathcal{S}$ . For each  $\alpha \in A$  there exists a unital  $*$ -homomorphism  $i_\alpha : X_\alpha \rightarrow \varinjlim \mathcal{S}$  which will be called the  $\alpha$ -th limit homomorphism of  $\mathcal{S}$ .

If  $A'$  is a directed subset of the indexing set  $A$ , then the subsystem  $\{X_\alpha, i_\alpha^\beta, A'\}$  of  $\mathcal{S}$  is denoted  $\mathcal{S}|A'$ .

**Definition 1.** Let  $\tau \geq \omega$  be a cardinal number. A direct system  $\mathcal{S} = \{X_\alpha, i_\alpha^\beta, A\}$  of unital  $C^*$ -algebras  $X_\alpha$  and unital  $*$ -homomorphisms  $i_\alpha^\beta : X_\alpha \rightarrow X_\beta$  is called a direct  $C^*_\tau$ -system [4] if the following conditions are satisfied:

- (a)  $A$  is a  $\tau$ -complete set (this means that for each chain  $C$  of elements of the directed set  $A$  with  $|C| \leq \tau$ , there exists an element  $\sup C$  in  $A$ ; see [6] for details).
- (b) Density of  $X_\alpha$  is at most  $\tau$  (i.e.  $d(X_\alpha) \leq \tau$ ),  $\alpha \in A$ .

- (c) The  $\alpha$ -th limit homomorphism  $i_\alpha: X_\alpha \rightarrow \varinjlim \mathcal{S}$  is an injective  $*$ -homomorphism for each  $\alpha \in A$ .
- (d) If  $B = \{\alpha_t: t \in T\}$  is a chain of elements of  $A$  with  $|T| \leq \tau$  and  $\alpha = \sup B$ , then the limit homomorphism  $\varinjlim \{i_{\alpha_t}^\alpha: t \in T\}: \varinjlim (\mathcal{S}|B) \rightarrow X_\alpha$  is an isomorphism.

We recall the concept of a  $\mathcal{C}$ -invertibility introduced in [5]. For a given class  $\mathcal{C}$  of unital  $C^*$ -algebras the  $\mathcal{C}$ -invertibility of a unital  $*$ -homomorphism  $p: Y \rightarrow Z$  means that for any unital  $*$ -homomorphism  $g: Y \rightarrow X$ , with  $X \in \mathcal{C}$ , there exists a unital  $*$ -homomorphism  $h: Z \rightarrow X$  such that  $g = h \circ p$ . It is easy to see that if  $\mathcal{C}$  consists of separable  $C^*$ -algebras and there exists a  $\mathcal{C}$ -invertible unital  $*$ -homomorphism  $p: C^*(\mathbf{F}_\infty) \rightarrow Z_C$  such that  $Z_C \in \mathcal{C}$ , where  $C^*(\mathbf{F}_\infty)$  denotes the group  $C^*$ -algebra of the free group on countable number of generators, then  $Z_C$  is an universal element in the class  $\mathcal{C}$ . Indeed, since every element  $X$  of  $\mathcal{C}$  can be represented as the image of  $C^*(\mathbf{F}_\infty)$  under a surjective  $*$ -homomorphism  $g: C^*(\mathbf{F}_\infty) \rightarrow X$ , the  $\mathcal{C}$ -invertibility of  $p$  guarantees that there exists a surjective  $*$ -homomorphism  $h: Z_C \rightarrow X$  such that  $g = h \circ p$ .

## 2 Bounded and real ranks

### 2.1 Real rank - simple characterizations

The definition of the real rank [3] (as well as of its prototype – topological stable rank [18]) of unital  $C^*$ -algebras is based on the following standard result from classical dimension theory (see [11], [15], [16], [10]) characterizing the Lebesgue dimension  $\dim$  of compact spaces: *The Lebesgue dimension  $\dim X$  of a compact space  $X$  is the least integer  $n$  such that the set  $\{f: C(X, \mathbf{R}^{n+1}): \mathbf{0} \notin f(X)\}$  is dense in the space<sup>1</sup>  $C(X, \mathbf{R}^{n+1})$  of all continuous maps of  $X$  in to the Euclidean space  $\mathbf{R}^{n+1}$ .*

A map  $f \in C(X, \mathbf{R}^{n+1})$  may be identified with the  $(n+1)$ -tuple  $(\pi_1 \circ f, \dots, \pi_{n+1} \circ f)$ , where  $\pi_k: \mathbf{R}^{n+1} \rightarrow \mathbf{R}$  is a projection onto the  $k$ -th coordinate,  $k = 1, \dots, n+1$ . The condition  $\mathbf{0} = (0, \dots, 0) \notin f(X)$  may be equivalently expressed as the condition  $\sum_{k=1}^{n+1} f_k^2(x) \neq 0$  for any  $x \in X$ . This, in turn, can be equivalently rephrased as the invertibility of the element  $\sum_{k=1}^{n+1} f_k^2$ . These two observations lead us to the definition of the real rank:

**Definition 2.** [3] *The real rank  $rr(X)$  of a unital  $C^*$ -algebra is the least integer  $n$  such that each  $(n+1)$ -tuple  $(x_1, \dots, x_{n+1})$  of elements from  $X_{sa}$  can be arbitrarily closely approximated by another  $(n+1)$ -tuple  $(y_1, \dots, y_{n+1})$  of self-adjoint elements so that the element  $\sum_{k=1}^{n+1} y_k^2$  is invertible.*

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<sup>1</sup> Compact-open topology is being considered.

We have the following description of the real rank:

**Proposition 4.** [8] *The following conditions are equivalent for a unital  $C^*$ -algebra  $X$ ;*

- (1)  $rr(X) \leq n$ ;
- (2) for each  $(n + 1)$ -tuple  $(x_1, \dots, x_{n+1})$  in  $X_{sa}$  and for each  $\epsilon > 0$ , there exists an  $(n + 1)$ -tuple  $(y_1, \dots, y_{n+1})$  in  $X_{sa}$  such that  $\sum_{k=1}^{n+1} y_k^2$  is invertible and  $\|x_k - y_k\| < \epsilon$  for each  $k = 1, 2, \dots, n + 1$ .
- (3) for each  $(n + 1)$ -tuple  $(x_1, \dots, x_{n+1})$  in  $X_{sa}$ , with  $\|x_k\| = 1$  for each  $k = 1, \dots, n + 1$ , and for each  $\epsilon > 0$ , there exists an  $(n + 1)$ -tuple  $(y_1, \dots, y_{n+1})$  in  $X_{sa}$  such that  $\sum_{k=1}^{n+1} y_k^2$  is invertible and  $\|x_k - y_k\| < \epsilon$  for each  $k = 1, \dots, n + 1$ .

Let  $\mathcal{K}(\mathbf{C}^m)$  denote the collection of compact subsets of  $\mathbf{C}^m$  and  $M \in \mathbf{N} \cup \{\infty\}$ . A generalized joint  $M$ -spectrum (or simply spectrum) on an unital  $C^*$ -algebra  $X$  is a collection  $\{\tilde{\sigma}_m : m \leq M\}$  of maps  $\tilde{\sigma}_m : X^m \rightarrow \mathcal{K}(\mathbf{C}^m)$  satisfying conditions (I)–(III) below:

- (I)  $\tilde{\sigma}_m(x_1, \dots, x_m)$  is a non-empty compact subset of  $\mathbf{C}^m$ .
- (II) If  $x \in X$ , then  $\tilde{\sigma}_1(x) = \sigma(x)$ , where  $\sigma(x)$  denotes the usual spectrum of the element  $x$ .
- (III)  $p(\tilde{\sigma}_m(x_1, \dots, x_m)) = \tilde{\sigma}_n(p(x_1, \dots, x_m))$  for every polynomial mapping  $p = (p_1, \dots, p_n) : \mathbf{C}^m \rightarrow \mathbf{C}^n$ .

We also consider the following two properties:

- (IV) There exists a constant  $K > 0$  (called a polynomial spectral constant) such that for any rational  $\delta > 0$ , for any  $m$ -tuple  $(x_1, \dots, x_m)$  in  $X$  (resp., in  $X_{sa}$ ) and for any polynomial mapping  $p = (p_1, \dots, p_m) : \mathbf{C}^m \rightarrow \mathbf{C}^m$  (resp., having real coefficients) with  $\|p - id_{\mathbf{C}^m}\| \leq K \cdot \delta$ , there exists an  $m$ -tuple  $(y_1, \dots, y_m)$  in  $X$  (resp., in  $X_{sa}$ ) such that  $\tilde{\sigma}(y_1, \dots, y_m) = p(\tilde{\sigma}_m(x_1, \dots, x_m))$  and  $\|x_k - y_k\| \leq \delta$  for each  $k = 1, \dots, m$ .
- (V) There exists  $K > 0$  (called a general spectral constant) such that for any rational  $\delta > 0$ , for any  $m$ -tuple  $(x_1, \dots, x_m)$  in  $X$  (resp., in  $X_{sa}$ ) and for any map  $f$  from  $\tilde{\sigma}_m(x_1, \dots, x_m)$  into  $\mathbf{C}^m$  (resp., into  $\mathbf{R}^m \subseteq \mathbf{C}^m$ ) with  $\|f - id_{\tilde{\sigma}_m(x_1, \dots, x_m)}\| \leq K \cdot \delta$ , there exists an  $m$ -tuple  $(y_1, \dots, y_m)$  in  $X$  (resp., in  $X_{sa}$ ) such that  $\tilde{\sigma}(y_1, \dots, y_m) = f(\tilde{\sigma}_m(x_1, \dots, x_m))$  and  $\|x_k - y_k\| \leq \delta$  for each  $k = 1, \dots, m$ .

