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On homologically locally connected spaces

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We provide some properties and characterizations of homologically UV^n-maps and lc^n_G-spaces. We show that there is a parallel between recently introduced by Cauty [3] algebraic ANR's and homologically lc^n_G-metric spaces, and this parallel is similar to the parallel between ordinary ANR's and LC^n-metric spaces. We also show that there is a similarity between the properties of LC^n-spaces and lc^n_G-spaces. Some open questions are raised.

1. Introduction

All spaces are assumed to be paracompact and all single-valued maps are continuous. Everywhere below singular homology $H_n(X;G)$, reduced in dimension 0, with a coefficient group G is considered. By default, if not explicitly stated otherwise, G is a commutative ring with a unit e. The following homology counterpart of the well known notion of a UV^n-set was introduced in [12]: A closed set $A \subset X$ is said to be *homologically* $UV^n(G)$ if every neighborhood U of A in X contains another neighborhood V such that the inclusion $V \hookrightarrow U$ induces trivial homomorphisms $H_k(V;G) \to H_k(U;G)$ (notation $A \overset{H_k}{\hookrightarrow} X$) for all $k \leq n$. Obviously, every UV^n-subset of X is $UV^n(G)$ for any G (below we call the UV^n-sets *homotopically* UV^n in order to distinguish them from homologically $UV^n(G)$-sets). It can be shown that if A is homologically $UV^n(\mathbb{Z})$,

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where $\mathbb{Z}$ is the group of the integers, then it is $UV^n(G)$ for any G (see, for example [11, Proposition 4.8]). Moreover, following the proof of Proposition 7.1.3 from [10], one can show that A is homologically $UV^n(G)$ in a given metric ANR-space X if and only if it is homologically $UV^n(G)$ in any metric ANR-space that contains A as a closed set. We also say that a surjective map $f : X \to Y$ is *homologically* $UV^n(G)$ if all fibers $f^{-1}(y)$ of f are homologically $UV^n(G)$-subsets of X. In particular, X is called *homologically locally connected in dimension n with respect to* G if the identity map on X is an $UV^n(G)$-map, notation X is lc^n_G. In this paper we provide some properties and characterizations of homologically UV^n-maps and lc^n_G-spaces. We show that there is a parallel between recently introduced by Cauty [3] algebraic ANR's and homologically lc^n_G-metric spaces, and this parallel is similar to the parallel between ordinary ANR's and LC^n-metric spaces. We also show that there is an analogy between the properties of LC^n-spaces and lc^n_G-spaces.

More precisely, Section 2 contains some definitions and properties of algebraic ANR's about the existence and extensions of homotopies between close chain morphisms. Section 3 is devoted to closed homotopically UV^n-surjections. We show that the well known properties of closed UV^n-maps (concerning extensions of partial realizations and approximate lifting of maps) have chain morphisms' analogues for homologically UV^n-maps. We also provide some Dugundji type extensions and approximate extensions of chain morphisms. Obviously, all results established in Section 3 are valid and for lc^n_G-spaces. In Section 4 we characterize lc^n_G-spaces in terms of existence of homotopies between any close chain morphisms (Proposition 4.1). The following two questions are discussed in that section:

Question 1.1. Is any metric n-dimensional lc^n_G-space an absolute neighborhood algebraic retract?

Question 1.2. Let $f : Y \to X$ be a closed $UV^n(G)$-surjection between metric spaces. Is it true that X is an lc^n_G-space?

In case of LC^n-spaces and UV^n-maps the above two questions have positive answers. We still do not know the answer to these questions, but we introduce approximate versions of algebraic ANR-spaces and lc^n_G-spaces and show that any space satisfying the hypotheses of Question 1.1 (resp., Question 1.2) is an approximate ANR (resp., approximate lc^n_G).

The main technical tool are chain morphisms between chain complexes. By a chain complex $C = \{C_k\}_{k\geq 0}$ we mean a sequence of G-modules C_k and boundary homomorphisms $\partial_k : C_k \to C_{k-1}$ such that all compositions $\partial_k \circ \partial_{k+1}$ are trivial. A chain morphism $\varphi : C \to C'$ between two chain complexes is a sequence of homomorphisms $\varphi_k : C_k \to C'_k$ such that $\varphi_k \circ \partial_{k+1} = \partial'_{k+1} \circ \varphi_{k+1}$, and φ_0 commutes with the augmentations in C and C'. We consider two types of chain complexes, oriented chain complexes $C(K) = \{C_k(K;G)\}_{k\geq 0}$, where K is a simplicial complex, and singular chain complexes $S(X;G) = \{S_k(X;G)\}_{k\geq 0}$, where X is a given topological space and $S_k(X;G)$ is the group of all singular k-chains with coefficients from G. If $\sigma : \triangle^k \to X$ is a singular k-simplex ($\triangle^k$ is the standard k-simplex), we denote by $|\sigma|$ the *carrier* $\sigma(\triangle^k)$ of σ. Similarly, we put $|c| = \bigcup_i |\sigma_i|$ for any chain $c \in S_k(X;G)$, where $c = \sum_i g_i\sigma_i$ is the irreducible representation of c. We agree that $|c| = \varnothing$ if $c = 0$. For any open cover $\mathcal{U}$ of X let $S(X,\mathcal{U};G)$ stand for the subgroup of $S(X;G)$ generated by all singular simplexes σ with $|\sigma| \subset U$ for some $U \in \mathcal{U}$. Note that every map $f : X \to Y$ induces a chain homomorphism $f_\sharp : S(X;G) \to S(Y;G)$ such that $f_\sharp(\sigma) = f \circ \sigma$ for every singular simplex $\sigma \in S(X;G)$.

By a sub-complex of $S(X;G)$ we mean any sub-family $\mathrm{S} \subset S(X;G)$ such that $\partial c \in \mathrm{S}$ for any chain $c \in \mathrm{S}$. For example, $S(X,\mathcal{U};G)$ and $S(A;G)$ are sub-complexes of $S(X;G)$ for any open cover $\mathcal{U}$ of X and any subset $A \subset X$. We also consider the n-dimensional sub-complex $S^{(n)}(X;G) = \{S_k(X;G)\}_{k\leq n}$ of $S(X;G)$. A face τ of a singular simplex $\sigma : \triangle^k \to X$ from $S(X;G)$ is the restriction of the map σ onto a face of the standard simplex $\triangle^k$. In particular, a vertex of σ is the singular 0-simplex $f_v : v \to X$ with v being a vertex of $\triangle^k$. If Y is a space and $\mathcal{U}$ is an open cover of Y, we say two chain morphisms $\varphi, \psi : S(X;G) \to S(Y;G)$

(resp., $\varphi, \psi : C(K;G) \to S(Y;G)$) are *$\mathcal{U}$-close* provided for each simplex $\sigma \in S(Y;G)$ (resp., $\sigma \in K$) there exists $U_\sigma \in \mathcal{U}$ with $|\varphi(\tau)| \cup |\psi(\tau)| \subset U_\sigma$ for all faces τ of σ. If φ is $\mathcal{U}$-close to the trivial morphism, then φ is said to be *$\mathcal{U}$-small*. We also say that φ is *correct* provided $\varphi(\sigma)$ (resp., $\varphi(v)$) is a singular 0-simplex in $S(Y;G)$ for each 0-simplex $\sigma \in S(X;G)$ (resp., for each vertex v of K).

2. Algebraic ANR's

Let K be a simplicial complex, X be a space and $\mathcal{U}$ an open cover of X. According to [3], a chain morphism $\varphi : C(L;G) \to S(X;G)$ is a *partial algebraic realization of* $C(K;G)$ in $\mathcal{U}$ provided L is a sub-complex of K containing the vertex set $K^{(0)}$ of K and for every simplex $\sigma \in K$ there exists $U_\sigma \in \mathcal{U}$ such that $|\varphi(\tau)| \subset U_\sigma$ for all faces τ of σ with $\tau \in L$. If $L = K$, then φ is called a *full algebraic realization of $C(K;G)$ in $\mathcal{U}$*. Obviously, $\varphi : C(K;G) \to S(X;G)$ is a full algebraic realization of $C(K;G)$ in $\mathcal{U}$ if and only if φ is $\mathcal{U}$-small.

Cauty [3] introduced an important class of metric spaces more general than metric ANR's.

Definition 2.1. [3] A metric space X is said to be an *absolute neighborhood algebraic G-retract* (briefly, *algebraic ANR_G*) if for every open cover $\mathcal{U}$ of X there is another open cover $\mathcal{V}$ refining $\mathcal{U}$ such that, for any simplicial complex K, any correct partial algebraic realization of $C(K;G)$ in $\mathcal{V}$ extends to a full algebraic realization of $C(K;G)$ in $\mathcal{U}$. Any acyclic algebraic ANR_G is called an *algebraic absolute G-retract* (br., *algebraic AR_G*).

Here are some properties of algebraic ANR_G's.

Definition 2.2. [3] A closed subset A of a space X is said to be a *neighborhood algebraic G-retract* of X if there are a neighborhood U of A in X, an open cover $\mathcal{U}$ of U and a chain morphism $\mu : S(U,\mathcal{U};G) \to S(A;G)$ such that:

(i) $\mu(c) = c$ for all $c \in S(A;G) \cap S(U,\mathcal{U};G)$;
(ii) For every $x \in A$ and its neighborhood V_x in A there exists a neighborhood $W_x \subset U$ with $\mu\big(S(W_x;G) \cap S(U,\mathcal{U};G)\big) \subset S(V_x;G)$.

If $U = X$ and $\mathcal{U} = \{X\}$, A is called an *algebraic G-retract of X*.

Theorem 2.3. *[3] A metric space X is an algebraic ANR_G if and only if X is a neighborhood algebraic G-retract of every metric space containing X as a closed set.*

Next result is an analogue of the ANR's properties concerning close maps and small homotopies. Recall that if $\varphi, \phi : S(Y;G) \to S(X;G)$ are two chain morphisms, then $D : S(Y;G) \to S(X;G)$ is a chain homotopy between φ and ψ provided there exists a sequence of homomorphisms $D_k : S_k(Y;G) \to S_{k+1}(X;G)$, $k \geq 0$, such that $\partial D_0(\sigma) = \varphi(\sigma) - \psi(\sigma)$ for any singular 0-simplex $\sigma \in S_0(Y;G)$ and $\partial D_k(\sigma) = \varphi(\sigma) - \psi(\sigma) - D_{k-1}(\partial\sigma)$ provided $\sigma \in S_k(Y;G)$ is a singular k-simplex with $k \geq 1$. We say that D is *$\mathcal{U}$-small*, where $\mathcal{U}$ is an open cover of X if for any $\sigma \in S(Y;G)$ there exists $U_\sigma \in \mathcal{U}$ such that $|D(\tau)| \cup |\varphi(v)| \cup |\psi(v)| \subset U_\sigma$ for all singular simplexes $\tau \in S(Y;G)$, which are faces τ of σ, and all vertexes v of σ.

