V. M. Valov, K. L. Kozlov, Spectral representations of topological groups and near-openly generated groups, *Sbornik: Mathematics*, 2020, Volume 211, Issue 2, 258–274

DOI: https://doi.org/10.1070/SM9193

Sbornik: *Mathematics* **211**:2 258–274 *Matematicheskiĭ Sbornik* **211**:2 106–124
DOI: https://doi.org/10.1070/SM9193

Spectral representations of topological groups and near-openly generated groups

V. M. Valov and K. L. Kozlov

Abstract. Near-openly generated groups are introduced. They form a topological and multiplicative subclass of $\mathbb{R}$-factorizable groups. Dense and open subgroups, quotients and the Raikov completion of a near-openly generated group are near-openly generated. Almost connected pro-Lie groups, Lindelöf almost metrizable groups and the spaces $C_p(X)$ of all continuous real-valued functions on a Tychonoff space with pointwise convergence topology are near-openly generated.

We provide characterizations of near-openly generated groups using methods of inverse spectra and topological game theory.

Bibliography: 24 titles.

Keywords: topological group, (nearly open) homomorphism, inverse spectrum, topological game, $\mathbb{R}$-factorizable group.

§ 1. Introduction

Inverse spectra are a useful device in topology and topological algebra. They provide a technique for approximating complicated spaces by simple ones. Pontryagin constructed a Lie group series for compact groups — transfinite continuous spectra of groups G_α and open homomorphisms such that the kernels of bonding homomorphisms $p_\alpha^{\alpha+1}$ and G_0 are compact Lie groups. Other examples of groups which are defined and investigated by means of inverse spectra are pro-Lie groups [1] and almost connected pro-Lie groups [2]. More applications of inverse spectra in the category of groups can be found in [3]–[9].

Everywhere below by a *spectrum* we mean an inverse spectrum. A spectrum $S = \{X_\alpha, p_\alpha^\beta, A\}$ is called an *almost continuous σ-spectrum* if it satisfies the following conditions:

(1) all the spaces X_α are second-countable and the bonding maps p_α^β are surjective;

(2) the directed set A is σ-complete (every increasing sequence in A has a supremum in A);

The research of V. M. Valov was carried out with the support of the Natural Sciences and Engineering Research Council of Canada — NSERC (grant no. 261914-13). The research of K. L. Kozlov was carried out with the support of the Russian Foundation for Basic Research (grant no. 17-51-18051 Болг_а).

AMS 2010 *Mathematics Subject Classification*. Primary 22A05, 22A25.

(3) for every increasing sequence $\{\alpha_n\} \subset A$ the space X_β is a dense subset of $\varprojlim\{X_{\alpha_n}, p_{\alpha_n}^{\alpha_{n+1}}, n \geqslant 1\}$, where $\beta = \sup\{\alpha_n\}$.

If a space X is embedded in $\varprojlim S$ in such a way that $p_\alpha(X) = X_\alpha$ for each α, where $p_\alpha\colon \varprojlim S \to X_\alpha$ is the αth limit projection, then we say that X is the *almost limit* of the spectrum S, the notation being $X = \mathrm{a} - \varprojlim S$. In case X is the almost limit of a spectrum S and for every continuous real-valued function f on X there exists $\alpha \in A$ and a continuous function f_α on X_α such that $f = f_\alpha \circ p_\alpha$, S is said to be a *factorizing spectrum*.

Recall that a continuous map $f\colon X \to Y$ is *nearly open* [10] (*skeletal*) if $f(U) \subset \operatorname{Int} \overline{f(U)}$ ($\operatorname{Int} \overline{f(U)} \neq \varnothing$, respectively) for every open $U \subset X$, where $\overline{f(U)}$ denotes the closure of $f(U)$ in Y. Nearly open maps were introduced in [11] (see also [12], where nearly open maps were called *d-open*). In [13] skeletal maps were called *ad-open*.

Definition 1. A topological group G is *near-openly generated* if $G = \mathrm{a} - \varprojlim S_G$, where $S_G = \{G_\alpha, p_\alpha^\beta, A\}$ is a factorizing almost continuous σ-spectrum consisting of second-countable topological groups G_α and continuous nearly open homomorphisms p_α^β.