Few observations are in order:

- (A) Concerning property (I), it is not usually requested (in a much general setting though) the spectrum of every tuple of non-commuting elements to be non-empty.

- (B) Properties (I)–(III) are standard axioms [20] of joint spectra (for commuting tuples) in Banach algebras. Property (III) is known as the spectral mapping property of  $\tilde{\sigma}$ .
- (C) It is easy to see [20] that properties (II) and (III) imply the inclusion  $\tilde{\sigma}_m(x_1, \dots, x_m) \subseteq \prod\{\sigma(x_k) : k = 1, \dots, m\}$  for any  $m$ -tuple  $(x_1, \dots, x_m)$  in  $X$ .
- (D) Note that if, in property (III), the  $m$ -tuple  $(x_1, \dots, x_m)$  consists of self-adjoint elements and the polynomials  $p_k$ ,  $k = 1, \dots, m$ , have real coefficients, then the  $m$ -tuple
 
$$p(x_1, \dots, x_m) = (p_1(x_1, \dots, x_m), \dots, p_m(x_1, \dots, x_m))$$
 also consists of self-adjoint elements.
- (E) We do not specify whether the  $m$ -tuple  $(y_1, \dots, y_m)$  in property (IV) is obtained as the image of the  $m$ -tuple  $(x_1, \dots, x_m)$  under the polynomial  $p$ . Nevertheless, in light of (D), we require that all  $y_k$ 's are self-adjoint provided that all  $x_k$ 's are self-adjoint and all the coefficients of the polynomials  $p_k$  are real.
- (F) Similar comment with respect to property (V). If all  $x_k$ 's are self-adjoint and  $f(\tilde{\sigma}_m(x_1, \dots, x_m)) \subseteq \mathbf{R}^m \subseteq \mathbf{C}^m$ , then all  $y_k$ 's are also assumed to be self-adjoint.
- (G) If  $T$  is a compact space and  $(x_1, \dots, x_m)$  is an  $m$ -tuple in  $C(T)$ , let define  $\tilde{\sigma}_m(x_1, \dots, x_m) = \Delta\{x_k : k = 1, \dots, m\}(T)$ , where  $\Delta\{x_k : k = 1, \dots, m\}(t) = (x_1(t), \dots, x_m(t))$  for each  $t \in T$ . So obtained generalized joint spectrum has all the properties (I)–(V) with  $K = 1$ .
- (H) For any  $C^*$ -algebra  $X$  the usual spectrum  $\sigma(x)$ ,  $x \in X$ , provides an example of a generalized joint 1-spectrum. This follows from spectral mapping theorem and considerations related to the functional calculus.

The real rank can be characterized in terms of generalized joint spectra:

**Proposition 5.** [8] *If there exists a generalized joint  $(n + 1)$ -spectrum on a unital  $C^*$ -algebra  $X$ , then the following conditions are equivalent:*

- (a)  $rr(X) \leq n$ ;
- (b) *For every  $(n + 1)$ -tuple  $(x_1, \dots, x_{n+1})$  of self-adjoint elements in  $X$  and for every  $\epsilon > 0$  there exists an  $(n + 1)$ -tuple  $(y_1, \dots, y_{n+1})$  of self-adjoint elements such that*
  - (i)  $\|x_k - y_k\| < \epsilon$  for each  $k = 1, \dots, n + 1$ ;
  - (ii)  $\mathbf{0} \notin \tilde{\sigma}_{n+1}(y_1, \dots, y_{n+1})$ .

## 2.2 Bounded rank

The analogy between the definition of the dimension  $\dim$  and the real rank is quite formal and does not go far beyond the obvious observation that  $rr(X) = \dim \Omega(X)$ ,

where  $\Omega(X)$  is the spectrum of the commutative unital  $C^*$ -algebra  $X$ , and a few other straightforward extensions of certain basic facts from classical dimension theory to a non-commutative situation. But there are difficulties in finding proper algebraic interpretations in terms of the concept of the real rank of a non-commutative unital  $C^*$ -algebra  $X$  of such an immediate geometric consequence of the condition  $\dim \Omega(X) \leq n$  as the possibility not only to remove the image of any  $f: \Omega(X) \rightarrow \mathbf{R}^{n+1}$  from  $\mathbf{0}$ , but even to push this image by an  $\epsilon$ -move outside the open ball  $O(\mathbf{0}, \epsilon)$  of radius  $\epsilon > 0$ . As it turns out in the presence of some form of functional calculus (as in the commutative or the real rank zero cases – see Corollaries 3 and 4) there exists a satisfactory analog of the above mentioned geometric fact. As for the general case, the situation remains unclear and we, as a consequence, are unable to answer the question:

**Question 6.** Is it true that  $rr(\prod\{X_t: t \in T\}) \leq n$  for any collection of unital  $C^*$ -algebras  $X_t$  such that  $rr(X_t) \leq n$  for each  $t \in T$ ?

The corresponding topological fact –  $\dim \beta(\oplus\{\Omega(X_t): t \in T\}) \leq n$  – is easy to establish<sup>2</sup>. Perhaps the easiest way of proving the desired inequality in terms of the above given characterization of  $\dim$  is first to approximate (as close as we wish) each of the restrictions  $f_t: \Omega(X_t) \rightarrow \mathbf{R}^{n+1}$ ,  $t \in T$ , of an arbitrarily given map  $f: \beta(\oplus\{\Omega(X_t): t \in T\}) \rightarrow \mathbf{R}^{n+1}$ , by maps  $g_t: \Omega(X_t) \rightarrow \mathbf{R}^{n+1} \setminus \{\mathbf{0}\}$  and then to move each of the images  $g_t(\Omega(X_t))$  outside of the open ball  $O$  (centered at  $\mathbf{0}$ ) of the appropriate radius by small moves (independent of  $t \in T$ ). This way we get a map  $g: \oplus\{\Omega(X_t): t \in T\} \rightarrow \mathbf{R}^{n+1} \setminus O$ . Since the image  $g(\oplus\{\Omega(X_t): t \in T\})$  has a compact closure,  $g$  can be extended to a map  $\tilde{g}: \beta(\oplus\{\Omega(X_t): t \in T\}) \rightarrow \mathbf{R}^{n+1} \setminus O$  which would be the required approximation of  $f$  whose image misses  $\mathbf{0}$ .

Analysis of this elementary reasoning leads us to the concept of the *bounded rank* (to be more specific – bounded rank with respect to a given positive constant).

**Definition 3.** [8] Let  $K > 0$ . We say that an  $m$ -tuple  $(y_1, \dots, y_m)$  of self-adjoint elements of a unital  $C^*$ -algebra  $X$  is  $K$ -unessential if for every rational  $\delta > 0$  there exists an  $m$ -tuple  $(z_1, \dots, z_m)$  of self-adjoint elements of  $X$  satisfying the following conditions:

- (a)  $\|y_k - z_k\| \leq \delta$  for each  $k = 1, \dots, m$ ,
- (b) The element  $\sum_{k=1}^m z_k^2$  is invertible and  $\left\| \left( \sum_{k=1}^m z_k^2 \right)^{-1} \right\| \leq \frac{1}{K \cdot \delta^2}$ .

1-unessential tuples are referred as unessential.

**Remark 2.1.** Obviously if  $K_1 \leq K_2$ , then every  $K_2$ -unessential  $m$ -tuple is  $K_1$ -unessential.

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<sup>2</sup>  $\beta K$  stands for the Stone-Čech compactification of a space  $K$ .

**Definition 4.** [8] Let  $K > 0$ . We say that the bounded rank of a unital  $C^*$ -algebra  $X$  with respect to  $K$  does not exceed  $n$  (notation:  $br_K(X) \leq n$ ) if for any  $(n + 1)$ -tuple  $(x_1, \dots, x_{n+1})$  of self-adjoint elements of  $X$  and for any  $\epsilon > 0$  there exists a  $K$ -unessential  $(n + 1)$ -tuple  $(y_1, \dots, y_{n+1})$  in  $X$  such that  $\|x_k - y_k\| < \epsilon$  for each  $k = 1, \dots, n + 1$ . For simplicity,  $br_1(X)$  is denoted by  $br(X)$  and it is called a bounded rank.