Proposition 2.4. *If X is a metric algebraic ANR_G , then for any open cover $\mathcal{U}$ of X there is an open cover $\mathcal{V}$ of X refining $\mathcal{U}$ such that, for any two correct $\mathcal{V}$-close chain morphisms $\varphi, \phi : S(Y;G) \to S(X;G)$, where Y is an arbitrary space, and any $\mathcal{V}$-small chain homotopy $D : S(A;G) \to S(X;G)$ between $\varphi|S(A;G)$ and $\phi|S(A;G)$ with A being a closed set in Y, there exists a $\mathcal{U}$-small chain homotopy $\Phi : S(Y;G) \to S(X;G)$ between φ and ϕ such that $\Phi(c) = D(c)$ for all $c \in S(A;G)$.*

Proof. Suppose X is an algebraic ANR_G and $\mathcal{U}$ is an open cover of X. Embed X as closed subset of a normed space E and let $\mu : S(W, \mathcal{W}; G) \to S(X; G)$ be an algebraic neighborhood G-retraction, where W is a neighborhood of X in E and $\mathcal{W}$ an open cover of W. For every $x \in X$ choose $U_x \in \mathcal{U}$ and $W_x \in \mathcal{W}$ containing x. We can assume that U_x and W_x satisfy the inclusion $\mu(S(W_x; G)) \subset S(U_x; G)$ for every $x \in X$. Let V_x be a convex open subset of W such that $x \in V_x \cap X \subset U_x$ and $V_x \subset W_x$. Denote $T = \bigcup_{x \in X} V_x$, $\widetilde{\mathcal{V}}_1 = \{V_x : x \in X\}$ and take an open cover $\widetilde{\mathcal{V}}$ of T consisting of convex sets such that $\widetilde{\mathcal{V}}$ is a star refinement of $\widetilde{\mathcal{V}}_1$. Let $\mathcal{V} = \{\widetilde{V} \cap X : \widetilde{V} \in \widetilde{\mathcal{V}}\}$.

Now, let $\varphi, \phi : S(Y; G) \to S(X; G)$ be two correct $\mathcal{V}$-close chain morphisms and $D : S(A; G) \to S(X; G)$ be a $\mathcal{V}$-small chain homotopy between $\varphi|S(A; G)$ and $\phi|S(A; G)$, where Y is a space and $A \subset Y$ is closed.

Everywhere below we say that a set $B \subset T$ is $\widetilde{\mathcal{V}}_1$-*small* if B is contained in some element of $\widetilde{\mathcal{V}}_1$.

Claim 1. *For every singular simplex $\sigma \in S(Y; G)$ there is a non-empty convex $\widetilde{\mathcal{V}}_1$-small set $\Lambda_\sigma \subset T$ such that:*

- *$\Lambda_\tau \subset \Lambda_\sigma$ and $|\varphi(\tau)| \cup |\phi(\tau)| \subset \Lambda_\sigma$ for all faces τ of σ;*
- *If $\sigma \in S(A; G)$, then Λ_σ contains also $|D(\tau)|$, τ is a face of σ.*

Indeed, since φ and ϕ are $\mathcal{V}$-close, for every singular simplex $\sigma \in S(Y; G)$ there exists $\widetilde{V}_\sigma \in \widetilde{\mathcal{V}}$ such that $|\varphi(\tau)| \cup |\phi(\tau)| \subset \widetilde{V}_\sigma$ for all singular simplexes $\tau \in S(Y; G)$ which are faces of σ. In particular, $\widetilde{V}_\sigma$ contains the non-empty sets $|\varphi(v)| \cup |\phi(v)|$, v is a vertex of σ (recall that $|\varphi(v)|$ and $|\phi(v)|$ are both non-empty because φ and ϕ are correct). Similarly, using that D is $\mathcal{V}$-small, for any $\sigma \in S(A; G)$ we can find $\widetilde{V}^D_\sigma \in \widetilde{\mathcal{V}}$ containing all $|D(\tau)| \cup |\varphi(v)| \cup |\phi(v)|$, where τ is a face of σ and v is a vertex of σ. On the other hand, $|\varphi(v)| \cup |\phi(v)| \subset \widetilde{V}_\sigma$. Therefore, $\widetilde{V}_\sigma \cap \widetilde{V}^D_\sigma \neq \varnothing$ for any singular simplex $\sigma \in S(A; G)$. Next, for any singular simplex $\sigma \in S(Y; G)$ let

$$\Gamma_\sigma = \bigcap\{\widetilde{V}_s \in \widetilde{\mathcal{V}} : \sigma \text{ is a face of a singular simplex } s \in S(Y; G)\}$$

and

$$\Gamma^D_\sigma = \bigcap\{\widetilde{V}^D_s \in \widetilde{\mathcal{V}} : s \in S(A; G) \text{ is a simplex and } \sigma \text{ is a face of } s\}.$$

Obviously, $\Gamma_\sigma \neq \varnothing$ for any $\sigma \in S(Y; G)$, and $\Gamma^D_\sigma \neq \varnothing$ provided $\sigma \in S(A; G)$. We assume that $\Gamma^D_\sigma = \varnothing$ if $\sigma \notin S(A; G)$. Consider the sets $\Omega_\sigma = \Gamma_\sigma \bigcup\{\Gamma^D_\tau : \tau \text{ is a face of } \sigma\}$, $\sigma \in S(Y; G)$. Observe that if a singular simplex $\tau \in S(A; G)$ is a face of some $\sigma \in S(Y; G)$, then $\Gamma_\sigma \cap \Gamma^D_\tau \neq \varnothing$ because this set contains all $|\varphi(v)| \cup |\phi(v)|$ with v being a vertex of τ. Moreover, each of the sets Γ_σ and Γ^D_τ are $\widetilde{\mathcal{V}}$-small sets. So, $\Omega_\sigma \subset \mathrm{St}(\Gamma_\sigma, \widetilde{\mathcal{V}})$, $\sigma \in S(Y; G)$, where $\mathrm{St}(\Gamma_\sigma, \widetilde{\mathcal{V}})$ denotes the star of the set Γ_σ with respect to $\widetilde{\mathcal{V}}$. Thus, Ω_σ is a $\widetilde{\mathcal{V}}_1$-small subset of T (recall that $\widetilde{\mathcal{V}}$ is a star refinement of $\widetilde{\mathcal{V}}_1$). This implies that the convex hull Λ_σ of Ω_σ is a convex non-empty $\widetilde{\mathcal{V}}_1$-small subset of T. It follows from the definitions of Γ_σ and Γ^D_σ that Ω_σ contains Ω_τ for all faces τ of σ, so $\Lambda_\tau \subset \Lambda_\sigma$. Because for any singular simplex $\sigma \in S(Y; G)$ (resp., $\sigma \in S(A; G)$) and a face τ of σ the set Γ_σ contains $|\varphi(\tau)| \cup |\phi(\tau)|$ (resp., Γ^D_σ contains $|D(\tau)|$), the sets Λ_σ have the same property. This completes the proof of Claim 1.

We are going to construct first a chain homotopy $\widetilde{D} : S(Y; G) \to S(T, \widetilde{\mathcal{V}}_1; G)$ between φ and ϕ such that $\widetilde{D}(c) = D(c)$ for all $c \in S(A; G)$. To this end, observe that both $\varphi(\sigma)$ and $\phi(\sigma)$ are singular 0-simplexes in $S(X; G)$ for any singular 0-simplex $\sigma \in S(Y; G)$. Then $\varphi(\sigma) - \phi(\sigma)$ is a 0-cycle in $S_0(\Lambda_\sigma; G)$. So, there is a singular 1-chain $c^1_\sigma \in S_1(\Lambda_\sigma; G)$ with $\partial c^1_\sigma = \varphi(\sigma) - \phi(\sigma)$ (recall that $H_0(\Lambda_\sigma; G) = 0$ because Λ_σ is a convex set). We define $D'_0(\sigma) = c^1_\sigma$ if $\sigma \notin S_0(A; G)$ and $D'_0(\sigma) = D_0(\sigma)$ if $\sigma \in S_0(A; G)$. Thus, $|D'_0(\sigma)| \subset \Lambda_\sigma$ for all singular 0-simplexes σ of $S(Y; G)$. So, we can extend D'_0 linearly to a homomorphism $\widetilde{D}_0 : S_0(Y; G) \to S_1(T, \widetilde{\mathcal{V}}_1; G)$.

Assume that homomorphisms $\widetilde{D}_k : S_k(Y; G) \to S_{k+1}(T, \widetilde{\mathcal{V}}_1; G)$ were defined for all $k \le m$ such that:

(1) $\partial\widetilde{D}_k(c^k) = \varphi(c^k) - \phi(c^k) - \widetilde{D}_{k-1}(\partial c^k)$ for all $c^k \in S_k(Y;G)$;
(2) $\widetilde{D}_k(c^k) = D_k(c^k)$ for all $c^k \in S(A;G)$;
(3) $|\widetilde{D}_i(\tau)| \subset \Lambda_\sigma$ for any singular k-simplex $\sigma \in S_k(Y;G)$ and any singular i-simplex τ, which is a face of σ.