The aim of this paper is to describe the class of near-openly generated topological groups. This class is topological (that is, invariant under homeomorphisms) and has nice properties. It is multiplicative. Dense and open subgroups, quotients and the Raikov completion of a near-openly generated group are near-openly generated. Almost connected pro-Lie groups (in particular compact groups) and Lindelöf almost metrizable groups [7] are near-openly generated. Another example of such groups are the spaces $C_p(X)$ of all continuous real-valued functions on a Tychonoff space X with pointwise convergence topology. It is worth noting that the class of near-openly generated groups is a subclass of $\mathbb{R}$-factorizable groups for which the problems of whether it is multiplicative and topological are unsolved.

Theorem 2 provides a topological characterization of near-openly generated groups.

The following are equivalent for a topological group G:

(1) *G is near-openly generated*;

(2) *G is* I-*favourable*;

(3) *G has a σ-lattice of skeletal maps*;

(4) *G has a σ-lattice of nearly open homomorphisms.*

I-favourable spaces [14] and σ-lattices of maps on a given space are defined in § 2. Note that, by Theorem 4, I-favourability of an ω-narrow group is a local property.

It follows from the results in § 2 that near-openly generated groups can be described as dense subgroups of the limits of almost continuous σ-spectra of topological groups and continuous nearly open homomorphisms (not necessarily factorizing). At the same time, near-openly generated groups are exactly those groups whose underlying spaces are almost limits of almost continuous σ-spectra with skeletal bonding maps (such spaces are called skeletally generated spaces [15]).

We also establish characterization of near-openly generated groups (Theorem 3) using their isomorphic embeddings in products of second-countable topological groups (see § 2 for the definitions of π-regular and regular embeddings).

The following are equivalent for a topological group G:

(1) G *is near-openly generated*;

(2) G *is topologically isomorphic to a subgroup of a product of second-countable topological groups and any such embedding is* π*-regular*;

(3) G *is topologically isomorphic to a subgroup of a product of second-countable topological groups and any such embedding is regular.*

The paper is organized as follows. Section 2 contains preliminary information about I-favourable and skeletally generated spaces. Section 3 contains the proofs of Theorems 2–4 mentioned above. We also provide the main properties of near-openly generated groups and examples of such groups. The last section, § 4, contains necessary information about spectra and σ-lattices. Proposition 10 is a spectral theorem of the form spectrum–lattice. The idea is similar to the classical spectral theorem for factorizing spectra [16] and the theorem on intersection of lattices [17]. We also provide a spectral representation of $\mathbb{R}$-factorizable groups [18].

All spaces are assumed to be Tychonoff and the maps are continuous. For information about topological groups and spectra see [10] and [16], respectively.

§ 2. I-favourable and skeletally generated spaces

I-favourable spaces were introduced in [14]. Two players are playing the so-called *open-open game* in a space $(X, \mathscr{T}_X)$. The players take countably many turns, a round consists of player I choosing a nonempty open set $U \subset X$ and player II choosing a nonempty open set $V \subset U$; player I wins if the union of II's open sets is dense in X, otherwise player II wins. A space X is called I-*favourable* if player I has a winning strategy. This means that there exists a function $\mu\colon \bigcup_{n\geqslant 0} \mathscr{T}_X^n \to \mathscr{T}_X$ such that for each game

$$\mu(\varnothing), B_0, \mu(B_0), B_1, \mu(B_0, B_1), B_2, \dots, B_n, \mu(B_0, \dots, B_n), B_{n+1}, \dots$$

such that $\mu(\varnothing) \neq \varnothing$ and $B_{k+1} \subset \mu(B_0, B_1, \dots, B_k) \neq \varnothing$ and $\varnothing \neq B_k \in \mathscr{T}_X$ for all $k \geqslant 0$ the union $\bigcup_{n\geqslant 0} B_n$ is dense in X.