We record the following statement for future references.

**Proposition 7.** [8]  $rr(X) \leq br_K(X)$  for any unital  $C^*$ -algebra  $X$  and for any  $K > 0$ .

*Proof.* Let  $br_K(X) = n$  and  $(x_1, \dots, x_{n+1})$  be an  $(n + 1)$ -tuple of self-adjoint elements in  $X$ . Let also  $\epsilon > 0$ . Since  $br_K(X) = n$ , there exists a  $K$ -unessential  $(n + 1)$ -tuple  $(y_1, \dots, y_{n+1})$  such that  $\|x_k - y_k\| < \frac{\epsilon}{2}$  for each  $k = 1, \dots, n + 1$ . This in turn means that for a rational number  $\delta$  with  $\delta \leq \frac{\epsilon}{2}$  there is an  $(n + 1)$ -tuple  $(z_1, \dots, z_{n+1})$  of self-adjoint elements such that

- (a)  $\|y_k - z_k\| \leq \delta$  for each  $k = 1, \dots, n + 1$ ,
- (b) The element  $\sum_{k=1}^{n+1} z_k^2$  is invertible and  $\left\| \left( \sum_{k=1}^{n+1} z_k^2 \right)^{-1} \right\| \leq \frac{1}{K \cdot \delta^2}$ .

Clearly  $\|x_k - z_k\| \leq \|x_k - y_k\| + \|y_k - z_k\| < \frac{\epsilon}{2} + \delta \leq \epsilon$ ,  $k = 1, \dots, n + 1$ . According to (b),  $\sum_{k=1}^{n+1} z_k^2$  is invertible which shows that  $rr(X) \leq n$ .  $\square$

The next two propositions provide a relation between the real and bounded ranks in the presence of a generalized joint spectrum.

**Proposition 8.** [8] If there exists a generalized joint  $(n + 1)$ -spectrum on a unital  $C^*$ -algebra  $X$  satisfying conditions (IV), then the following are equivalent:

- (a)  $rr(X) \leq n$ ;
- (b)  $br_{\frac{K^2}{4}}(X) \leq n$ , where  $K$  is a polynomial spectral constant of the joint spectrum.

**Proposition 9.** [8] If there exists a generalized joint  $(n + 1)$ -spectrum on the unital  $C^*$ -algebra  $X$  satisfying condition (V), then the following are equivalent:

- (a)  $rr(X) \leq n$ ;
- (b)  $br_{K^2}(X) \leq n$ , where  $K$  is a general spectral constant of the joint spectrum.

**Proposition 10.** [8] Let  $(y_1, \dots, y_m)$  be a commuting  $m$ -tuple of self-adjoint elements of the unital  $C^*$ -algebra  $X$ . Then  $(y_1, \dots, y_m)$  is  $K$ -unessential for any positive  $K \leq 1$  provided  $\sum_{i=1}^m y_i^2$  is invertible.

**Corollary 3.** [8] Let  $X$  be a commutative unital  $C^*$ -algebra and  $0 < K \leq 1$ . Then  $br_K(X) = rr(X) = \dim \Omega(X)$ , where  $\Omega(X)$  is the spectrum of  $X$ .

*Proof.* By Proposition 7,  $rr(X) \leq br_K(X)$ . The opposite inequality  $br_K(X) \leq rr(X)$  follows from Proposition 10. The remaining part is well known (see [3, Proposition 1.1]).  $\square$

**Corollary 4.** [8] *Let  $X$  be a unital  $C^*$ -algebra and  $0 < K \leq 1$ . Then  $br_K(X) = 0$  if and only if  $rr(X) = 0$ .*

*Proof.* According to Proposition 7,  $br_K(X) = 0$  yields  $rr(X) = 0$ . Conversely, if  $rr(X) = 0$ , then, by Proposition 10,  $br_K(X) = 0$ .  $\square$

We also have the following property of the bounded rank:

**Proposition 11.** [8] *Let  $K > 0$  and  $p: X \rightarrow Y$  be a surjective  $*$ -homomorphism of unital  $C^*$ -algebras. Then  $br_K(Y) \leq br_K(X)$ .*

Comparing to the real rank, the bounded rank has the advantage that the corresponding analog of Question 6 has a positive answer. This is actually one of the main reasons of introducing the concept of bounded rank.

**Proposition 12.** [8] *Let  $\{X_t: t \in T\}$  be a family of unital  $C^*$ -algebras such that  $br_K(X_t) \leq n$  for each  $t \in T$ . Then for every  $K > 0$  we have  $br_K(\prod\{X_t: t \in T\}) \leq n$ .*

*Proof.* Let  $(x_1, \dots, x_{n+1})$  be an  $(n+1)$ -tuple of self-adjoint elements of the product  $X = \prod\{X_t: t \in T\}$ , where  $x_k = \{x_k^t: t \in T\}$  for each  $k = 1, \dots, n+1$ , and let  $\epsilon > 0$ . Our goal is to find a  $K$ -inessential  $(n+1)$ -tuple  $(y_1, \dots, y_{n+1})$  in  $X$  such that  $\|x_k - y_k\| < \epsilon$ . For a given  $t \in T$  consider the  $(n+1)$ -tuple  $(x_1^t, \dots, x_{n+1}^t)$  of self-adjoint elements in  $X_t$ . Since  $br_K(X_t) \leq n$ , there exists a  $K$ -inessential  $(n+1)$ -tuple  $(y_1^t, \dots, y_{n+1}^t)$  in  $X_t$  such that  $\|x_k^t - y_k^t\|_{X_t} < \frac{\epsilon}{2}$  for each  $k = 1, \dots, n+1$ . Consider the  $(n+1)$ -tuple  $(y_1, \dots, y_{n+1})$ , where  $y_k = \{y_k^t: t \in T\}$  for each  $k = 1, \dots, n+1$ . Note that  $y_k \in X$  for each  $k = 1, \dots, n+1$ . Indeed  $\|y_k^t\|_{X_t} \leq \|y_k^t - x_k^t\|_{X_t} + \|x_k^t\|_{X_t} \leq \frac{\epsilon}{2} + \sup\{\|x_k^t\|_{X_t}: t \in T\}$  and  $\sup\{\|y_k^t\|_{X_t}: t \in T\} \leq \frac{\epsilon}{2} + \sup\{\|x_k^t\|_{X_t}: t \in T\} < \infty$ . Also note that  $\|x_k - y_k\| = \sup\{\|x_k^t - y_k^t\|_{X_t}: t \in T\} \leq \frac{\epsilon}{2} < \epsilon$ . It only remains to show that the  $(n+1)$ -tuple  $(y_1, \dots, y_{n+1})$  is  $K$ -inessential in  $X$ . Indeed, let  $\delta > 0$  be rational. Since the  $(n+1)$ -tuple  $(y_1^t, \dots, y_{n+1}^t)$  is  $K$ -inessential in  $X_t$ , there exists an  $(n+1)$ -tuple  $(z_1^t, \dots, z_{n+1}^t)$  in  $X_t$  such that

- (a) $_t$   $\|y_k^t - z_k^t\|_{X_t} \leq \delta$  for each  $k = 1, \dots, n+1$ .
- (b) $_t$   $\left\| \left( \sum_{k=1}^{n+1} (z_k^t)^2 \right)^{-1} \right\| \leq \frac{1}{K \cdot \delta^2}$ .

Next, consider the  $(n+1)$ -tuple  $(z_1, \dots, z_{n+1})$ , where  $z_k = \{z_k^t: t \in T\}$  for each  $k = 1, \dots, n+1$ . As above,  $z_k \in X$  and obviously

$$\|y_k - z_k\| = \sup\{\|y_k^t - z_k^t\|_{X_t}: t \in T\} \leq \delta, k = 1, \dots, n+1.$$

Finally, note that  $\left\| \left( \sum_{k=1}^{n+1} z_k^2 \right)^{-1} \right\| = \left\| \left\{ \left( \sum_{k=1}^{n+1} (z_k^t)^2 \right)^{-1} : t \in T \right\} \right\| =$   
 $\sup \left\{ \left\| \left( \sum_{k=1}^{n+1} (z_k^t)^2 \right)^{-1} \right\|_{X_t} : t \in T \right\} \leq \frac{1}{K \cdot \delta^2}$  as required.  $\square$

### 2.3 Spectral decompositions of unital $C^*$ -algebras of bounded rank $n$

In this subsection we investigate the behaviour of the bounded rank with respect to direct systems and prove the existence of universal elements in  $\mathcal{BR}_n^K$ , where  $\mathcal{BR}_n^K$  denotes the class of all separable unital  $C^*$ -algebras of bounded rank with respect to  $K$  at most  $n$ . A motivation for such a result lies, once again, in the classical dimension theory. It is a well-known observation (see, for instance, [10, Theorem 1.3.15]) that the Menger cube  $\mu^n$  contains a topological copy of any at most  $n$ -dimensional metrizable compact space. This means that every commutative separable unital  $C^*$ -algebra of real rank at most  $n$  is a quotient of the  $C^*$ -algebra  $C(\mu^n)$ . We extend this result to the non-commutative case.