To define $\widetilde{D}_{m+1}$, let σ be a singular $(m+1)$-simplex in $S_{m+1}(Y;G)$. Then $|\varphi(\sigma)| \cup |\phi(\sigma)| \subset \Lambda_\sigma$ and, according to (3), Λ_σ contains also $|\widetilde{D}_m(\partial\sigma)|$. Therefore, the chain $\gamma_\sigma = \varphi(\sigma) - \phi(\sigma) - \widetilde{D}_m(\partial\sigma)$ belongs to $S_{m+1}(\Lambda_\sigma;G)$. It is easily seen that γ_σ is a cycle, and since Λ_σ is convex, there is a chain $c_\sigma^{m+2} \in S_{m+2}(\Lambda_\sigma;G)$ with $\partial c_\sigma^{m+2} = \gamma_\sigma$. We define $D'_{m+1}(\sigma) = c_\sigma^{m+2}$ if $\sigma \notin S_{m+1}(A;G)$ and $D'_{m+1}(\sigma) = D(\sigma)$ if $\sigma \in S_{m+1}(A;G)$. According to the properties of Λ_σ, we always have $|D'_{m+1}(\sigma)| \subset \Lambda_\sigma$, $\sigma \in S_{m+1}(Y;G)$. Therefore, we can extend D'_{m+1} to a homomorphism $\widetilde{D}_{m+1} : S_{m+1}(Y;G) \to S_{m+2}(T,\widetilde{\mathcal{V}}_1;G)$. Obviously, $\widetilde{D}_{m+1}$ satisfies conditions $(1)-(3)$. This completes the construction of $\widetilde{D}$.

Finally, note that $S(T,\widetilde{\mathcal{V}}_1;G) \subset S(W,\mathcal{W};G)$ because $T \subset W$ and $\widetilde{\mathcal{V}}_1$ refines $\mathcal{W}$. Hence, for any $k \geq 0$ the homomorphism $\Phi_k : S_k(Y;G) \to S_{k+1}(X;G)$, $\Phi_k = \mu_{k+1} \circ \widetilde{D}_k$, is well defined. According to (1) we have $\partial\Phi_k(c^k) = \varphi_k(c^k) - \phi_k(c^k) - \Phi_{k-1}(\partial c^k)$ for all $c^k \in S_k(Y;G)$. Therefore, Φ is a chain homotopy between φ and ϕ and, since $\mu(S(W_x;G)) \subset S(U_x;G)$ for every $x \in X$, Φ is $\mathcal{U}$-small. Because D is $\mathcal{V}$-small, $c \in S(A;G)$ implies $D(c) \in S(X;G) \cap S(T,\widetilde{\mathcal{V}}_1;G)$. So, $\Phi(c) = D(c)$ for all $c \in S(A;G)$ (recall that μ is an algebraic retraction). Thus, Φ extends D. □

Corollary 2.5. *Let X be a metric algebraic ANR_G. Then for every open cover $\mathcal{U}$ of X there is an open cover $\mathcal{V}$ of X refining $\mathcal{U}$ such that any two correct $\mathcal{V}$-close morphisms $\varphi,\phi : S(Y;G) \to S(X;G)$, where Y is a metric space, are $\mathcal{U}$-homotopic.*

Proposition 2.4 and Corollary 2.5 remain true if the singular complex $S(Y;G)$ and the space $A \subset Y$ are replaced by a simplicial complex $C(K;G)$ and a sub-complex L of K, respectively.

Proposition 2.6. *If X is a metric algebraic ANR_G, then for any open cover $\mathcal{U}$ of X there is an open cover $\mathcal{V}$ of X refining $\mathcal{U}$ such that, for any two correct $\mathcal{V}$-close chain morphisms $\varphi,\phi : C(K;G) \to S(X;G)$, where K is a simplicial complex, and any $\mathcal{V}$-small chain homotopy $D : C(L;G) \to S(X;G)$ between $\varphi|C(L;G)$ and $\phi|C(L;G)$ with L being a sub-complex of K, there exists a chain $\mathcal{U}$-small homotopy $\Phi : C(K;G) \to S(X;G)$ between φ and ϕ extending D.*

Corollary 2.7. *Let X be a metric algebraic ANR_G. Then for every open cover $\mathcal{U}$ of X there is an open cover $\mathcal{V}$ of X refining $\mathcal{U}$ such that any two correct $\mathcal{V}$-close morphisms $\varphi,\phi : C(K;G) \to S(X;G)$, where K is a simplicial complex, are $\mathcal{U}$-homotopic.*

3. Homologically UV^n-maps

In this section we establish some properties of closed $UV^n(G)$-maps. We start this section with the following statement, which an analogue of the corresponding result for homotopically UV^n-maps, see for example [5].

Proposition 3.1. *Let $f : X \to Y$ be a closed homologically $UV^n(G)$ surjection between paracompact spaces. Then for every open cover $\mathcal{U}$ of Y there is an open cover $\mathcal{V}$ of Y refining $\mathcal{U}$ such that any correct partial algebraic realization of $C(K;G)$ in $f^{-1}(\mathcal{V})$, where K is an $(n+1)$-dimensional simplicial complex, extends to a full algebraic realization $\phi : C(K;G) \to S(X;G)$ of $C(K;G)$ in $f^{-1}(\mathcal{U})$.*

Proof. Denote $\mathcal{U}$ by $\mathcal{U}_{n+1}$ and for every $y \in Y$ fix $U_{n+1}(y) \in \mathcal{U}_{n+1}$ containing y. We are going to construct by induction open covers $\mathcal{U}_k$ of Y for each $k = n, n-1, .., 0$ following the proof of [5, Theorem 3.1]. Since

f is a closed map and each fiber $f^{-1}(y)$ is a homologically $UV^n(G)$-set in X, for every $y \in Y$ there exists a neighborhood $V_{n+1}(y)$ of y such that $f^{-1}(V_{n+1}(y)) \overset{H_n}{\hookrightarrow} f^{-1}(U_{n+1}(y))$. Let $\mathcal{U}_n$ be an open star-refinement of the cover $\{V_{n+1}(y) : y \in Y\}$. If $\mathcal{U}_{k+1}$ is already defined, we repeat the above construction to obtain for each $y \in Y$ its neighborhood $V_{k+1}(y)$ with $f^{-1}(V_{k+1}(y)) \overset{H_k}{\hookrightarrow} f^{-1}(U_{k+1}(y))$, and then take $\mathcal{U}_k$ to be an open star-refinement of the cover $\{V_{k+1}(y) : y \in Y\}$. We proceed recursively until construct $\mathcal{U}_0$.

Let show that $\mathcal{V} = \mathcal{U}_0$ is the required cover. Suppose K is an $(n+1)$-dimensional simplicial complex and $\varphi : C(L;G) \to S(X;G)$ a correct partial algebraic realization of $C(K;G)$ in $f^{-1}(\mathcal{U}_0)$, where L is a sub-complex of K containing all vertices of K. For every $k \geq 1$ we are going to construct a homomorphism $\phi_k : C_k(K;G) \to S_k(X;G)$ extending $\varphi_k : C_k(L;G) \to S_k(X;G)$ such that $\partial^X \circ \phi_k = \phi_{k-1} \circ \partial$ and satisfying the following condition (4_k) (here, ∂ and ∂^X are the boundary operators in $C(K;G)$ and $S(X;G)$, respectively).

(4_k) For every k-simplex σ in K there exists $U^k_\sigma \in \mathcal{U}_k$ such that $|\phi_i(\tau)| \subset f^{-1}(U^k_\sigma)$ for any $i \leq k$ and any i-dimensional face τ of σ.

To construct ϕ_1, let $\sigma = (v_0, v_1)$ be a 1-simplex from K and take $U^0_\sigma \in \mathcal{U}_0$ such that $f^{-1}(U^0_\sigma)$ contains $|\varphi(v_i)|$, $i = 0,1$. Moreover, if $\sigma \in L$ we can suppose that $f^{-1}(U^0_\sigma)$ also contains $|\varphi(\sigma)|$. Since $\mathcal{U}_0$ is a star refinement of $\{V_1(y) : y \in Y\}$, $U^0_\sigma \subset f^{-1}(V_1(y_\sigma))$ for some $y_\sigma \in Y$. So, $\varphi(\partial\sigma) = \varphi(v_1) - \varphi(v_0)$ is a singular 0-cycle in $f^{-1}(V_1(y_\sigma))$ (recall that each $\varphi(v_i)$, $i = 1,2$, is a singular 0-simplex and $\epsilon(\varphi(v_1) - \varphi(v_0)) = 0$, where $\epsilon : S_0(X;G) \to G$ is the augmentation of $S(X;G)$). Since $f^{-1}(V_1(y_\sigma)) \overset{H_1}{\hookrightarrow} f^{-1}(U_1(y_\sigma))$, there exists a 1-chain $c^1_\sigma \in f^{-1}(U_1(y_\sigma))$ such that $\partial^X(c^1_\sigma) = \varphi(\partial\sigma)$. We define $\phi'_1(\sigma) = c^1_\sigma$ if $\sigma \notin L$ and $\phi'_1(\sigma) = \varphi(\sigma)$ if $\sigma \in L$, and extend ϕ'_1 linearly to a homomorphism $\phi_1 : C_1(K;G) \to S_1(X;G)$. Then condition (4_1) is satisfied with $U^1_\sigma = U_1(y_\sigma)$ because $|\phi(v_0)| \cup |\phi(v_1)| \cup |\phi_1(\sigma)| \subset f^{-1}(U_1(y_\sigma))$.