If there exists an almost continuous σ-spectrum S with skeletal bonding maps such that $X = \mathrm{a} - \varprojlim S$, we say that X is *skeletally generated*: see [15]. Let us mention that in [19] a space X is called skeletally generated if X is an almost limit of a factorizing almost continuous σ-spectrum. It follows from Proposition 1 below that the two definitions are equivalent.

Recall that a subspace X of a space Y is π*-regularly embedded in* Y [13] if there exists a function (a π*-regular operator*) $\mathrm{e}\colon \mathscr{T}_X \to \mathscr{T}_Y$ between the topologies of X and Y such that:

(e1) $\mathrm{e}(\varnothing) = \varnothing$ and $\mathrm{e}(U) \cap X$ is a dense subset of U;

(e2) $\mathrm{e}(U) \cap \mathrm{e}(V) = \varnothing$ for any $U, V \in \mathscr{T}_X$, provided $U \cap V = \varnothing$.

A similar function with the additional property that $\mathrm{e}(U) \cap \mathrm{e}(V) = \mathrm{e}(U \cap V)$ for all $U, V \in \mathscr{T}_X$ was also called a π*-regular extension operator* in [20]). If e satisfies condition (e2) and $\mathrm{e}(U) \cap X = U$ for all $U \in \mathscr{T}_X$, the operator e is called *regular* (see [21]).

Following [17], [22], Definition 8, and [13], we say that a family Ψ of maps from a space X to second-countable spaces is a σ*-lattice on* X if Ψ satisfies the following conditions.

(L1) If $\{\varphi_n\} \subset \Psi$ is such that $\varphi_{n+1} \prec \varphi_n$ for all n, then Ψ contains the diagonal product $\varphi = \triangle_{n\geqslant 1}\varphi_n$ (here $\varphi_{n+1} \prec \varphi_n$ means that there is a continuous map $\varphi_n^{n+1}\colon \varphi_{n+1}(X) \to \varphi_n(X)$ such that $\varphi_n = \varphi_n^{n+1} \circ \varphi_{n+1}$).

(L2) For any map $f\colon X \to f(X)$ with $f(X)$ having a countable weight there is $\varphi \in \Psi$ such that $\varphi \prec f$ (that is, there is a map $h\colon \varphi(X) \to f(X)$ such that $f = h \circ \varphi$).

The following characterizations of skeletally generated spaces (using our definition) were established in [15], Theorem 1.1.

Theorem 1 (see [15]). *For a space X the following are equivalent*:
(1) *X is* I-*favourable*;
(2) *every embedding of X in another space is π-regular*;
(3) *X is skeletally generated.*

We can extend the list of conditions characterizing skeletally generated spaces. A family of sets $\{B_t\}_{t<\lambda}$ indexed by ordinals such that $B_t \subset B_{t+1}$, $t+1 < \lambda$, and $B_\alpha = \bigcup\{B_t\colon t < \alpha\}$ for limit ordinals $\alpha < \lambda$ is called an *increasing transfinite family of sets*.

Proposition 1. *For a space X the following are equivalent*:
(1) *X is skeletally generated*;
(2) *X has a σ-lattice of skeletal maps*;
(3) *there exists a factorizing almost continuous σ-spectrum S with skeletal bonding maps such that $X = \mathrm{a} - \varprojlim S$.*

Proof. (1) $\Rightarrow$ (2). Suppose that X is skeletally generated. Then, X has countable cellularity since it is I-favourable (see [14], Theorem 1.1). If X is metrizable, then X is second-countable and the identity map forms a σ-lattice with the required properties. So, let X be of uncountable weight. We consider X as a C-embedded subset of a product $\Pi = \prod_{\alpha\in A} X_\alpha$ of second-countable spaces. Then there exists a π-regular operator $\mathrm{e}\colon \mathscr{T}_X \to \mathscr{T}_\Pi$.