The following decomposition theorem was initially established for  $n = 0$  in [5].

**Proposition 13.** [8] *Let  $K > 0$ . The following conditions are equivalent for any unital  $C^*$ -algebra  $X$ :*

- (a)  $br_K(X) \leq n$ .
- (b)  $X$  can be represented as the direct limit of a direct  $C_\omega^*$ -system  $\{X_\alpha, i_\alpha^\beta, A\}$  satisfying the following properties:
  - (a) The indexing set  $A$  is cofinal and  $\omega$ -closed in the  $\omega$ -complete set  $\exp_\omega Y$  for some (any) countable dense subset  $Y$  of  $X$ .
  - (b)  $X_\alpha$  is a  $C^*$ -subalgebra of  $X$  such that  $br_K(X_\alpha) \leq n, \alpha \in A$ .

The proof of Proposition 13 provided in [8] works for the real rank as well.

**Corollary 5.** [8] *Let  $K > 0$ . Every countable subset of a unital  $C^*$ -algebra  $X$  with  $br_K(X) \leq n$  is contained in a unital separable  $C^*$ -subalgebra  $X_0$  such that  $br_K(X_0) \leq n$ .*

Next, for any  $K > 0$ , we construct a universal separable unital  $C^*$ -algebra  $Z_n^K$  of bounded rank  $n$ . Universal in the sense that any other separable unital  $C^*$ -algebra with bounded rank  $\leq n$  is its quotient. Recall that  $C^*(\mathbf{F}_\infty)$  denotes the group  $C^*$ -algebra of the free group on countable number of generators.

**Theorem 14.** [8] *Let  $K > 0$ . The class  $\mathcal{BR}_n^K$  of all separable unital  $C^*$ -algebras with  $br_K \leq n$  contains an universal element  $Z_n^K$ . More precisely, there is a  $\mathcal{BR}_n^K$ -*

invertible unital  $*$ -homomorphism  $p: C^*(\mathbf{F}_\infty) \rightarrow Z_n^K$ , where  $Z_n^K$  is a separable unital  $C^*$ -algebra such that  $br_K(Z_n^K) = n$ .

*Proof.* Let  $\mathcal{A} = \{f_t: C^*(\mathbf{F}_\infty) \rightarrow X_t, t \in T\}$  denote the set of all unital  $*$ -homomorphisms, defined on  $C^*(\mathbf{F}_\infty)$ , such that  $br_K(X_t) \leq n$ . Next, consider the product  $\prod\{X_t: t \in T\}$ . Since  $br_K(X_t) \leq n$  for each  $t \in T$ , it follows from Proposition 12 that  $br_K(\prod\{X_t: t \in T\}) \leq n$ . The  $*$ -homomorphisms  $f_t, t \in T$ , define the unital  $*$ -homomorphism  $f: C^*(\mathbf{F}_\infty) \rightarrow \prod\{X_t: t \in T\}$  such that  $\pi_t \circ f = f_t$  for each  $t \in T$  (here  $\pi_t: \prod\{X_t: t \in T\} \rightarrow X_t$  denotes the corresponding canonical projection  $*$ -homomorphism). By Proposition 13,  $\prod\{X_t: t \in T\}$  can be represented as the limit of the  $C_\omega^*$ -system  $\mathcal{S} = \{C_\alpha, i_\alpha^\beta, A\}$  such that  $C_\alpha$  is a separable unital  $C^*$ -algebra with  $br_K(C_\alpha) \leq n$  for each  $\alpha \in A$ . Suppressing injective unital  $*$ -homomorphisms  $i_\alpha^\beta: C_\alpha \rightarrow C_\beta$ , we can, for notational simplicity, assume that  $C_\alpha$ 's are unital  $C^*$ -subalgebras of  $\prod\{X_t: t \in T\}$ . Let  $\{a_k: k \in \omega\}$  be a countable dense subset of  $C^*(\mathbf{F}_\infty)$ . It is easily seen that for each  $k \in \omega$  there exists an index  $\alpha_k \in A$  such that  $f(a_k) \in C_{\alpha_k}$ . By [6, Corollary 1.1.28], there exists an index  $\alpha_0 \in A$  such that  $\alpha_0 \geq \alpha_k$  for each  $k \in \omega$ . Then  $f(a_k) \in C_{\alpha_k} \subseteq C_{\alpha_0}$  for each  $k \in \omega$  (see also Corollary 5). This observation coupled with the continuity of  $f$  guarantees that  $f(C^*(\mathbf{F}_\infty)) = f(cl\{a_k: k \in \omega\}) \subseteq cl\{f(\{a_k: k \in \omega\})\} \subseteq clC_{\alpha_0} = C_{\alpha_0}$ .

Let  $Z_n^K = C_{\alpha_0}$  and  $p$  denote the unital  $*$ -homomorphism  $f$  considered as the homomorphism of  $C^*(\mathbf{F}_\infty)$  into  $Z_n^K$ . Note that  $f = i \circ p$ , where  $i: Z_n^K = C_{\alpha_0} \hookrightarrow \prod\{X_t: t \in T\}$  stands for the inclusion.

By construction,  $br_K(Z_n^K) \leq n$ . Let us show that  $p: C^*(\mathbf{F}_\infty) \rightarrow Z_n^K$  is  $\mathcal{BR}_n^K$ -invertible in the sense of Introduction. In our situation, for any unital  $*$ -homomorphism  $g: C^*(\mathbf{F}_\infty) \rightarrow X$ , where  $X$  is a separable unital  $C^*$ -algebra with  $br_K(X) \leq n$ , we need to establish the existence of a unital  $*$ -homomorphism  $h: Z_n^K \rightarrow X$  such that  $g = h \circ p$ . Indeed, by definition of the set  $\mathcal{A}$ , we conclude that  $g = f_t$  for some index  $t \in T$  (in particular,  $X = X_t$  for the same index  $t \in T$ ). Next observe that  $g = f_t = \pi_t \circ f = \pi_t \circ i \circ p$ . This allows us to define the required unital  $*$ -homomorphism  $h: Z_n^K \rightarrow X$  as the composition  $h = \pi_t \circ i$ . Hence,  $p$  is  $\mathcal{BR}_n^K$ -invertible which yields the universality of  $Z_n^K$ .  $\square$

It is interesting to note that not only  $Z_n^K$  is universal in the above sense, but for every unital  $*$ -homomorphism  $g: C^*(\mathbf{F}_\infty) \rightarrow \mathbf{C}$  the pushout  $Z_n^K \star_{\mathbf{C}} \mathbf{C}$ , generated by  $p$  and  $g$ , is also  $\mathcal{BR}_n^K$ -universal. To see this take any separable unital  $C^*$ -algebra  $X$  such that  $br_K(X) \leq n$  and consider the unital  $*$ -homomorphism  $h: \mathbf{C} \rightarrow X$ . Since  $p$  is  $\mathcal{BR}_n^K$ -invertible, there exists a unital  $*$ -homomorphism  $\tilde{h}: Z_n^K \rightarrow X$  such that  $\tilde{h} \circ p = h \circ g$ . The homomorphisms  $h$  and  $\tilde{h}$  uniquely determine the unital  $*$ -homomorphism  $\varphi: Z_n^K \star_{\mathbf{C}} \mathbf{C} \rightarrow X$  as required.

### 3 Infinite ranks

As in dimension theory, some  $C^*$ -algebras of infinite real rank have infinite rank in a very strong sense of this word, while others do not. In order to distinguish these types of infinite ranks we introduce the concept of weakly (strongly) infinite real rank. First, let us provide an alternative characterization of real rank in terms of infinite sequences.