Suppose that the homomorphisms ϕ_i have been already constructed for some $k > 1$ and all $i \leq k$, and let σ be a $(k+1)$-simplex of K. Choose $U_0 \in \mathcal{U}_0$ and $U^k_0 \in \mathcal{U}_k$ such that $f^{-1}(U_0)$ contains all $|\varphi(v)|$ with $v \in \sigma^{(0)}$ and $U_0 \subset U^k_0$. In case $\sigma \in L$, we can suppose that $|\varphi(\sigma)|$ is also contained in $f^{-1}(U_0)$. Then, by (4_k), for every k-simplex τ, which is a face of σ, there exists $U^k_\tau \in \mathcal{U}_k$ such that $|\phi_i(s)| \subset f^{-1}(U^k_\tau)$ for all i-dimensional faces s of τ, $i \leq k$. In particular, $f^{-1}(U^k_\tau)$ contains $|\varphi(v)|$ for all vertexes v of τ. Because $|\varphi(v)| \neq \varnothing$ for all $v \in K^{(0)}$ (recall that φ is correct), we have $|\varphi(v)| \subset f^{-1}(U^k_0) \cap f^{-1}(U^k_\tau) \neq \varnothing$ for all faces τ of σ and all $v \in \tau^{(0)}$. Consequently, $f^{-1}(\mathrm{St}(U^k_0, \mathcal{U}_k)) \neq \varnothing$ and $|\phi_k(\partial(\sigma))| \subset f^{-1}(\mathrm{St}(U^k_0, \mathcal{U}_k))$, where $\mathrm{St}(U^k_0, \mathcal{U}_k)$ is the star of U^k_0 with respect to $\mathcal{U}_k$. Since $\mathcal{U}_k$ is a star-refinement of $\{V_{k+1}(y) : y \in Y\}$, $|\phi_k(\partial(\sigma))| \subset f^{-1}(V_{k+1}(y_\sigma))$ for some $y_\sigma \in Y$. Hence, $\phi_k(\partial(\sigma))$ is a singular k-cycle in $f^{-1}(V_{k+1}(y_\sigma))$. Finally, since $f^{-1}(V_{k+1}(y_\sigma)) \overset{H_k}{\hookrightarrow} f^{-1}(U_{k+1}(y_\sigma))$, there exists a $(k+1)$-chain $c^{k+1}_\sigma \in f^{-1}(U_{k+1}(y_\sigma))$ such that $\partial^X(c^{k+1}_\sigma) = \phi_k(\partial(\sigma))$. We define $\phi'_{k+1}(\sigma) = c^{k+1}_\sigma$ if $\sigma \notin L$ and $\phi'_{k+1}(\sigma) = \varphi(\sigma)$ if $\sigma \in L$, and then extend ϕ'_{k+1} to a homomorphism $\phi_{k+1} : C_{k+1}(K;G) \to S_{k+1}(X;G)$ by linearity. Condition (4_{k+1}) is satisfied with $U^{k+1}_\sigma = U_{k+1}(y_\sigma)$.

In this way we complete the inductive step. The required full algebraic realization of $C(K;G)$ in $f^{-1}(\mathcal{U})$ is the chain morphism $\phi : C(K;G) \to S(X;G)$ with $\phi = \{\phi_k\}_{k=0}^{n+1}$. □

Proposition 3.1 implies the following "approximate lifting" of chain morphisms, see [4, Theorem 16.7] for the homotopical version.

Corollary 3.2. *Let $f : X \to Y$ be as Proposition 3.1. Then every open cover $\mathcal{U}$ of Y has an open refinement $\mathcal{V}$ covering Y such that: If K is an $(n+1)$-dimensional simplicial complex, L its sub-complex and $\varphi_L : C(L;G) \to S(X;G)$, $\phi : C(K;G) \to S(Y;G)$ are two correct chain morphisms such that $\phi|C(L;G) = f_\sharp \circ \varphi_L$ and ϕ is $\mathcal{V}$-small, then there is a chain morphism $\varphi_K : C(K;G) \to S(X;G)$ extending φ_L with ϕ and $f_\sharp \circ \varphi_K$ being $\mathcal{U}$-close.*

Proof. Let $\mathcal{U}$ be an open cover of Y and choose $\mathcal{U}_1$ to be a star-refinement of $\mathcal{U}$. Then, there exists a corresponding refinement $\mathcal{V}$ of $\mathcal{U}_1$ satisfying Proposition 3.1. Let $\phi : C(K;G) \to S(Y;G)$ and $\varphi_L : C(L;G) \to S(X;G)$ be chain morphisms satisfying the hypotheses of the corollary. We can suppose that L contains all vertexes of K. Indeed, otherwise for any $v \in K^{(0)} \setminus L$ let $\phi(v) = g \cdot \sigma^0$ with $g \in G$ and σ^0 being a singular 0-simplex in Y. Then $|\sigma^0|$ is a point in Y, and define $\varphi_L(v) = g \cdot \tau^0$, where τ^0 is a singular 0-simplex in X with $f(|\tau^0|) = |\sigma^0|$.

Because $\phi|C(L;G) = f_\sharp \circ \varphi_L$ and ϕ is $\mathcal{V}$-small, φ_L is a correct partial algebraic realization of $C(K;G)$ in $f^{-1}(\mathcal{V})$. So, φ_L extends to a full algebraic realization $\varphi_K : C(K;G) \to S(X;G)$ of $C(K;G)$ in $f^{-1}(\mathcal{U}_1)$. Hence, for every $\sigma \in K$ there exists $U'_\sigma \in \mathcal{U}_1$ such that $|\varphi_K(\tau)| \subset f^{-1}(U'_\sigma)$ for all faces τ of σ. On the other hand, since ϕ is $\mathcal{V}$-small, there exists $V_\sigma \in \mathcal{V}$ such that $|\phi(v)| \neq \varnothing$ and $|\phi(v)| \cup |\phi(\tau)| \subset V_\sigma$, τ is a face of σ and $v \in \sigma^{(0)}$. Hence, for all $v \in \sigma^{(0)}$ we have $|\phi(v)| \subset V_\sigma \cap U'_\sigma$. So, $\mathrm{St}(\mathrm{V}_\sigma;\mathcal{U}_1) \neq \varnothing$ and it contains $|\phi(\tau)| \cup f(|\varphi_K(\tau)|)$ for all faces τ of σ. Finally, since $\mathcal{V}$ refines $\mathcal{U}_1$ and $\mathcal{U}_1$ is a star refinement of $\mathcal{U}$, there is $U_\sigma \in \mathcal{U}$ containing all $|\phi(\tau)| \cup f(|\varphi_K(\tau)|)$, τ being a face of σ. Therefore, ϕ is $\mathcal{U}$-close to $f_\sharp \circ \varphi_K$. □

Let K be a singular sub-complex of $S(P;G)$, X be a space and $\mathcal{U}$ an open cover of X. If L is a sub-complex of K containing all 0-singular simplexes of K, we say that a chain morphism $\varphi : L \to S(X;G)$ is a *partial singular realization* of K in $\mathcal{U}$ if for every singular simplex $\sigma \in K$ there exists $U_\sigma \in \mathcal{U}$ such that $|\varphi(\tau)| \subset U_\sigma$ for all faces τ of σ with $\tau \in L$. If $L = K$, then φ is called a *full singular realization of K in $\mathcal{U}$*. If in the above definition $\varphi(\sigma)$ is a singular 0-simplex in $S(X;G)$ for every singular 0-simplex σ of K, then φ is said to be a *correct partial singular realization* of K in $\mathcal{U}$.

The proofs of Proposition 3.1 and Corollary 3.2 remain true if φ is a correct partial singular realization of $K = S^{(n+1)}(P;G)$ in $f^{-1}(\mathcal{V})$. In this case φ extends to a full singular realization of K in $f^{-1}(\mathcal{U})$. So, we have the following "singular analogues" of Proposition 3.1 and Corollary 3.2.

Proposition 3.3. *Let $f : X \to Y$ be as in Proposition 3.1. Then for every open cover $\mathcal{U}$ of Y there is an open cover $\mathcal{V}$ of Y refining $\mathcal{U}$ such that any correct partial singular realization of a singular complex $S^{(n+1)}(P;G)$ in $f^{-1}(\mathcal{V})$ extends to a full singular realization of $S^{(n+1)}(P;G)$ in $f^{-1}(\mathcal{U})$.*

Corollary 3.4. *Let $f : X \to Y$ be as Proposition 3.1. Then every open cover $\mathcal{U}$ of Y has an open refinement $\mathcal{V}$ covering Y such that: If $S^{(n+1)}(P;G)$ is an $(n+1)$-dimensional singular complex, L its sub-complex and $\varphi : L \to S(X;G)$, $\phi : S^{(n+1)}(P;G) \to S(Y;G)$ are two correct chain morphisms such that $\phi|L = f_\sharp \circ \varphi$ and ϕ is $\mathcal{V}$-small, then there is a chain morphism $\widetilde{\varphi} : S^{(n+1)}(P;G) \to S(X;G)$ extending φ with ϕ and $f_\sharp \circ \widetilde{\varphi}$ being $\mathcal{U}$-close.*

Another application of Proposition 3.3 is the following chain analogue of Dugundji's extension theorem for LC^n-spaces [6]. For that reason we introduce the following notion: a chain morphism $\varphi : S(Z;G) \to S(Y;G)$ is said to be *continuous* if for every $z \in Z$ and any neighborhood U of $|\varphi(z)|$ in Y there is a neighborhood V of z in Z with $|\varphi(z')| \subset U$ for all $z' \in V$ (here, z and z' are treated as singular 0-simplexes in $S(Z;G)$). For example, if $f : Z \to Y$ is a continuous map, then the chain morphism $f_\sharp : S(Z;G) \to S(Y;G)$ is continuous.

Proposition 3.5. *Let $f : X \to Y$ be as in Proposition 3.1. Then each open cover $\mathcal{U}$ of Y admits an open refinement $\mathcal{V}$ with the property: If A is a closed subset of a metric space M and $\varphi : S(A;G) \to S(X;G)$ is a continuous correct $f^{-1}(\mathcal{V})$-small chain morphism, then there exist a neighborhood W of A in M, an open cover ω of W and a $f^{-1}(\mathcal{U})$-small chain morphism $\widetilde{\phi} : S^{(n+1)}(W,\omega;G) \to S(X;G)$ such that $\widetilde{\phi}(c) = \varphi(c)$ for all $c \in S^{(n+1)}(W,\omega;G) \cap S(A;G)$.*

Proof. Every open cover $\mathcal{U}$ of Y has a refinement $\mathcal{V}'$ satisfying Proposition 3.3, and let $\mathcal{V}$ be an open star refinement of $\mathcal{V}'$. Suppose $\varphi : S(A;G) \to S(X;G)$ is a continuous correct $f^{-1}(\mathcal{V})$-small morphism, where A is a closed subset of a metric space (M,ρ). According to [6], there is a locally finite canonical open cover α of

$M \setminus A$. This means that for every $a \in A$ and its neighborhood $O(a)$ in M there exists another neighborhood $\Gamma(a)$ in M such that $\mathrm{St}(\Gamma(\mathrm{a}), \alpha) \subset \mathrm{O}(\mathrm{a})$. For every $\Lambda \in \alpha$ we choose a point $a_\Lambda \in A$ with

(5) $\rho(a_\Lambda, \Lambda) < 2\sup_{z\in\Lambda} \rho(z, A)$.