For any set $B \subset A$ fix a standard open base $\mathscr{B}_B$ of cardinality $\max\{|B|, \aleph_0\}$ for $\prod_{\alpha\in B} X_\alpha$ and consider the projection $\pi_B\colon \Pi \to \prod_{\alpha\in B} X_\alpha$. We say that a set $B \subset A$ is e-*admissible* if

$$\pi_B^{-1}(\pi_B(\overline{\mathrm{e}(\pi_B^{-1}(U)\cap X)})) = \overline{\mathrm{e}(\pi_B^{-1}(U)\cap X)}$$

for all $U \in \mathscr{B}_B$. Because for every open set $W \subset \Pi$ there is a countable set $B_W \subset A$ with $\pi_{B_W}^{-1}(\pi_{B_W}(\overline{W})) = \overline{W}$, one can show that the family $\mathscr{A}$ of all e-admissible subsets of A has the following properties, where $X_B = \pi_B(X)$ (see the proof of Proposition 3.1, (ii) in [23], or [19], Proposition 3.7):

- for every open $V \subset X$ and every $B \in \mathscr{A}$ we have $\pi_B(\mathrm{e}(V)) \subset \overline{\pi_B(V)}$ and the restriction map $p_B = \pi_B|_X\colon X \to X_B$ is skeletal;
- the union of every increasing transfinite family $\{B_t\} \subset \mathscr{A}$ belongs to $\mathscr{A}$;
- for any set $C \subset A$ there is $B_C \in \mathscr{A}$ of cardinality $|B_C| = \max\{|C|, \aleph_0\}$ with $C \subset B_C$.

Claim 1. *For every surjective map $f\colon X \to M$, where M is a second-countable space, there is a countable* e-*admissible set $B \subset A$ such that $p_B \prec f$.*

Indeed, let $\{h_k\}$ be a sequence of continuous functions on M determining the topology of M. Since X is C-embedded in Π, each $h_k \circ f$ can be continuously extended to a function f_k on Π. So, there is a countable set $B \subset A$ such that all the f_k can be factored through $\Pi_B = \prod_{\alpha\in B} X_\alpha$. Consequently, for any k there is a function g_k on Π_B with $g_k \circ \pi_B = f_k$. The maps g_k determine a map $g\colon X_f = \pi_B(X) \to M$ such that $f = g \circ \pi_B|_X$. Because every countable subset of B is contained in a countable e-admissible set, we may assume that B is also e-admissible.

Let us show that the family Ψ of all maps p_B such that $B \subset A$ is countable and e-admissible form a σ-lattice of skeletal maps for X. Note that every $p_B \in \Psi$ is skeletal. By Claim 1, Ψ satisfies condition (L2). To show that Ψ also satisfies (L1), suppose $\{B_n\}$ is a sequence of countable e-admissible subsets of A such that $p_{B_{n+1}} \prec p_{B_n}$ for all n. Note that the last relation does not imply that each B_n is a subset of B_{n+1}, but we have surjective maps $q_n^{n+k}\colon X_{B_{n+k}} \to X_{B_n}$, $n, k \geqslant 1$, such that $q_{n-1}^n \circ q_n^{n+1} = q_{n-1}^{n+1}$ for each n. So, there is an inverse sequence $S = \{X_{B_n}, q_n^{n+1}, n \geqslant 1\}$. Because $p_B \prec \triangle_n p_{B_n} \prec p_B$, where $B = \bigcup_{n=1}^\infty B_n$, X_B is a dense set in $\varprojlim S$ and there are maps $q_n\colon X_B \to X_{B_n}$ with $q_n \circ p_B = p_{B_n}$, $n \geqslant 1$. This means that if $\mathscr{B}_n$ is a standard open base for $\Pi_{B_n} = \prod_{\alpha\in B_n} X_\alpha$, then the open family $\mathscr{B} = \{q_n^{-1}(V \cap X_{B_n})\colon V \in \mathscr{B}_n, n \geqslant 1\}$ is a base for X_B. It is clear that each $W \in \mathscr{B}$ is the intersection of X_B with an element of the standard open base for Π_B. Because each B_n is e-admissible, we have

$$\pi_{B_n}^{-1}(\pi_{B_n}(\overline{\mathrm{e}(p_{B_n}^{-1}(V \cap X_{B_n}))})) = \overline{\mathrm{e}(p_{B_n}^{-1}(V \cap X_{B_n}))}.$$