**Proposition 15.** [9] *Let  $X$  be a unital  $C^*$ -algebra. Then the following conditions are equivalent:*

- (i)  $rr(X) \leq n$ .
- (ii) *for each  $(n + 1)$ -tuple  $(x_1, \dots, x_{n+1})$  in  $X_{sa}$  and for each  $\epsilon > 0$ , there exists an  $(n + 1)$ -tuple  $(y_1, \dots, y_{n+1})$  in  $X_{sa}$  such that  $\sum_{k=1}^{n+1} y_k^2$  is invertible and  $\|x_k - y_k\| < \epsilon$  for each  $k = 1, 2, \dots, n + 1$ .*
- (iii) *for any sequence of self-adjoint elements  $\{x_i : i \in \mathbf{N}\} \subseteq X_{sa}$  and for any sequence of positive real numbers  $\{\epsilon_i : i \in \mathbf{N}\}$  there exists a sequence  $\{y_i : i \in \mathbf{N}\} \subseteq X_{sa}$  such that*
  - (a)  $\|x_i - y_i\| < \epsilon_i$ , for each  $i \in \mathbf{N}$ ,
  - (b) *for any subset  $D \subseteq \mathbf{N}$ , with  $|D| = n + 1$ , the element  $\sum_{i \in D} y_i^2$  is invertible.*
- (iv) *for any sequence of self-adjoint elements  $\{x_i : i \in \mathbf{N}\} \subseteq X_{sa}$  and for any  $\epsilon > 0$  there exists a sequence  $\{y_i : i \in \mathbf{N}\} \subseteq X_{sa}$  such that*
  - (a)  $\|x_i - y_i\| < \epsilon$ , for each  $i \in \mathbf{N}$ ,
  - (b) *for any subset  $D \subseteq \mathbf{N}$ , with  $|D| = n + 1$ , the element  $\sum_{i \in D} y_i^2$  is invertible.*
- (v) *for any sequence of self-adjoint elements  $\{x_i : i \in \mathbf{N}\} \subseteq X_{sa}$  such that  $\|x_i\| = 1$  for each  $i \in \mathbf{N}$  and for any  $\epsilon > 0$  there exists a sequence  $\{y_i : i \in \mathbf{N}\} \subseteq X_{sa}$  such that*
  - (a)  $\|x_i - y_i\| < \epsilon$ , for each  $i \in \mathbf{N}$ ,
  - (b) *for any subset  $D \subseteq \mathbf{N}$ , with  $|D| = n + 1$ , the element  $\sum_{i \in D} y_i^2$  is invertible.*

Proposition 15 provides a basis for the next definition.

**Definition 5.** [9] *We say that a unital  $C^*$  algebra  $X$  has a weakly infinite real rank if for any sequence of self-adjoint elements  $\{x_i : i \in \mathbf{N}\} \subseteq X_{sa}$  and any  $\epsilon > 0$  there is a sequence  $\{y_i : i \in \mathbf{N}\} \subseteq X_{sa}$  such that  $\|x_i - y_i\| < \epsilon$  for every  $i \in \mathbf{N}$  and the element  $\sum_{i \in D} y_i^2$  is invertible for some finite set  $D$  of indices. If  $X$  does not have weakly infinite real rank, then we say that  $X$  has strongly infinite real rank.*

The bounded version can be defined similarly. If  $K > 0$ , we say that a sequence of self-adjoint elements of a unital  $C^*$ -algebra is  $K$ -unessential provided it contains a finite  $K$ -unessential (in the sense of Definition 3) subset.

**Definition 6.** [9] Let  $K > 0$ . We say that a unital  $C^*$  algebra  $X$  has a weakly infinite bounded rank with respect to  $K$  if for any sequence of self-adjoint elements  $\{x_i : i \in \mathbf{N}\} \subset X_{sa}$  and any  $\epsilon > 0$  there is a  $K$ -inessential sequence  $\{y_i : i \in \mathbf{N}\} \subset X_{sa}$  such that  $\|x_i - y_i\| < \epsilon$  for every  $i \in \mathbf{N}$ . If  $X$  does not have weakly infinite bounded rank, then we say that  $X$  has strongly infinite bounded rank.

Here are some properties of infinite ranks:

**Proposition 16.** [9] Every unital  $C^*$ -algebra of a finite real rank has weakly infinite real rank.

We will see below that there exists a  $C^*$ -algebra of infinite real rank, but not finite real rank.

**Proposition 17.** [9] Let  $f : X \rightarrow Y$  be a surjective  $*$ -homomorphism of unital  $C^*$ -algebras.

- If  $X$  has weakly infinite real rank, then so does  $Y$ ;
- If  $X$  has weakly infinite bounded rank with respect to some  $K > 0$ , then so does  $Y$ .

The next proposition provides a relation between both infinite ranks.

**Proposition 18.** [9] Let  $K > 0$ . If the unital  $C^*$ -algebra  $X$  has weakly infinite bounded rank with respect to  $K$ , then it has weakly infinite real rank.

**Corollary 6.** [9] If a unital  $C^*$ -algebra has strongly infinite real rank, then it has strongly infinite bounded rank with respect to any positive constant.

We are going to characterize the infinite ranks of  $C(X)$ , where  $X$  is a compact Hausdorff space, in terms of the dimension of  $X$ . We already observed that if  $X$  is a finite-dimensional compact space, then, according to Corollary 3,  $rr(C(X)) = br_1(C(X)) = \dim X$  for any positive  $K \leq 1$ . Our next goal is to extend this result to the infinite-dimensional situation.

First, recall that a compact Hausdorff space  $X$  is called *weakly infinite-dimensional* [1] if for any sequence  $\{(F_i, H_i) : i \in \mathbf{N}\}$  of pairs of closed disjoint subsets of  $X$  there are partitions  $L_i$  between  $F_i$  and  $H_i$  such that  $\bigcap_{i=1}^{\infty} L_i = \emptyset$ . Here,  $L_i \subset X$  is called a partition between  $F_i$  and  $H_i$  if  $L_i$  is closed in  $X$  and  $X \setminus L_i$  is decomposed as the union  $U_i \cup V_i$  of disjoint open sets with  $F_i \subset U_i$  and  $H_i \subset V_i$ . Since  $X$  is compact,  $\bigcap_{i=1}^{\infty} L_i = \emptyset$  is equivalent to  $\bigcap_{i=1}^k L_i = \emptyset$  for some  $k \in \mathbf{N}$ . If  $X$  is not weakly infinite-dimensional, then it is *strongly infinite-dimensional*.

A standard example of a weakly infinite dimensional, but not finite-dimensional, metrizable compactum can be obtained by taking the one-point compactification

$\alpha(\oplus\{\mathbf{I}^n : n \in \mathbf{N}\})$  of the discrete union of increasing dimensional cubes. The Hilbert cube  $\mathbf{Q}$  is, of course, strongly infinite-dimensional.

**Theorem 19.** [9] *Let  $X$  be a compact Hausdorff space and  $0 < K \leq 1$ . Then the following conditions are equivalent:*

- (a)  $C(X)$  has weakly infinite bounded rank with respect to  $K$ ;
- (b)  $C(X)$  has weakly infinite real rank;
- (c)  $X$  is weakly infinite-dimensional.

*Proof.* (a) $\implies$ (b). This implication follows from Proposition 18 (which is valid for any – not necessarily commutative – unital  $C^*$ -algebras).

(b) $\implies$ (c). Suppose that  $C(X)$  has a weakly infinite real rank. Take an arbitrary sequence  $\{(B_i, K_i) : i \in \mathbf{N}\}$  of pairs of disjoint closed subsets of  $X$  and define functions  $f_i : X \rightarrow [-1, 1]$  such that  $f_i(B_i) = -1$  and  $f_i(K_i) = 1$  for every  $i \in \mathbf{N}$ . Then, according to our hypothesis, there is a sequence  $\{g_i : i \in \mathbf{N}\} \subset C(X)$  of real-valued functions and an integer  $k$  with  $\|f_i - g_i\| < 1$ ,  $i \in \mathbf{N}$ , and  $\sum_{i=1}^k g_i^2(x) > 0$  for each  $x \in X$ . If  $C_i$  denotes the set  $g_i^{-1}(0)$ , the last inequality means that  $\bigcap_{i=1}^k C_i = \emptyset$ . Therefore, in order to prove that  $X$  is weakly infinite-dimensional, it only remains to show each  $C_i$  is a separator between  $B_i$  and  $K_i$ . To this end, we fix  $i \in \mathbf{N}$  and observe that  $\|f_i - g_i\| < 1$  implies the following inclusions:  $g_i(B_i) \subseteq [-2, 0)$ ,  $g_i(K_i) \subseteq (0, 2]$  and  $g_i(X) \subseteq [-2, 2]$ . So,  $X \setminus C_i = U_i \cup V_i$ , where  $U_i = g_i^{-1}([-2, 0))$  and  $V_i = g_i^{-1}((0, 2])$ . Moreover,  $B_i \subseteq U_i$  and  $K_i \subseteq V_i$ , i.e.  $C_i$  separates  $B_i$  and  $K_i$ .

(c) $\implies$ (a). Let us show that the weak infinite-dimensionality of  $X$  forces  $C(X)$  to have a weakly infinite bounded rank with respect to  $K$ . To this end, take any sequence  $\{f_i : i \in \mathbf{N}\} \subset C(X)$  of real-valued functions and any positive number  $\epsilon$ . It suffices to find another sequence  $\{g_i : i \in \mathbf{N}\}$  of real-valued functions in  $C(X)$  such that  $\|f_i - g_i\| \leq \epsilon$  for every  $i \in \mathbf{N}$  and  $\sum_{i=1}^m g_i^2(x) > 0$  for every  $x \in X$  and some  $m \in \mathbf{N}$ . Indeed, if  $\sum_{i=1}^m g_i^2(x) > 0$  for every  $x \in X$ , then the function  $\sum_{i=1}^m g_i^2$  is invertible. This, according to Proposition 10, is equivalent to the  $K$ -unessentiality of the  $m$ -tuple  $(g_1, \dots, g_m)$ . On the other hand,  $\sum_{i=1}^m g_i^2(x) > 0$  for each  $x \in X$  if and only if  $\bigcap_{i=1}^m g_i^{-1}(0) = \emptyset$ . Further, since  $X$  is compact, the existence of  $m \in \mathbf{N}$  with  $\bigcap_{i=1}^m g_i^{-1}(0) = \emptyset$  is equivalent to  $\bigcap_{i=1}^\infty g_i^{-1}(0) = \emptyset$ . Therefore, our proof is reduced to constructing, for each  $i \in \mathbf{N}$ , a function  $g_i$  which is  $\epsilon$ -close to  $f_i$  and such that the intersection of all  $g_i^{-1}(0)$ 's,  $i \in \mathbf{N}$ , is empty.