Since φ is correct and continuous, for every $a \in A$ we find $V_a \in \mathcal{V}$ and $\varepsilon_a > 0$ with $|\varphi(z)| \subset f^{-1}(V_a)$ for all $z \in A \cap B_\rho(a, \varepsilon_a)$, where $B_\rho(a, \varepsilon_a)$ is the open ball in M with radius ε_a and center a. Using the notations above, let $\Gamma(a)$ be the corresponding to $O(a) = B_\rho(a, \varepsilon_a/3)$ neighborhood of a, and let $W = \bigcup_{a\in A} \Gamma(a)$ and $\omega = \{\Gamma(a) : a \in A\}$.

Claim 2. *If $\Lambda \cap \Gamma(a) \neq \varnothing$ for some $\Lambda \in \alpha$ and $a \in A$, then $a_\Lambda \in B_\rho(a, \varepsilon_a)$.*

Indeed, for any such Λ we have $\Lambda \subset B_\rho(a, \varepsilon_a/3)$. Consequently, $2\sup_{z\in\Lambda} \rho(z, A) < 2\varepsilon_a/3$ and, according to (5), there is $z_\Lambda \in \Lambda$ with $\rho(a_\Lambda, z_\Lambda) < 2\varepsilon_a/3$. So, $\rho(a_\Lambda, a) \leq \rho(a_\Lambda, z_\Lambda) + \rho(z_\Lambda, a) < \varepsilon_a$.

Obviously, $L = S_0(W; G) \cup S^{(n+1)}(A, \omega; G)$ is a sub-complex of the complex $S^{(n+1)}(W, \omega; G)$ containing all singular 0-simplexes in W. For any singular 0-simplex $z \in S_0(W; G)$ we define $\phi_0'(z) = \varphi(z)$ if $z \in A$ and $\phi'(z) = \varphi(a_{\Lambda(z)})$ if $z \in W \setminus A$, where $\Lambda(z)$ is an arbitrary element of α containing z. Next, extend ϕ_0' to a homomorphism $\phi_0 : S_0(W; G) \to S(X; G)$ by linearity. Obviously, ϕ_0 can be extended to a homomorphism $\phi : L \to S(X; G)$ by defining $\phi(\sigma) = \varphi(\sigma)$ for all $\sigma \in S^{(n+1)}(A, \omega; G) \setminus S_0(W; G)$.

Claim 3. *ϕ is a correct partial singular realization of $S^{(n+1)}(W, \omega; G)$ in $f^{-1}(\mathcal{V}')$.*

The correctness of ϕ follows from the correctness of φ. To show that ϕ is a partial singular realization of $S^{(n+1)}(W, \omega; G)$ in $f^{-1}(\mathcal{V}')$, let $\sigma \in S^{(n+1)}(W, \omega; G)$ be a singular simplex and z its vertex. Then $|\sigma| \subset \Gamma(a) \in \omega$ for some $a \in A$, so z is a point from $\Gamma(a)$ (we identify z with $|z|$). Since $\Gamma(a) \subset B_\rho(a, \varepsilon_a/3)$, according to Claim 2 and the choice of ε_a, we have $|\phi(z)| = |\varphi(a_{\Lambda(z)})| \subset f^{-1}(V_a)$ if $z \in W \setminus A$. The inclusion $|\phi(z)| \subset f^{-1}(V_a)$ holds also if $z \in A$. Therefore, $|\phi(z)| \subset f^{-1}(V_a)$ for all vertices z of σ. Take $V_\sigma \in \mathcal{V}'$ with $\mathrm{St}(V_\mathrm{a}, \mathcal{V}) \subset V_\sigma$ (recall that $\mathcal{V}$ is a star refinement of $\mathcal{V}'$) and let $\tau \in L$ be a face of σ. If τ is 0-simplex, then $|\phi(\tau)|$ (being a subset of $f^{-1}(V_a)$) is contained in $f^{-1}(V_\sigma)$. If τ is a singular simplex of dimension ≥ 1, then $\tau \in S^{(n+1)}(A, \omega; G)$. Because φ is $f^{-1}(\mathcal{V})$-small, there is $V_\tau \in \mathcal{V}$ such that $|\varphi(z)| \cup |\varphi(\tau)| \subset f^{-1}(V_\tau)$ for all vertices z of τ. So, $|\varphi(z)| \subset f^{-1}(V_a) \cap f^{-1}(V_\tau)$ for any vertex z of τ. Consequently, $\mathrm{St}(V_\mathrm{a}, \mathcal{V}) \neq \varnothing$ and contains $|\phi(\tau)|$. Hence, $|\phi(\tau)| \subset f^{-1}(V_\sigma)$ for all faces τ of σ. Thus, ϕ is a correct partial singular realization of $S^{(n+1)}(W, \omega; G)$ in $f^{-1}(\mathcal{V}')$.

Finally, by Proposition 3.3, ϕ extends to a full singular realization $\widetilde{\phi}$ of $S^{(n+1)}(W, \omega; G)$ in $f^{-1}(\mathcal{U})$. Therefore, $\widetilde{\phi}$ is $f^{-1}(\mathcal{U})$-small and $\widetilde{\phi}(c) = \varphi(c)$ for any $c \in S^{(n+1)}(W, \omega; G) \cap S(A; G)$. □

If in Proposition 3.5 $\dim M \leq n + 1$, then we have the following "approximate extension" version of Proposition 3.5

Proposition 3.6. *Suppose f is as in Proposition 3.1. Then for every open cover $\mathcal{U}$ of Y there exists an open cover $\mathcal{V}$ of Y with the following property: If $\varphi : S(A; G) \to S(X; G)$ is a continuous correct $f^{-1}(\mathcal{V})$-small chain morphism, where A is a closed subset of a metric space M with $\dim M \leq n+1$, then there exist an open set $W \subset M$ containing A, an open cover α of W and a correct chain morphism $\phi : S(W, \alpha; G) \to S(X; G)$ such that $\phi|S(A, \alpha; G)$ and $\varphi|S(A, \alpha; G)$ are $f^{-1}(\mathcal{U})$-close.*

Proof. Let $\mathcal{U}$ be an open cover of Y and $\mathcal{U}_1$ be a star refinement of $\mathcal{U}$. Take another open cover $\mathcal{V}_1$ of Y satisfying the hypotheses of Proposition 3.1 with respect to $\mathcal{U}_1$ (i.e., any correct partial algebraic realization of $C(K; G)$ in $f^{-1}(\mathcal{V}_1)$ can be extended to a full algebraic realization of $C(K; G)$ in $f^{-1}(\mathcal{U}_1)$, where K is a simplicial complex with $\dim K \leq n + 1$). Let $\mathcal{V}$ be a locally finite open star-refinement of $\mathcal{V}_1$. Since

φ is continuous and correct, for every $a \in A$ there are $V_a \in \mathcal{V}$ and a neighborhood O_a of a in A with $|\varphi(z)| \subset f^{-1}(V_a)$ for any $z \in O_a$. Take a locally finite open cover $\Gamma = \{\Lambda_t : t \in T\}$ of A refining the cover $\{O_a : a \in A\}$ such that the nerve of Γ is of dimension $\leq n+1$. Because M is a metric space, we can extend each $\Lambda_t \in \Gamma$ to an open set $\widetilde{\Lambda}_t \subset M$ such that for any finitely many $\Lambda_{t_1}, .., \Lambda_{t_k}$ we have $\bigcap \widetilde{\Lambda}_{t_i} \neq \varnothing$ if and only if $\bigcap \Lambda_{t_i} \neq \varnothing$. The last relation implies that the nerve K of $\widetilde{\Gamma} = \{\widetilde{\Lambda}_t : t \in T\}$ is also of dimension $\leq n+1$. Let $W = \bigcup\{\widetilde{\Lambda}_t : t \in T\}$. For every $\widetilde{\Lambda}_t \in \widetilde{\Gamma}$ choose a point $a(t) \in \Lambda_t$ and define $\psi_0'(\widetilde{\Lambda}_t) = \varphi(a(t))$, where $\widetilde{\Lambda}_t$ is considered as a vertex of K and $a(t)$ as a singular 0-simplex from $S(A;G)$. Then extend ψ_0' to a homomorphism $\psi_0 : C_0(K;G) \to S_0(X;G)$.

Since $\mathcal{V}$ is a star-refinement of $\mathcal{V}_1$, ψ_0 is a correct partial algebraic realization of $C(K;G)$ in $\mathcal{V}_1$. Indeed suppose $\sigma = (\widetilde{\Lambda}_{t_0}, .., \widetilde{\Lambda}_{t_m})$ is a simplex from K. Then for every $i = 0, .., m$ there is $V_i \in \mathcal{V}$ such that $|\varphi(a)| \in f^{-1}(V_i)$ for all $a \in \Lambda_{t_i}$. So, $f^{-1}(V_0) \cap f^{-1}(V_i) \neq \varnothing$ and $|\varphi(a(t_i))| \subset f^{-1}(V_i)$, $i = 0, .., m$. Hence, $\mathrm{St}(\mathrm{f}^{-1}(\mathrm{V}_0), \mathrm{f}^{-1}(\mathcal{V}))$ contains $\bigcup|\varphi(a(t_i))|$. Consequently, $\bigcup|\varphi(a(t_i))| \subset f^{-1}(V')$ for some $V' \in \mathcal{V}_1$ (recall that $\mathcal{V}$ is a star-refinement of $\mathcal{V}_1$). Thus, ψ_0 is a correct partial algebraic realization of $C(K;G)$ in $\mathcal{V}_1$.

So, ψ_0 extends to a full algebraic realization $\psi : C(K;G) \to S(X;G)$ of $C(K;G)$ in $f^{-1}(\mathcal{U}_1)$. Let $\kappa : W \to |K|$ be a canonical map, where the polytope $|K|$ is equipped with the Whitehead topology. According to [7, Proposition 8.6.6], there are an open cover $\mathcal{S}$ of $|K|$ such that each $|s|$, $s \in K$, is contained in some $P_s \in \mathcal{S}$, and a chain equivalence $\gamma : S(|K|, \mathcal{S}; G) \to C^{\Omega}(K;G)$. Here $C^{\Omega}(K;G)$ is the chain complex whose simplexes are finite arrays $[\widetilde{\Lambda}_0, \widetilde{\Lambda}_1, .., \widetilde{\Lambda}_k]$, where all $\widetilde{\Lambda}_i$, not necessarily distinct, are vertices of K spanning a simplex from K. There exists also a natural chain morphism $\theta : C^{\Omega}(K;G) \to C(K;G)$ such that $\theta([\widetilde{\Lambda}_0, \widetilde{\Lambda}_1, .., \widetilde{\Lambda}_k])$ is the simplex $(\widetilde{\Lambda}_0, \widetilde{\Lambda}_1, .., \widetilde{\Lambda}_k) \in K$ if all $\widetilde{\Lambda}_i$ are distinct, and 0 otherwise. Let α be the intersection of the covers $\widetilde{\Gamma}$ and $\kappa^{-1}(\mathcal{S})$, and let $\phi_1 : S(W, \alpha; G) \to C(K;G)$ and $\phi : S(W, \alpha; G) \to C(K;G)$ be the chain morphisms $\phi_1 = \theta \circ \gamma \circ \kappa_\sharp$ and $\phi = \psi \circ \phi_1$, respectively.