Since $\pi_B \prec \pi_{B_n}$ for all n, we obtain that if $W = q_n^{-1}(V \cap X_{B_n})$, then

$$\mathrm{e}(p_B^{-1}(W)) = \mathrm{e}(p_{B_n}^{-1}(V \cap X_{B_n})) \quad \text{and} \quad \pi_B^{-1}(\pi_B(\overline{\mathrm{e}(p_B^{-1}(W))})) = \overline{\mathrm{e}(p_B^{-1}(W))}.$$

The last equalities imply that p_B is e-admissible.

(2) ⇒ (3) follows from Proposition 9.

(3) ⇒ (1). This implication is trivial.

Proposition 1 is proved.

Corollary 1 (see [15]). *A space X is* I*-favourable if and only if X is π-regularly embedded in a product of second-countable spaces.*

The first item of the next proposition is from [15], while the others were established in [14], Corollary 4.

Proposition 2 (see [14] and [15]). *Let X be an* I*-favourable space. Then*:
- *every open subset of X is* I*-favourable*;
- *every dense subspace of X is* I*-favourable*;
- *every space containing X as a dense subspace is* I*-favourable*;
- *every image of X under a skeletal map is* I*-favourable*;
- *a product of* I*-favourable spaces is* I*-favourable.*

A subspace Y of X is said to be *z-embedded* in X if for every zero-set F in Y there exists a zero-set Φ in X such that $F = Y \cap \Phi$.

Proposition 3. *If Y is π-regularly embedded in the product Π of second-countable spaces*, *then Y is z-embedded in Π.*

Proof. Let $\mathrm{e}\colon \mathscr{T}_Y \to \mathscr{T}_\Pi$ be a π-regular operator and $F \in Y$ be a zero-set. We represent F as the intersection of a decreasing sequence $\{U_i\}$ of open sets $U_i \subset Y$ with $\overline{U}_{i+1} \subset U_i$ for each $i \geqslant 1$. Since Π is a product of second-countable spaces, all the $\overline{\mathrm{e}(U_i)}$ are zero-sets in Π. Obviously, $F \subset \Phi = \bigcap \overline{\mathrm{e}(U_i)}$. If there is a point $x \in (\Phi \cap Y) \setminus F$, then $x \notin \overline{U}_k$ for some k. So, we can find a neighbourhood $V \subset Y$ of x with $V \cap \overline{U}_k = \varnothing$. Consequently, $x \in \overline{\mathrm{e}(V)}$ and $\mathrm{e}(V) \cap \mathrm{e}(U_k) = \varnothing$, which contradicts the inclusion $x \in \overline{\mathrm{e}(U_k)}$. Hence $F = \Phi \cap X$.

The proposition is proved.

We also need the next proposition, which was established in [15], Proposition 3.3. Recall that a transfinite spectrum $S = \{X_\alpha, p^\beta_\alpha,\ \alpha \leqslant \beta < \tau\}$ is almost continuous if for any limit ordinal β the space X_β is a (dense) subset of $\varprojlim\{X_\alpha, p^{\alpha'}_\alpha,\ \alpha \leqslant \alpha' < \beta\}$.

Proposition 4 (see [15], Proposition 4.3). *Let $S = \{X_\alpha, p^\beta_\alpha,\ \alpha \leqslant \beta < \tau\}$ be a transfinite almost continuous spectrum with nearly open bonding maps such that $X = \mathrm{a}-\varprojlim S$. Then*:

(1) *X is regularly embedded in $\prod_{\alpha<\tau} X_\alpha$*;

(2) *if, additionally, each X_α is regularly embedded in a space Y_α, then X is regularly embedded in $\prod_{\alpha<\tau} Y_\alpha$.*

§ 3. Near-openly generated groups

3.1. Characterizations and properties. A topological group homeomorphic to an I-favourable space is called an I-*favourable group*. Following [10], a topological group G is said to be $\mathbb{R}$-*factorizable* if for every continuous real-valued function f on G there is a continuous homomorphism $\pi\colon G \to K$ onto a second-countable topological group K and a continuous function h on K such that $f = h \circ \pi$. We also say that G is an ω-*narrow group* [10] if G is topologically isomorphic to a subgroup of a product of second-countable groups.