For every  $i \in \mathbf{N}$  let  $c_i = \inf\{f_i(x) : x \in X\}$  and  $d_i = \sup\{f_i(x) : x \in X\}$ . We can suppose, without loss of generality, that each interval  $(c_i, d_i)$  is not empty and contains 0. For every  $i$  we choose  $\eta_i > 0$  such that  $\eta_i < \frac{\epsilon}{2}$  and  $L_i = [-\eta_i, \eta_i] \subset (c_i, d_i)$ ,  $i \in \mathbf{N}$ . Let  $Q = \prod_{i=1}^\infty [c_i, d_i]$ ,  $Q_0 = \prod_{i=1}^\infty L_i$  be the topological products of all  $[c_i, d_i]$ 's and  $L_i$ 's, respectively. Consider the diagonal product  $f = \Delta\{f_i : i \in \mathbf{N}\} : X \rightarrow Q$  and note that  $H = \bigcap_{i=1}^\infty H_i$ , where  $H = f^{-1}(Q_0)$  and  $H_i = f_i^{-1}(L_i)$  for each  $i \in \mathbf{N}$ . We also consider the sets

$$F_i^- = f_i^{-1}([c_i, -\eta_i]) \text{ and } F_i^+ = f_i^{-1}([\eta_i, d_i]), \quad i \in \mathbf{N}.$$

Since  $H$  is weakly infinite-dimensional (as a closed subset of  $X$ ), by [1, Theorem 19, §10.4], there is a continuous map  $p = (p_1, p_2, \dots): H \rightarrow Q_0$  and a pseudointerior point  $b = \{b_i: i \in \mathbf{N}\} \in Q_0$  (i.e. each  $b_i$  lies in the interior of the interval  $L_i$ ) such that

$$b \notin p(H), \quad F_i^- \cap H \subset p_i^{-1}(\{-\eta_i\}), \text{ and } F_i^+ \cap H \subset p_i^{-1}(\{\eta_i\}), \quad i \in \mathbf{N}.$$

Since each  $b_i$  is an interior point of  $L_i = [-\eta_i, \eta_i]$ , there exists homeomorphisms  $s_i: L_i \rightarrow L_i$  which leaves the endpoints  $-\eta_i$  and  $\eta_i$  fixed and such that  $s_i(b_i) = 0$ . Let  $s = \Delta\{s_i: i \in \mathbf{N}\}: Q_0 \rightarrow Q_0$  and  $q = s \circ p$ . Obviously  $s(b) = \mathbf{0}$  and  $\mathbf{0} \notin q(H)$ , where  $\mathbf{0}$  denotes the point of  $Q_0$  having all coordinates 0. Further observe that if  $q_i = \pi_i \circ q$ , where  $q_i: Q_0 \rightarrow L_i$  denotes the natural projection onto the  $i$ -th coordinate, then

$$F_i^- \cap H \subset q_i^{-1}(\{-\eta_i\}) \text{ and } F_i^+ \cap H \subset q_i^{-1}(\{\eta_i\}), \quad i \in \mathbf{N}.$$

Therefore, each  $q_i, i \in \mathbf{N}$ , is a function from  $H$  into  $L_i$  satisfying the following condition:  $q_i(F_i^- \cap H) = f_i(F_i^- \cap H_i) = -\eta_i$  and  $q_i(F_i^+ \cap H) = f_i(F_i^+ \cap H_i) = \eta_i$ . Let  $h_i: H_i \rightarrow L_i$  be an extension of  $q_i, i \in \mathbf{N}$ . Note that the restrictions of  $h_i$  and  $f_i$  onto the sets  $F_i^- \cap H_i$  and  $F_i^+ \cap H_i$  coincide. Finally, define  $g_i: X \rightarrow [c_i, d_i]$  by letting

$$g_i(x) = \begin{cases} h_i(x), & \text{if } x \in H_i; \\ f_i(x), & \text{if } x \in X - H_i. \end{cases}$$

To finish the proof of the "if" part, we need to show that  $g_i(x)$  is  $\epsilon$ -close to  $f_i(x)$  for each  $i \in \mathbf{N}$  and  $x \in X$ , and that  $\bigcap_{i=1}^\infty g_i^{-1}(0) = \emptyset$ . Since  $g_i$  and  $f_i$  are identical outside  $H_i$ , the first condition is satisfied for  $x \notin H_i$ . If  $x \in H_i$ , then both  $f_i(x)$  and  $g_i(x)$  belong to  $L_i$ , so again  $|f_i(x) - g_i(x)| < \epsilon$ . To prove the second condition, observe first that  $x \notin H$  implies  $x \notin H_j$  for some  $j$ . Hence,  $g_j(x) = f_j(x) \notin L_j$ , so  $g_j(x) \neq 0$ . If  $x \in H$ , then  $g_i(x) = q_i(x)$  for all  $i$  and, because  $\mathbf{0} \notin q(H)$ , at least one  $g_i(x)$  must be different from 0. Thus,  $\bigcap_{i=1}^\infty g_i^{-1}(0) = \emptyset$ . □

It follows from Theorem 14 that  $rr(C^*(\mathbf{F}_\infty)) > n$  for each  $n$ . Our results imply much stronger observation.

**Corollary 7.** [9] *The group  $C^*$ -algebra  $C^*(\mathbf{F}_\infty)$  of the free group on countable number of generators has strongly infinite real rank.*

*Proof.* It is well known that every separable unital  $C^*$ -algebra is an image of  $C^*(\mathbf{F}_\infty)$  under a surjective  $*$ -homomorphism. In particular, there exists a surjective  $*$ -homomorphism  $f: C^*(\mathbf{F}_\infty) \rightarrow C(Q)$ , where  $Q$  denotes the Hilbert cube. It is

well known (see, for instance, [1, §10.5]) that the Hilbert cube  $Q$  is strongly infinite dimensional. By Proposition 19,  $C(Q)$  has strongly infinite real rank. Finally, by Proposition 17, real rank of  $C^*(\mathbf{F}_\infty)$  must also be strongly infinite.  $\square$

Next statement provides a formal example of a unital  $C^*$ -algebra of weakly infinite, but not finite real rank.

**Proposition 20.** [9] *Let  $X = \alpha(\oplus\{I^n : n \in \mathbf{N}\})$  be the one-point compactification of the discrete topological sum of increasing-dimensional cubes. In other words,  $C(X) = \prod\{C(I^n) : n \in \mathbf{N}\}$  (here  $\prod$  stands for the direct product of indicated  $C^*$ -algebras). Then  $C(X)$  has weakly infinite, but not finite real rank.*

*Proof.* Obviously  $X$  is countably dimensional and hence, by [1, Corollary 1, §10.5], it is weakly infinite dimensional. By Proposition 19,  $C(X)$  has weakly infinite real rank. It only remains to note that  $rr(X) > n$  for any  $n \in \mathbf{N}$ .  $\square$

In conclusion let us note that there exist non-commutative  $C^*$ -algebras with similar properties (compare with Corollary 7).

**Corollary 8.** [9] *There exist non-commutative unital  $C^*$ -algebras of weakly infinite, but not finite real rank.*

*Proof.* Let  $X$  be as in Proposition 20 and  $A$  be a non-commutative unital  $C^*$ -algebra of a finite real rank. Then the product  $C(X) \oplus A$  has weakly infinite real rank. It is clear that  $C(X) \oplus A$  is non-commutative and does not have a finite real rank.  $\square$

## 4 $C^*$ -algebras with the approximate $n$ -th root property

All topological spaces in this section are assumed to be (at least) completely regular. For a space  $X$  and an integer  $n \geq 2$ , we consider the following conditions ( $\|\cdot\|$  denotes the supremum norm):

- (\*) $_n$  For each bounded continuous function  $f: X \rightarrow \mathbf{C}$  and each  $\varepsilon > 0$ , there exists a continuous function  $g: X \rightarrow \mathbf{C}$  such that  $\|f - g^n\| < \varepsilon$ .
- (\*\*) $_n$  For each bounded continuous function  $f: X \rightarrow \mathbf{C}$  and each  $\varepsilon > 0$ , there exist bounded continuous functions  $g_1, \dots, g_n: X \rightarrow \mathbf{C}$  such that  $f = \prod_{i=1}^{i=n} g_i$  and  $\|g_i - g_j\| < \varepsilon$  for each  $i, j$ .