Let show that $\varphi|S(A, \alpha; G)$ and $\phi|S(A, \alpha; G)$ are $f^{-1}(\mathcal{U})$-close. Indeed, since φ is $f^{-1}(\mathcal{V})$-small, for any singular simplex $\sigma \in S(A, \alpha; G)$ there is $V_\sigma \in \mathcal{V}$ with $|\varphi(\tau)| \subset f^{-1}(V_\sigma)$ for all faces τ of σ. On the other hand, $\sigma_1 = \kappa_\sharp(\sigma)$ is a singular simplex from $S(|K|, \mathcal{S}; G)$ such that, according to the definition of γ (see [7, p. 339]), $\gamma(\sigma_1)$ is a "simplex" $s = [\widetilde{\Lambda}_0, \widetilde{\Lambda}_1, .., \widetilde{\Lambda}_k]$ from $C^{\Omega}(K;G)$ satisfying the following condition: if τ is a face of σ, then $\kappa_\sharp(\tau)$ is a face of σ_1 and the vertices of $\gamma(\kappa_\sharp(\tau))$ are also vertices of $\gamma(\kappa_\sharp(\sigma))$. In particular, for any vertex v of σ we have $\gamma(\kappa_\sharp(v)) = \gamma(\kappa_\sharp(|v|))$ is one of the vertexes $\widetilde{\Lambda}_i$ such that $|v|$ is a point from Λ_i. So, for every face τ of σ either $\phi_1(\tau) = 0$ or $\phi_1(\tau)$ is a simplex from K whose vertices are contained in the set $\{\widetilde{\Lambda}_i; i = 0, 1, ..k\}$, but definitely the union of all $\phi(\tau)$, τ is a face of σ, is non-empty. Hence, there exists a simplex $\delta \in K$ containing $\phi_1(\tau)$ for all faces τ of σ such that the vertices of δ are in the set $\{\widetilde{\Lambda}_i; i = 0, 1, ..k\}$. Since ψ is $f^{-1}(\mathcal{U}_1)$-small, we can find $U_\delta \in \mathcal{U}_1$ containing all $|\phi(\tau)| \subset f^{-1}(U_\delta)$, τ is a face of σ. We fix a vertex v^* of σ. Then $\phi_1(v^*) = \widetilde{\Lambda}_j$ for some j with $|v^*| \in \Lambda_j$, and let $V_a \in \mathcal{V}$ such that $|\varphi(z)| \in f^{-1}(V_a)$ for all $z \in \Lambda_j$. So, according to the definition of ψ, $\phi(v^*) = \psi(\widetilde{\Lambda}_j) = \varphi(z^*)$ for some $z^* \in \Lambda_j$. Consequently, $|\phi(v^*)| = |\varphi(z^*)| \in f^{-1}(V_a)$. Hence, $|\phi(v^*)| \in f^{-1}(V_a) \cap f^{-1}(U_\sigma)$ and $|\varphi(v^*)| \in f^{-1}(V_a) \cap f^{-1}(V_\sigma)$. Therefore, since $\mathcal{V}$ is refining $\mathcal{U}_1$, for all faces τ of σ we have

$$|\varphi(\tau)| \cup |\phi(\tau)| \subset f^{-1}(V_\sigma) \cup f^{-1}(U_\sigma) \subset \mathrm{St}(\mathrm{f}^{-1}(\mathrm{V_a}), \mathrm{f}^{-1}(\mathcal{U}_1)).$$

Because $\mathrm{St}(\mathrm{f}^{-1}(\mathrm{V_a}), \mathrm{f}^{-1}(\mathcal{U}_1))$ is contained in $f^{-1}(U)$ for some $U \in \mathcal{U}$, we finally obtain that $\varphi|S(A, \alpha; G)$ and $\phi|S(A, \alpha; G)$ are $f^{-1}(\mathcal{U})$-close. □

Next proposition is an analogue of Proposition 2.4.

Proposition 3.7. *Let $f : X \to Y$ be as in Proposition 3.1, Z an arbitrary space and $A \subset Z$ a closed subset. Then for every open cover $\mathcal{U}$ of Y there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that for any two correct $f^{-1}(\mathcal{V})$-close chain morphisms $\varphi, \phi : S^{(n)}(Z;G) \to S(X;G)$ and any $f^{-1}(\mathcal{V})$-small chain homotopy $\Phi : S(A;G) \to S(X;G)$ between $\varphi|S^{(n)}(A;G)$ and $\phi|S^{(n)}(A;G)$ there exists a $f^{-1}(\mathcal{U})$-small homotopy $D : S^{(n)}(Z;G) \to S(X;G)$ between φ and ϕ extending Φ.*

Proof. We follow the proof of [2, Lemma 5.4]. As in the proof of Proposition 3.1, for every $k = n, n-1, .., 0$ we construct open covers $\mathcal{U}_k$ and $\mathcal{V}_k$ of Y such that $\mathcal{U} = \mathcal{U}_n$, $\mathcal{U}_k$ star-refines $\mathcal{V}_{k+1}$ for $k = n-1, .., 0$ and for each $V \in \mathcal{V}_k$ there exists $U \in \mathcal{U}_k$ with $f^{-1}(V) \overset{H_k}{\hookrightarrow} f^{-1}(U)$ if $k = n, .., 1$ and $\mathrm{St}(\mathrm{f}^{-1}(\mathrm{V}), \mathrm{f}^{-1}(\mathcal{V}_0)) \overset{\mathrm{H}_0}{\hookrightarrow} \mathrm{f}^{-1}(\mathrm{U})$ if $k = 0$. We claim that $\mathcal{V} = \mathcal{V}_0$ is the required cover. Indeed, suppose $\varphi, \phi : S^{(n)}(Z; G) \to S(X; G)$ are two $f^{-1}(\mathcal{V})$-close correct chain morphisms and $\Phi : S(A; G) \to S(X; G)$ is a $f^{-1}(\mathcal{V})$-small chain homotopy between $\varphi|S^{(n)}(A; G)$ and $\phi|S^{(n)}(A; G)$. We are going to construct homomorphisms $D_k : S_k(Z; G) \to S_{k+1}(X; G)$ such that:

(6) $\partial D_k(c^k) = \varphi(c^k) - \phi(c^k) - D_{k-1}(\partial c^k)$ for all $c^k \in S_k(Z; G)$;
(7) $D_k(c^k) = \Phi(c^k)$ for all $c^k \in S(A; G)$;
(8) For any singular k-simplex $\sigma \in S_k(Z; G)$ there is $U_\sigma \in \mathcal{U}_k$ such that $f^{-1}(U_\sigma)$ contains $|D_i(\tau)| \cup |\phi(\tau)|$ for all $i \leq k$ and all i-dimensional faces τ of σ.

Because φ and ϕ are $f^{-1}(\mathcal{V})$-close, for any $z \in Z$ there is $V_z \in \mathcal{V}$ with $\varphi(z), \phi(z)$ being singular 0-simplexes in $f^{-1}(V_z)$ (we identify each $z \in Z$ with the singular 0-simplex $\sigma \in S_0(Z; G)$ such that $|\sigma| = \{z\}$). So, $\varphi(z) - \phi(z)$ is a singular 0-cycle in $f^{-1}(V_z)$. Since $\mathrm{St}(\mathrm{f}^{-1}(\mathrm{V_z}), \mathrm{f}^{-1}(\mathcal{V})) \overset{\mathrm{H}_0}{\hookrightarrow} \mathrm{f}^{-1}(\mathrm{U_z})$ for some $U_z \in \mathcal{U}_0$, there is $c^1_z \in S_1(f^{-1}(U_z); G)$ with $\partial c^1_z = \varphi(z) - \phi(z)$. For every $z \in Z$ we define $D'_0(z) = c^1_z$ if $z \notin A$ and $D'_0(z) = \Phi(z)$ if $z \in A$, and extend D'_0 linearly to a homomorphism $D_0 : S_0(Z; G) \to S_1(X; G)$. Obviously, $|D_0(z)| \cup |\phi(z)| \subset f^{-1}(U_z)$ if $z \notin A$. If $z \in A$, then there is $V'_z \in \mathcal{V}$ with $|\Phi(z)| \cup |\phi(z)| \subset f^{-1}(V'_z)$ (recall that Φ is $f^{-1}(\mathcal{V})$-small). So, $|\phi(z)| \subset f^{-1}(V'_z) \cap f^{-1}(V_z)$, which shows that $\mathrm{St}(\mathrm{f}^{-1}(\mathrm{V_z}), \mathrm{f}^{-1}(\mathcal{V})) \neq \varnothing$ and contains $|\Phi(z)| \cup |\phi(z)|$. Thus, $|\Phi(z)| \cup |\phi(z)| \subset f^{-1}(U_z)$ for all $z \in Z$. Therefore, D_0 satisfies conditions $(6) - (8)$.