Proposition 5. *Any* I-*favourable topological group is* $\mathbb{R}$-*factorizable.*

Proof. Any I-favourable space has countable cellularity [14]. Therefore, G is an ω-narrow group (see [10], Theorem 3.4.7). So, G is topologically isomorphic to a subgroup of a product Π of second-countable groups. By Theorem 1 the embedding of G in Π is π-regular and, by Proposition 3, it is a z-embedding in the $\mathbb{R}$-factorizable group Π. Therefore, according to Theorem 8.2.6 in [10], G is an $\mathbb{R}$-factorizable group.

Theorem 2. *The following are equivalent for a topological group G*:

(1) *G is near-openly generated*;

(2) *G is* I-*favourable*;

(3) *G has a σ-lattice of skeletal maps*;

(4) *G has a σ-lattice of nearly open homomorphisms.*

Proof. (1) $\Rightarrow$ (2). The space of a near-openly generated group is skeletally generated. So, by Theorem 1, G is I-favourable.

(2) $\Rightarrow$ (3). Suppose G is I-favourable. Consequently, it is skeletally generated, and according to Proposition 1, G has a σ-lattice consisting of skeletal maps.

(3) ⇒ (4). Suppose G has a σ-lattice Ψ_1 of skeletal maps. Then, by Proposition 1 and Theorem 1, G is I-favourable. So, G is an $\mathbb{R}$-factorizable group (see Proposition 5). Hence the family Ψ_2 of all continuous homomorphisms from G onto second-countable groups is a σ-lattice. Thus, by Proposition 11 the intersection Ψ of these two σ-lattices is a σ-lattice consisting of continuous homomorphisms which are skeletal maps. Finally, because a skeletal homomorphism is nearly open (see [10], Lemma 4.3.29), Ψ is a σ-lattice of homomorphisms which are nearly open maps.

(4) ⇒ (1) follows from Proposition 9.

The theorem is proved.

Proposition 2 and Theorem 2 provide the following properties of near-openly generated groups.

Proposition 6. *Let G be a near-openly generated group. Then:*

- *every dense subgroup of G is also near-openly generated;*
- *every topological group containing G as a dense subgroup (in particular, the Raikov completion of G) is near-openly generated;*
- *every coset space of G is skeletally generated (in particular, any quotient group of G is near-openly generated);*
- *a product of near-openly generated groups is near-openly generated.*

The next proposition provides another interesting property of near-openly generated groups.

Proposition 7. *Let G be a near-openly generated group with $G = a-\varprojlim S$, where S is a factorizing almost continuous σ-spectrum. Then the spectrum S has a cofinal subspectrum which is an almost continuous σ-spectrum consisting of groups and nearly open homomorphisms.*

Proof. Since there exists a σ-lattice of nearly open homomorphisms on G, we apply Proposition 10 to finding a cofinal subspectrum of S with the required properties. The proposition is proved.

Our next aim is to establish the external characterization of near-openly generated groups which we stated in Theorem 3.

Proposition 8. *Let G be a near-openly generated topological group of uncountable weight τ, $\lambda = \mathrm{cf}(\tau)$. Then there is an almost continuous transfinite spectrum $S_G = \{G_\gamma, p^\delta_\gamma, \gamma \leqslant \delta < \lambda\}$ of near-openly generated topological groups G_γ and nearly open homomorphisms p^δ_γ, such that $w(G_\gamma) < \tau$ for each γ and $G = \mathrm{a}-\varprojlim S_G$.*