**Definition 7.** *We say that the space  $C^*(X)$  of all bounded complex-valued functions on  $X$  has the approximate  $n$ -th root property if  $X$  satisfies condition (\*) $_n$ .*

We have the following result established in [12] (see [2] for another characterization of function algebras with the approximate  $n$ -th root property):

**Theorem 21.** *Let  $X$  be a compactum with  $\dim X \leq 1$  and  $n$  a positive integer. Then the following conditions are equivalent:*

- (1)  $C(X)$  has the approximate  $n$ -th root property.
- (2)  $X$  satisfies condition  $(**)_n$ .
- (3) the first Čech cohomology  $\check{H}^1(X; \mathbf{Z})$  is  $n$ -divisible, that is, each element of  $\check{H}^1(X; \mathbf{Z})$  is divided by  $n$ .

Let  $\mathcal{A}(n)$  denote the class of all completely regular spaces satisfying condition  $(*)_n$  and  $\mathcal{A}_1(n)$  is the subclass of  $\mathcal{A}(n)$  consisting of spaces  $X$  with  $\dim X \leq 1$ .

## 4.1 Properties of the classes $\mathcal{A}(n)$ and $\mathcal{A}_1(n)$

Everywhere in this subsection  $\mathcal{K}_n$  will denote one of the classes  $\mathcal{A}(n)$  and  $\mathcal{A}_1(n)$ ,  $n \geq 2$ .

**Proposition 22.** [7] *A compactum  $X$  belongs to  $\mathcal{K}_n$  if and only if  $X$  can be represented as the limit space of an inverse system  $\{X_\alpha, p_\alpha^\beta : \alpha, \beta \in A\}$  of metrizable compacta with each  $X_\alpha \in \mathcal{K}_n$ .*

We say that a class of spaces  $\mathcal{P}$  is *factorizable* if, for every map  $f: X \rightarrow Y$  of a compactum  $X \in \mathcal{P}$ , there exists a compactum  $Z \in \mathcal{P}$  of weight  $w(Z) \leq w(Y)$  and maps  $\pi: X \rightarrow Z$  and  $p: Z \rightarrow Y$  such that  $f = p \circ \pi$ .

**Proposition 23.** [7] *Any one of the classes  $\mathcal{A}(n)$  and  $\mathcal{A}_1(n)$ ,  $n \geq 2$ , is factorizable.*

**Corollary 9.** [7] *Every space  $X \in \mathcal{K}_n$  has a compactification  $Z \in \mathcal{K}_n$  with  $w(Z) = w(X)$ .*

*Proof.* Obviously,  $X \in \mathcal{K}_n$  implies  $\beta X \in \mathcal{K}_n$ . Let  $Y$  be an arbitrary compactification of  $X$  with  $w(Y) = w(X)$  and let  $f: \beta X \rightarrow Y$  be the extension of the identity on  $X$ . Then, by Proposition 23, there exists a compactum  $Z \in \mathcal{K}_n$  and maps  $g: \beta X \rightarrow Z$  and  $h: Z \rightarrow Y$  with  $h \circ g = f$  and  $w(Z) = w(X)$ . It remains only to observe that  $Z$  is a compactification of  $X$ .  $\square$

The final result in this subsection implies that each  $\mathcal{K}_n$ ,  $n \geq 2$ , has an universal element. Recall that a map  $h: X \rightarrow Y$  is said to be *invertible* for a given class  $\mathcal{P}$  (or simply,  *$\mathcal{P}$ -invertible*) if for every map  $g: Z \rightarrow Y$  with  $Z \in \mathcal{P}$  there exists a map  $\bar{g}: Z \rightarrow X$  such that  $g = h \circ \bar{g}$ .

**Theorem 24.** [7] *Let  $n$  be a positive integer and let  $\mathcal{K}_n$  denote one of the classes  $\mathcal{A}(n)$  and  $\mathcal{A}_1(n)$ . Then for every cardinal  $\tau \geq \omega$  there exists a compactum  $X_\tau \in \mathcal{K}_n$  of weight  $\leq \tau$  and a  $\mathcal{K}_n$ -invertible map  $f_{\mathcal{K}}: X_\tau \rightarrow \mathbf{I}^\tau$ .*

*Proof.* We consider the family of all maps  $\{h_\alpha: Y_\alpha \rightarrow \mathbf{I}^\tau\}_{\alpha \in \Lambda}$  such that each  $Y_\alpha$  is a closed subset of  $\mathbf{I}^\tau$  with  $Y_\alpha \in \mathcal{K}_n$ . Let  $Y$  be the disjoint sum of all  $Y_\alpha$  and the map  $h: Y \rightarrow \mathbf{I}^\tau$  coincides with  $h_\alpha$  on every  $Y_\alpha$ . We extend  $h$  to a map  $\bar{h}: \beta Y \rightarrow \mathbf{I}^\tau$ . Since  $\beta Y \in \mathcal{K}_n$ , by Proposition 23, there exists a compactum  $X$  of weight  $\leq \tau$  and maps  $p: \beta Y \rightarrow X$  and  $f: X \rightarrow \mathbf{I}^\tau$  such that  $X \in \mathcal{K}_n$  and  $f \circ p = \bar{h}$ .

Let us show that  $f$  is  $\mathcal{K}_n$ -invertible. Take a space  $Z \in \mathcal{K}_n$  and a map  $g: Z \rightarrow \mathbf{I}^\tau$ . Considering  $\beta Z$  and the extension  $\bar{g}: \beta Z \rightarrow \mathbf{I}^\tau$  of  $g$ , we can assume that  $Z$  is compact. We also can assume that the weight of  $Z$  is  $\leq \tau$  (otherwise we apply again Proposition 23 to find a compact space  $T \in \mathcal{K}_n$  of weight  $\leq \tau$  and maps  $g_1: Z \rightarrow T$  and  $g_2: T \rightarrow \mathbf{I}^\tau$  with  $g_2 \circ g_1 = g$ , and then consider the space  $T$  and the map  $g_2$  instead, respectively, of  $Z$  and  $g$ ). Therefore, without loss of generality, we can assume that  $Z$  is a closed subset of  $\mathbf{I}^\tau$ . According to the definition of  $Y$  and the map  $h$ , there is an index  $\alpha \in \Lambda$  such that  $Z = Y_\alpha$  and  $g = h_\alpha$ . The restriction  $p|_Z: Z \rightarrow X$  is a lifting of  $g$ , i.e.  $f \circ (p|_Z) = g$ .  $\square$

**Corollary 10.** [7] *Let  $\mathcal{K}_n$  be one of the classes  $\mathcal{A}(n)$  and  $\mathcal{A}_1(n)$ . Then for every  $\tau \geq \omega$  there exists a compactum  $X \in \mathcal{K}_n$  of weight  $\tau$  which contains every space from  $\mathcal{K}_n$  of weight  $\leq \tau$ .*

## 4.2 The approximate $n$ -root property of $C^*$ -algebras

It is easily seen that the modification of condition  $(*)_n$ , obtained by requiring both  $f$  and  $g$  to be of norm  $\leq 1$ , is equivalent to  $(*)_n$ . This observation leads us to consider the following classes of general (non-commutative)  $C^*$ -algebras.

**Definition 8.** [7] *We say that a  $C^*$ -algebra  $X$  satisfies the approximation  $n$ -th root property if for every  $a \in X$  with  $\|a\| \leq 1$  and every  $\varepsilon > 0$  there exists  $b \in X$  such that  $\|b\| \leq 1$  and  $\|a - b^n\| < \varepsilon$ .*

The class of all  $C^*$ -algebras with the approximate  $n$ -th root property is denoted by  $\mathcal{AP}(n)$ . Let  $\mathcal{AP}_1(n)$  be the subclass of  $\mathcal{AP}(n)$  consisting of  $C^*$ -algebras of bounded rank  $\leq 1$  (see Definition 4). We also consider the class  $\mathcal{HP}(n)$  of  $C^*$ -algebras  $X$  with the following property: for every invertible element  $a \in X$  with  $\|a\| \leq 1$  and every  $\varepsilon > 0$  there exists  $b \in X$  such that  $\|b\| \leq 1$  and  $\|a - b^n\| < \varepsilon$ . In the sequel,  $\mathcal{AP}(n)_s$  denotes the class of all separable  $C^*$ -algebras from  $\mathcal{AP}(n)$ . The notations  $\mathcal{AP}_1(n)_s$  and  $\mathcal{HP}(n)_s$  have the same meaning.

In this subsection we investigate the behavior of the classes  $\mathcal{AP}(n)$ ,  $\mathcal{AP}_1(n)$  and  $\mathcal{HP}(n)$  with respect to direct systems and then use the result to prove the existence of universal elements in the classes  $\mathcal{AP}(n)_s$ ,  $\mathcal{AP}_1(n)_s$  and  $\mathcal{HP}(n)_s$ . The results from the present subsection are actually a non-commutative analogues of the corresponding result from the previous subsection.