Suppose we already constructed the homomorphisms $D_i : S_i(Z; G) \to S_{i+1}(X; G)$, $i \leq k$, satisfying the above conditions, and let σ be a singular $(k+1)$-simplex from $S_{k+1}(Z; G)$. Since φ and ϕ are $f^{-1}(\mathcal{V})$-close, there exists $V_\sigma \in \mathcal{V}$ such that $|\phi(z)| \cup |\varphi(\tau)| \cup |\phi(\tau)| \subset f^{-1}(V_\sigma)$ for all faces τ and all vertexes z of σ. On the other hand, according to (8), for any k-singular face τ of σ there is $U^k_\tau \in \mathcal{U}_k$ with $f^{-1}(U^k_\tau)$ containing $|D_i(s)| \cup |\phi(z)|$ for all $i \leq k$ and all i-dimensional faces s of τ and $z \in \tau^{(0)}$. So, $|\phi(z)| \subset f^{-1}(V_\sigma) \cap f^{-1}(U^k_\tau)$ for all k-faces τ of σ and all $z \in \tau^{(0)}$. Hence, $\mathrm{St}(\mathrm{f}^{-1}(\mathrm{V}_\sigma); \mathrm{f}^{-1}(\mathcal{U}_\mathrm{k})) \neq \varnothing$ and contains $|\gamma_\sigma|$ and all $|D_i(s)| \cup |\phi(s)|$, $i \leq k$ and s is a i-dimensional face of σ, where $\gamma_\sigma = \varphi(\sigma) - \phi(\sigma) - D_k(\partial\sigma)$. Choose $V^{k+1}_\sigma \in \mathcal{V}_{k+1}$ and $U^{k+1}_\sigma \in \mathcal{U}_{k+1}$ such that $\mathrm{St}(\mathrm{f}^{-1}(\mathrm{V}_\sigma); \mathrm{f}^{-1}(\mathcal{U}_\mathrm{k})) \subset \mathrm{f}^{-1}(\mathrm{V}^{k+1}_\sigma)$ and $f^{-1}(V^{k+1}_\sigma) \overset{H_{k+1}}{\hookrightarrow} f^{-1}(U^{k+1}_\sigma)$. Finally, since γ_σ is a singular $(k+1)$-cycle in $f^{-1}(V^{k+1}_\sigma)$, we can find a $(k+2)$-chain $c^{k+2}_\sigma \in S_{k+2}(f^{-1}(U^{k+1}_\sigma); G)$ with $\partial c^{k+2}_\sigma = \gamma_k$. Define $D'_{k+1}(\sigma) = c^{k+2}_\sigma$ if $\sigma \notin S_{k+1}(A; G)$ and $D'_{k+1}(\sigma) = \Phi(\sigma)$ if $\sigma \in S_{k+1}(A; G)$, and extend D'_{k+1} linearly to a homomorphism $D_{k+1} : S_{k+1}(Z; G) \to S_{k+1}(X; G)$ satisfying conditions $(6) - (8)$.

In this way we construct the homomorphisms D_k for all $k \leq n$ satisfying conditions $(6) - (8)$. Then $D = \{D_k\}_{k \leq n}$ is the required homotopy between φ and ϕ extending Φ. □

We also have the following proposition, whose proof is similar to that one of Proposition 3.7.

Proposition 3.8. *Let $f : X \to Y$ be as in Proposition 3.1, K a simplicial complex with $\dim K \leq n$ and L a sub-complex of K. Then for every open cover $\mathcal{U}$ of Y there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that for any two correct $f^{-1}(\mathcal{V})$-close chain morphisms $\varphi, \phi : C(K; G) \to S(X; G)$ and any $f^{-1}(\mathcal{V})$-small chain homotopy $\Phi : C(L; G) \to S(X; G)$ between $\varphi|C(L; G)$ and $\phi|C(L; G)$ there exists a $f^{-1}(\mathcal{U})$-small homotopy $D : C(K; G) \to S(X; G)$ between φ and ϕ extending Φ.*

4. Homologically locally connected spaces

First, let us note that all results from Section 3 remain true in case X is an lc^n_G-space and $f : X \to X$ is the identity map. Some of these results characterize lc^n_G-spaces. For example, we have the following proposition.

Proposition 4.1. *A paracompact space X is lc^n_G if and only if each open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ such that for any two correct $\mathcal{V}$-close chain morphisms $\varphi, \phi : S^{(n)}(Z;G) \to S(X;G)$, where Z is an arbitrary space, there exists an $\mathcal{U}$-small homotopy $D : S^{(n)}(Z;G) \to S(X;G)$ between φ and ϕ.*

Proof. The necessity follows from Proposition 3.7. So, we need to prove only the sufficiency. Suppose X satisfies that condition, and let U_x be a neighborhood of a point $x \in X$ and $\mathcal{U} = \{U_x, X \setminus \overline{W}_x\}$, where W_x is a neighborhood of x with $\overline{W}_x \subset U_x$. Then there is an open cover $\mathcal{V}$ of X satisfying the hypotheses of the proposition. We can assume that $\mathcal{V}$ is a star-refinement of $\mathcal{U}$, and take $V_x \in \mathcal{V}$ containing x. Obviously, $\mathrm{St(V_x, \mathcal{V}) \subset U_x}$. Consider the correct chain morphisms $\varphi, \phi : S^{(n)}(V_x;G) \to S(X;G)$ defined by $\varphi(c) = c$ and $\phi(\sigma^k) = \sigma^k_x$ for all $c \in S^{(n)}(V_x;G)$ and all singular k-simplexes $\sigma^k \in S^{(n)}(V_x;G)$, where σ^k_x denotes the unique singular k-simplex with $|\sigma^k_x| = \{x\}$. Then there exists a $\mathcal{U}$-small homotopy $D : S^{(n)}(V_x;G) \to S(X;G)$ between φ and ϕ. Let $c^k = \sum g_i\sigma^k_i \in S^{(n)}(V_x;G)$ be a k-cycle, $k \le n$. Hence, $D(c^k)$ is a chain from $S_k(X;G)$ such that $\partial D(c^k) = c^k - \phi(c^k)$. Define $c^{k+1} = D(c^k) + (\sum g_i)\sigma^{k+1}_x$. So, $\partial c^{k+1} = c^k - (\sum g_i)\sigma^k_x + (\sum g_i)\partial\sigma^{k+1}_x$. When $k+1$ is an odd integer, we have $\partial\sigma^{k+1}_x = \sigma^k_x$. Therefore, in this case $\partial c^{k+1} = c^k$. For even integers $k+1$ we have $\partial\sigma^{k+1}_x = 0$ and $\partial\sigma^k_x = \sigma^{k-1}_x$. Then, since c^k is a cycle, $0 = \partial\phi(c^k) = (\sum g_i)\sigma^{k-1}_x$. Consequently, $\sum g_i = 0$ and $\partial c^{k+1} = c^k$. Therefore, $\partial c^{k+1} = c^k$ for all integers k.

It remains to see that $|c^{k+1}| \subset U_x$. To this end, let σ^k_j be a fixed singular simplex from the representation of c^k. Since D is $\mathcal{U}$-small, $|D(\sigma^k_j)| \cup |\phi(v)|$ is contained in an element of $\mathcal{U}$ for every vertex v of σ^k_j. But $|\phi(v)| = |\sigma^0_x| = \{x\}$, so $|D(\sigma^k_j)| \subset U_x$ for all j. Because $|(\sum g_i)\sigma^{k+1}_x| = \{x\}$, we finally conclude that $|c^{k+1}| \subset U_x$. □

Here is another property of lc^n_G-spaces, similar to the corresponding property for LC^n-spaces (see [8, Theorem 6.1]).

Proposition 4.2. *Let X be a paracompact lc^n_G-space. Then for each open cover $\mathcal{U}$ of X there exists a simplicial complex K of dimension $\le n+1$ together with a correct chain morphism $\Phi : C(K;G) \to S(X;G)$ such that for every correct continuous $\mathcal{V}$-small chain morphism $\varphi : S(Y;G) \to S(X;G)$, where Y is a paracompact space with $\dim Y \le n+1$, there exist an open cover Υ of Y and a chain morphism $\phi : S(Y,\Upsilon;G) \to C(K;G)$ such that $\varphi|S(Y,\Upsilon;G)$ and $(\Phi \circ \phi)$ are $\mathcal{U}$-close.*

Proof. Let $\mathcal{U}$ be a given open cover of X and $\mathcal{U}_1$ be a star open refinement of $\mathcal{U}$. Then there is an open cover $\mathcal{V}_1$ of X satisfying the hypotheses of Proposition 3.1 (with $X = Y, \mathcal{U} = \mathcal{U}_1$ and f being the identity). Let $\mathcal{V}$ be a locally finite star-refinement of $\mathcal{V}_1$ and K be the $(n+1)$-dimensional skeleton of the nerve of $\mathcal{V}$ (we consider K as a simplicial complex, not as a polytope). For each $V \in \mathcal{V}$ pick a point $x_v \in V$ and define $\Phi'_0(V) = x_v$ (V is considered here as a vertex of K), and extend Φ'_0 to a homomorphism $\Phi_0 : C_0(K;G) \to S_0(X;G)$. Because $\mathcal{V}$ is a star-refinement of $\mathcal{V}_1$, Φ_0 is a correct partial algebraic realization of $C(K;G)$ in $\mathcal{V}_1$. So, by Proposition 3.1, Φ_0 extends to a full algebraic realization $\Phi : C(K;G) \to S(X;G)$ in $\mathcal{U}_1$.

Suppose now that Y is a paracompact space of dimension $\le n+1$ and $\varphi : S(Y;G) \to S(X;G)$ is a continuous correct $\mathcal{V}$-small morphism. Then $\varphi(y)$ is a singular 0-simplex in $S(X;G)$, so $|\varphi(y)|$ is a point and it is contained in some $V_y \in \mathcal{V}$. Since φ is continuous, there is a neighborhood Λ_y of y in Y such that $|\varphi(z)| \subset V_y$ for all $z \in \Lambda_y$. In this way we obtain an open cover $\Gamma = \{\Lambda_y : y \in Y\}$ of Y. Because $\dim Y \le n+1$, we can suppose that Γ is locally finite and its nerve $\mathcal{N}_\Gamma$ is at most $(n+1)$-dimensional. So, there exists a simplicial map $\lambda : \mathcal{N}_\Gamma \to K$ defined by the assignment $\Lambda_y \mapsto V_y$, $y \in Y$, and let $\kappa : Y \to |\mathcal{N}_\Gamma|$ be a canonical map ($|\mathcal{N}_\Gamma|$ is equipped with the Whitehead topology). According to [7, Proposition 8.6.6], there are an open cover $\mathcal{S}$ of $|\mathcal{N}_\Gamma|$ such that each $|s|$, $s \in \mathcal{N}_\Gamma$, is contained in some $P_s \in \mathcal{S}$, and a chain equivalence $\gamma : S(|\mathcal{N}_\Gamma|, \mathcal{S};G) \to C^\Omega(\mathcal{N}_\Gamma;G)$. Here $C^\Omega(\mathcal{N}_\Gamma;G)$ is the chain complex whose simplexes are finite arrays $[\Lambda_0, \Lambda_1, .., \Lambda_k]$, where all Λ_i, not necessarily distinct, are vertices of $\mathcal{N}_\Gamma$ spanning a simplex from $\mathcal{N}_\Gamma$. There exists also a natural chain morphism $\theta : C^\Omega(\mathcal{N}_\Gamma;G) \to C(\mathcal{N}_\Gamma;G)$ such that $\theta([\Lambda_0, \Lambda_1, .., \Lambda_k])$ is the simplex $(\Lambda_0, \Lambda_1, .., \Lambda_k) \in C(\mathcal{N}_\Gamma; G)$ if all Λ_i are distinct, and 0 otherwise. Let Υ be the intersection of the covers Γ and $\kappa^{-1}(\mathcal{S})$, and let $\phi : S(Y, \Upsilon; G) \to C(K; G)$ be the chain morphism $\phi = \lambda_\sharp \circ \theta \circ \gamma \circ \kappa_\sharp$.