Proof. According to Theorem 2 and Lemma 4.3.29 in [10], it suffices to find a spectrum $S_G = \{G_\gamma, p^\delta_\gamma, \gamma \leqslant \delta < \lambda\}$ of I-favourable groups G_γ and skeletal homomorphisms p^δ_γ, where $\lambda = \mathrm{cf}(\tau)$, such that $w(G_\gamma) < \tau$ for each γ and $G = \mathrm{a}-\varprojlim S_G$. Since G is $\mathbb{R}$-factorizable by Proposition 5, G is topologically isomorphic to a subgroup of the product of a family of second-countable topological groups G_α, $\alpha \in A$, with $|A| = \tau$. For each α let $\overline{G}_\alpha$ be a metrizable compactification of G_α and A be the union of an increasing transfinite family $\{A_\delta\}_{\delta<\lambda}$ with $|A_\delta| < \tau$ for each $\delta < \lambda$. Then the closure $\overline{G}$ of G in the product $H = \prod_{\alpha\in A} \overline{G}_\alpha$, being a compactification of G, is I-favourable (see Proposition 2). So, there is a π-regular operator $\mathrm{e}\colon \mathscr{T}_{\overline{G}} \to \mathscr{T}_H$ (see Theorem 1).

For every subset $B \subset A$ let $\pi_B\colon H \to H_B = \prod_{\alpha\in B} \overline{G}_\alpha$ denote the projection, $\widetilde{G}_B = \pi_B(\overline{G})$ and let $G_B = p_B(G)$, where $p_B = \pi_B|_G$. We also fix a standard open base $\mathscr{B}_B$ for H_B of cardinality $\max\{|B|, \aleph_0\}$, $B \subset A$. For each $U \in \mathscr{B}_B$ there is a countable set $k(U) \subset A$ with

$$\pi_{k(U)}^{-1}(\pi_{k(U)}(\overline{\mathrm{e}(\pi_B^{-1}(U) \cap G)})) = \overline{\mathrm{e}(\pi_B^{-1}(U) \cap G)}.$$

This can be done because each $\overline{\mathrm{e}(\pi_B^{-1}(U) \cap G)}$ is a zero-set in H and every continuous function on H depends on countably many coordinates. Following the proof of Proposition 1, let $\mathscr{A}$ be the family of all e-admissible subsets of A. Recall that the restriction $\pi_B|_{\overline{G}}\colon \overline{G} \to \widetilde{G}_B$ is a skeletal map for each $B \in \mathscr{A}$ and for every $B \subset A$ there is a set $B_\infty \in \mathscr{A}$ containing B with $|B_\infty| = \max\{|B|, \aleph_0\}$ (see the proof of Proposition 1, (1) $\Rightarrow$ (2)).

Next, using transfinite induction we construct an increasing family $\{B_\delta\}_{\delta<\lambda}$ of e-admissible sets $B_\delta \subset A$ satisfying the following conditions:

(1) $A_\delta \subset B_\delta$ and $|B_\delta| = |A_\delta|$;

(2) $B_\delta = \bigcup_{\gamma<\delta} B_\gamma$ when δ is a limit ordinal.

So, A is the union of an increasing transfinite family $\{B_\delta\}_{\delta<\lambda}$ of e-admissible sets. Hence the spectrum $S_G = \{G_\gamma, p_\gamma^\delta, \gamma \leqslant \delta < \lambda\}$ is almost continuous and $G = \mathrm{a}-\varprojlim S_G$, where $G_\gamma = G_{B_\gamma}$ and p_γ^δ is the restriction of the projection $\pi_{B_\gamma}^{B_\delta}\colon H_{B_\delta} \to H_{B_\gamma}$ to G_{B_δ}. Since each set B_γ is e-admissible, the restrictions $\pi_{B_\gamma}|\overline{G}\colon \overline{G} \to \widetilde{G}_{B_\gamma}$ are skeletal and so are the homomorphisms $p_\gamma\colon G \to G_\gamma$ since G is a dense subset of $\overline{G}$. This implies that each G_γ is I-favourable (as a skeletal image of G), and the bonding homomorphisms p_γ^δ are also skeletal.

The proposition is proved.

Now we can establish an external characterization of near-openly generated groups.