**Proposition 25.** [7] *Let  $\mathcal{K}$  be one of the classes  $\mathcal{AP}(n)$ ,  $\mathcal{AP}_1(n)$  and  $\mathcal{HP}(n)$ . If  $X$  is the limit of a direct system  $\mathcal{S} = \{X_\alpha, i_\alpha^\beta, A\}$  consisting of unital  $C^*$ -algebras and unital  $*$ -inclusions with  $X_\alpha \in \mathcal{K}$  for each  $\alpha$ , then  $X \in \mathcal{K}$ .*

As in the commutative case (see Proposition 22), we can establish a decomposition theorem for the classes  $\mathcal{AP}(n)$ ,  $\mathcal{AP}_1(n)$  and  $\mathcal{HP}(n)$ .

**Proposition 26.** [7] *Let  $\mathcal{K}$  be one of the classes  $\mathcal{AP}(n)$ ,  $\mathcal{AP}_1(n)$  and  $\mathcal{HP}(n)$ . The following conditions are equivalent for any unital  $C^*$ -algebra  $X$ :*

- (1)  $X \in \mathcal{K}$ .
- (2)  $X$  can be represented as the direct limit of a direct  $C_\omega^*$ -system  $\{X_\alpha, i_\alpha^\beta, A\}$  satisfying the following properties:
  - (a) The indexing set  $A$  is cofinal and  $\omega$ -closed in the  $\omega$ -complete set  $\exp_\omega Y$  for some (any) dense subset  $Y$  of  $X$  such that  $|Y| = d(X)$ .
  - (b)  $X_\alpha$  is a (separable)  $C^*$ -subalgebra of  $X$  with  $X_\alpha \in \mathcal{K}$ ,  $\alpha \in A$ .

Finally, we can establish the existence of universal elements in the classes  $\mathcal{AP}(n)_s$ ,  $\mathcal{AP}_1(n)_s$  and  $\mathcal{HP}(n)_s$ .

**Theorem 27.** [7] *Let  $n \geq 2$  be a positive integer and let  $\mathcal{K}$  be one of the classes  $\mathcal{AP}(n)$ ,  $\mathcal{AP}_1(n)$  and  $\mathcal{HP}(n)$ . Then there exists a  $\mathcal{K}$ -invertible unital  $*$ -homomorphism  $p: C^*(\mathbf{F}_\infty) \rightarrow Z_{\mathcal{K}}$  of  $C^*(F_\infty)$  to a separable unital  $C^*$ - algebra  $Z_{\mathcal{K}} \in \mathcal{K}$ .*

*Proof.* Let  $\mathcal{B} = \{f_t: C^*(\mathbf{F}_\infty) \rightarrow X_t: t \in T\}$  denote the set of all unital  $*$ -homomorphisms on  $C^*(\mathbf{F}_\infty)$  such that  $X_t \in \mathcal{K}$ . We claim that the product  $\prod\{X_t: t \in T\}$  belongs to  $\mathcal{K}$ . This is obviously true if  $\mathcal{K}$  is either  $\mathcal{AP}(n)$  or  $\mathcal{HP}(n)$ . Since the bounded rank of this product is  $\leq 1$  provided each  $X_t$  is of bounded rank  $\leq 1$  (see Proposition 12), the claim holds for the class  $\mathcal{AP}_1(n)$  as well. The  $*$ -homomorphisms  $f_t$ ,  $t \in T$ , define the unital  $*$ -homomorphism  $f: C^*(\mathbf{F}_\infty) \rightarrow \prod\{X_t: t \in T\}$  such that  $\pi_t \circ f = f_t$  for each  $t \in T$ , where  $\pi_t: \prod\{X_t: t \in T\} \rightarrow X_t$  denotes the canonical projection  $*$ -homomorphism onto  $X_t$ . By Proposition 26,  $\prod\{X_t: t \in T\}$  can be represented as the limit of the  $C_\omega^*$ -system  $\mathcal{S} = \{C_\alpha, i_\alpha^\beta, A\}$  such that  $C_\alpha$  is a separable unital  $C^*$ -algebra with  $C_\alpha \in \mathcal{K}$  for each  $\alpha \in A$ . Suppressing the injective unital  $*$ -homomorphisms  $i_\alpha^\beta: C_\alpha \rightarrow C_\beta$ , we may assume, for notational simplicity, that  $C_\alpha$ 's are unital  $C^*$ -subalgebras of  $\prod\{X_t: t \in T\}$ . Let  $\{a_k: k \in \omega\}$

be a countable dense subset of  $C^*(\mathbf{F}_\infty)$ . For each  $k \in \omega$  there exists an index  $\alpha_k \in A$  such that  $f(a_k) \in C_{\alpha_k}$ . Since  $A$  is  $\omega$ -complete, there exists an index  $\alpha_0 \in A$  such that  $\alpha_0 \geq \alpha_k$  for each  $k \in \omega$ . Then  $f(a_k) \in C_{\alpha_k} \subseteq C_{\alpha_0}$  for each  $k \in \omega$ . This observation coupled with the continuity of  $f$  guarantees that  $f(C^*(\mathbf{F}_\infty)) = f(\text{cl}\{a_k : k \in \omega\}) \subseteq \text{cl}\{f(\{a_k : k \in \omega\})\} \subseteq \text{cl}C_{\alpha_0} = C_{\alpha_0}$ .

Let  $Z_{\mathcal{K}} = C_{\alpha_0}$  and define the unital  $*$ -homomorphism  $p: C^*(\mathbf{F}_\infty) \rightarrow Z_{\mathcal{K}}$  as  $f$ , regarded as a homomorphism of  $C^*(\mathbf{F}_\infty)$  into  $Z_{\mathcal{K}}$ . Note that  $f = i \circ p$ , where  $i: Z_{\mathcal{K}} = C_{\alpha_0} \hookrightarrow \prod\{X_t : t \in T\}$  stands for the inclusion.

By construction, we see  $Z_{\mathcal{K}} \in \mathcal{K}$ . Let us show that  $p: C^*(\mathbf{F}_\infty) \rightarrow Z_{\mathcal{K}}$  is  $\mathcal{K}$ -invertible. For a given unital  $*$ -homomorphism  $g: C^*(\mathbf{F}_\infty) \rightarrow X$ , where  $X$  is a unital  $C^*$ -algebra with  $X \in \mathcal{K}$ , we need to establish the existence of a unital  $*$ -homomorphism  $h: Z_{\mathcal{K}} \rightarrow X$  such that  $g = h \circ p$ . Indeed, by definition of the set  $\mathcal{B}$ , we conclude that  $g = f_t: C^*(\mathbf{F}_\infty) \rightarrow X_t = X$  for some index  $t \in T$ . Observe that  $g = f_t = \pi_t \circ f = \pi_t \circ i \circ p$ . This allows us to define the required unital  $*$ -homomorphism  $h: Z_{\mathcal{K}} \rightarrow X$  as the composition  $h = \pi_t \circ i$ . Hence,  $p$  is  $\mathcal{K}$ -invertible.  $\square$

It is well-known that every separable  $C^*$ -algebra is a surjective image of  $C^*(\mathbf{F}_\infty)$ . Therefore, if  $\mathfrak{R}$  is a class of separable  $C^*$ -algebras and  $p: C^*(\mathbf{F}_\infty) \rightarrow Y_{\mathfrak{R}}$  is a  $\mathfrak{R}$ -invertible homomorphism with  $Y_{\mathfrak{R}} \in \mathfrak{R}$ , then  $Y_{\mathfrak{R}}$  is universal for the class  $\mathfrak{R}$ . Hence, Theorem 27 implies the following

**Corollary 11.** [7] *Each of the classes  $\mathcal{AP}(n)_s$ ,  $\mathcal{AP}_1(n)_s$  and  $\mathcal{HP}(n)_s$  has a universal element.*

Let us note that there exists a non-commutative  $C^*$ -algebra which belongs to any one of the classes  $\mathcal{AP}(n)_s$ ,  $\mathcal{AP}_1(n)_s$  and  $\mathcal{HP}(n)_s$ . Indeed, let  $X = M(m)$  be the algebra of all  $m \times m$  complex matrixes, where  $m \geq 2$  is a fixed integer. By [3], the bounded rank of  $X$  is 0. Moreover, using the canonical Jordan form representation, one can show that if  $A \in X$  and  $n \geq 2$ , then  $A$  can be approximated by a matrix  $B \in X$  with  $C^n = B$  for some  $C \in X$ . Hence,  $X$  is a common element of  $\mathcal{AP}(n)_s$ ,  $\mathcal{AP}_1(n)_s$  and  $\mathcal{HP}(n)_s$ . This implies that the universal elements of  $\mathcal{AP}(n)_s$ ,  $\mathcal{AP}_1(n)_s$  and  $\mathcal{HP}(n)_s$  are also non-commutative.

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