It remains to show that $\varphi|S(Y, \Upsilon; G)$ and $(\Phi \circ \phi)$ are $\mathcal{U}$-close. We follow the final part of the proof of Proposition 3.6. Let $\sigma \in S(Y, \Upsilon; G)$ be a singular simplex. Since φ is $\mathcal{V}$-small, there is $V_\sigma \in \mathcal{V}$ containing $|\varphi(\tau)|$ for all faces τ of σ. On the other hand, $\sigma_1 = \kappa_\sharp(\sigma)$ is a singular simplex from $S(|\mathcal{N}_\Gamma|, \mathcal{S}; G)$ such that, according to the definition of γ (see [7, p. 339]), $\gamma(\sigma_1)$ is a "simplex" $s = [\Lambda_0, \Lambda_1, .., \Lambda_k]$ from $C^\Omega(\mathcal{N}; G)$ satisfying the following condition: if τ is a face of σ, then $\kappa_\sharp(\tau)$ is a face of σ_1 and the vertices of $\gamma(\kappa_\sharp(\tau))$ are also vertices of $\gamma(\kappa_\sharp(\sigma))$. In particular, for any vertex v of σ we have $\gamma(\kappa_\sharp(v)) = \gamma(\kappa_\sharp(|v|))$ is one of the vertexes Λ_i such that $|v|$ is a point from Λ_i. So, for every face τ of σ either $\phi(\tau) = 0$ or $\phi(\tau)$ is a simplex from K whose vertices are contained in the set $\{\lambda(\Lambda_i); i = 0, 1, ..k\}$, but definitely the union of all $\phi(\tau)$, τ is a face of σ, is non-empty. Hence, there exists a simplex $\delta \in K$ containing $\phi(\tau)$ for all faces τ of σ such that the vertices of δ are in the set $\{\lambda(\Lambda_i); i = 0, 1, ..k\}$. Since Φ is $\mathcal{U}_1$-small, we can find $U_\delta \in \mathcal{U}_1$ containing all $|\Phi(\phi(\tau))| \subset U_\delta$, τ is a face of σ. We fix a vertex v^* of σ. Then $\phi(v^*) = \lambda(\Lambda_j)$ for some $0 \le j \le k$ with $|v^*| \in \Lambda_j$, and $\varnothing \neq |\Phi(\phi(v^*))| \subset U_\delta$. But $\Phi(\lambda(\Lambda_j))$ is a singular 0-simplex from $S(X, G)$ whose carrier is a point $x^* \in \lambda(\Lambda_j)$. Consequently, according to the definition of the sets Λ_y, we have $|\varphi(v^*)| \in \lambda(\Lambda_j)$. Therefore, $x^* \in U_\delta \cap \lambda(\Lambda_j)$ and $|\varphi(v^*)| \in V_\sigma \cap \lambda(\Lambda_j)$ with $\lambda(\Lambda_j), V_\sigma \in \mathcal{V}$ and $U_\delta \in \mathcal{U}_1$. Since V_σ is contained in some element of $\mathcal{U}_1$, we have that

$$|\varphi(\tau)| \cup |\Phi(\phi(\tau))| \subset U_\delta \cup V_\sigma \cup \lambda(\Lambda_j) \subset \mathrm{St}\big(\lambda(\Lambda_\mathrm{j}), \mathcal{U}_1\big)$$

for all faces τ of σ. Finally, since $\mathcal{U}_1$ is a star refinement of $\mathcal{U}$, there is $U \in \mathcal{U}$ containing $\mathrm{St}\big(\lambda(\Lambda_\mathrm{j}, \mathcal{U}_1\big)$. □

Because every n-dimensional metric LC^n-space is an ANR, it is interesting if n-dimensional metric lc^n_G-spaces are algebraic ANR_G. We still do not know whether this is true, but we can show that any such space has a weaker property.

Definition 4.3. We say that a metric space X is an *approximate absolute neighborhood G-retract* (briefly, *algebraic* $AANR_G$) if for every embedding of X as a closed subset of a metric space Y and every open cover $\mathcal{U}$ of X there is a neighborhood W of X in Y, an open cover α of W and a chain morphism $\phi : S(W, \alpha; G) \to S(X; G)$ such that $\phi|S(X, \alpha; G)$ and the identity morphism on $S(X, \alpha; G)$ are $\mathcal{U}$-close. The morphism ϕ is called an algebraic approximate $\mathcal{U}$-retraction.

Proposition 4.4. *Any n-dimensional metric lc^n_G-space is an $AANR_G$.*

Proof. Let X be a metric lc^n_G-space and $\mathcal{U}$ be an open cover of X. By [9] X can be embedded as a closed subset of an $(n+1)$-dimensional metrizable AR-space Z. According to Proposition 3.6, there exist an open set $W_Z \subset Z$ containing X, an open cover β of W_Z and a chain morphism $\phi : S(W_Z, \beta; G) \to S(X; G)$ such that $\phi|S(X, \beta; G)$ and the identity on $S(X, \beta; G)$ are $\mathcal{U}$-close. Now, assume X is a closed subset of a metric space Y and $r : Y \to Z$ is a map extending the identity on X (such r exists because Z is an AR). Let $W = r^{-1}(W_Z)$ and $\alpha = r^{-1}(\beta)$. Then $\phi \circ r_\sharp : S(W, \alpha; G) \to S(X; G)$ is an algebraic approximate $\mathcal{U}$-retraction. □

It is well known that if $f : X \to Y$ is a closed homotopically UV^n-surjection between metric spaces, then Y is LC^n, see [1], [4], [5]. The question whether the homological version of this result is also true is very natural. It is easily seen that Y is lc^0_G provided f is a closed homologically UV^0_G-surjection between paracompact spaces. We can show that Y has the an "approximate version" of the lc^n_G-property if f is a closed homologically UV^n_G-surjection.

Definition 4.5. A space X has the *approximate* lc^n_G*-property* if for every $x \in X$ and its neighborhood U_x in X there exist two neighborhoods $V_x \subset W_x$ such that for every cycle $c^k \in S_k(V_x; G)$, $k \leq n$, there exists a cycle $\widetilde{c}^k \in S_k(W_x; G)$ whose vertices are the same as of c^k and homologous to zero in U_x.

Proposition 4.6. *Let $f : X \to Y$ be a closed homologically $UV^n(G)$-surjection between paracompact spaces. Then Y is approximately lc^n_G.*

Proof. Let $y \in Y$ and $U_y \subset Y$ be a neighborhood of y. Choose another two neighborhoods W_y and O_y of y such that $\overline{O}_y \subset W_y$ and $f^{-1}(W_y) \overset{H_k}{\hookrightarrow} f^{-1}(U_y)$ for all $k \leq n$. Then $\mathcal{U}_1 = \{W_y, Y \setminus \overline{O}_y\}$ is an open cover of Y and let $\mathcal{U}$ be a star-refinement of $\mathcal{U}_1$. There exists another open cover $\mathcal{V}$ of Y refining $\mathcal{U}$ and satisfying the hypotheses of Corollary 3.4. Let V_y be an element of $\mathcal{V}$ containing y. Obviously, $\mathrm{St(V_y,\mathcal{U}) \subset W_y}$. Consider $L = S_0(V_y; G)$ as a sub-complex of $S^{(n+1)}(V_y; G)$. Identifying the points of V_y and $f^{-1}(V_y)$ with the singular 0-simplexes in V_y and $f^{-1}(V_y)$, respectively, for every $z \in V_y$ we define $\varphi'(z) = x_z$ with $x_z \in f^{-1}(y)$, and extend φ' to a homomorphism $\varphi : L \to S_0(X; G)$. We can consider φ as a correct chain morphism from L into $S(X; G)$. Since the identity morphism $\phi : S^{(n+1)}(V_y; G) \to S(Y; G)$ is correct and $\mathcal{V}$-small and $f_\sharp \circ \varphi = \phi|L$, according to Corollary 3.4, φ can be extended to a chain morphism $\widetilde{\varphi} : S^{(n+1)}(V_y; G) \to S(X; G)$ such that ϕ and $f_\sharp \circ \widetilde{\varphi}$ are $\mathcal{U}$-close. Now, suppose c^k is a singular k-cycle in $S^{(n+1)}(V_y; G)$ for some $k \leq n$. Then $\eta^k = \widetilde{\varphi}(c^k)$ is a cycle in $S_k(X; G)$ such that $f_\sharp(\eta^k)$ is $\mathcal{U}$-close to c^k. Since $f_\sharp \circ \varphi = \phi|L$, the "vertexes" of $\widetilde{c}^k = f_\sharp(\eta^k)$ and c^k coincide, so $|\widetilde{c}^k| \subset \mathrm{St(V_y,\mathcal{U}) \subset W_y}$. Hence, $|\eta^k| \subset f^{-1}(W_y)$ and there exists a $(k + 1)$-chain $\eta^{k+1} \in S(f^{-1}(U_y); G)$ with $\partial\eta^{k+1} = \eta^k$. This implies that $\widetilde{c}^k$ is homologous to zero in U_y. Therefore Y is approximatively lc^n_G. □

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