Theorem 3. *The following are equivalent for a topological group G:*

(1) *G is near-openly generated;*

(2) *G is topologically isomorphic to a subgroup of a product of second-countable topological groups and any such embedding is π-regular;*

(3) *G is topologically isomorphic to a subgroup of a product of second-countable topological groups and any such embedding is regular.*

Proof. (1) $\Rightarrow$ (2). Suppose G is a near-openly generated group. Then, by Theorem 2, G is I-favourable. So, G is $\mathbb{R}$-factorizable (see Proposition 5). Consequently, according to Proposition 8.1.3 in [10], G is topologically isomorphic to a subgroup of a product of second-countable topological groups. Finally, by Theorem 1, any embedding of G into another space is π-regular.

(2) $\Rightarrow$ (3). Suppose G satisfies (2). Then G is π-regularly embedded in a product of second-countable spaces and, by Corollary 1, G is I-favourable. We are going to show, using transfinite induction with respect to $w(G)$, that every subgroup of a product of second-countable topological groups, which are topologically isomorphic to G, is regularly embedded in that product. Suppose $w(G)$ is countable and let G be a subgroup of a product $\Pi = \prod_{\alpha\in\Gamma} G_\alpha$ of second-countable groups. We can find a countable set $A \subset \Gamma$ and a countable base $\mathscr{B}$ of G such

that $\pi_A|_G\colon G \to \pi_A(G) = G_A$ is an isomorphism and $\pi_A^{-1}(\pi_A(V)) \cap G = V$ for all $V \in \mathscr{B}$. Since $\Pi_A = \prod_{\alpha\in A} G_\alpha$ is second-countable, there is a regular operator $\mathrm{e}_A\colon \mathscr{T}_{G_A} \to \mathscr{T}_{\Pi_A}$. Then

$$\mathrm{e}\colon \mathscr{T}_G \to \mathscr{T}_\Pi, \quad \mathrm{e}(U) = \bigcup\{\pi_A^{-1}(\mathrm{e}_A(\pi_A(V)))\colon V \in \mathscr{B} \text{ and } V \subset U\},$$

is a regular operator.

Suppose $w(G) = \tau$ is uncountable and our statement holds for every I-favourable group of weight $< \tau$. Let G be a subgroup of a product $\Pi = \prod_{\alpha\in\Gamma} G_\alpha$ of second-countable groups. For any sets $C \subset B \subset \Gamma$ let $\pi_B\colon \Pi \to \Pi_B = \prod_{\alpha\in B} G_\alpha$ and $\pi_C^B\colon \Pi_B \to \Pi_C$ denote the corresponding projections. Take a local base $\{U_\gamma\colon \gamma \in \Lambda\}$ at the neutral element in G of cardinality $|\Lambda| = \tau$ with each U_γ being of the form $U_\gamma = \pi_{B_\gamma}^{-1}(V_\gamma) \cap G$, where $B_\gamma \subset \Gamma$ is finite and V_γ is a neighbourhood of the neutral element in $\pi_{B_\gamma}(G)$. Then the set $A = \bigcup_{\gamma\in\Lambda} B_\gamma$ is a subset of Γ of cardinality τ and G is topologically isomorphic to $\pi_A(G)$. We identify G with the image $\pi_A(G)$. By Proposition 8, A can be covered by an increasing family A_δ, $\delta < \lambda = \mathrm{cf}(\tau)$, such that:

- $|A_\delta| < \tau$ for each δ;
- the spectrum $S_G = \{G_\delta, q_\eta^\delta, \eta \leqslant \delta < \lambda\}$, where $G_\delta = \pi_{A_\delta}(G)$ and $q_\eta^\delta = \pi_{A_\eta}^{A_\delta}|_{G_\delta}$, is almost continuous and consists of I-favourable groups and nearly open homomorphisms;
- $G = \mathrm{a}-\varprojlim S_G$.

According to our assumption, for each δ there exists a regular operator $\mathrm{e}_\delta\colon \mathscr{T}_{G_\delta} \to \mathscr{T}_{\Pi_{A_\delta}}$. Then Proposition 4 implies the existence of a regular operator $\mathrm{e}\colon \mathscr{T}_G \to \mathscr{T}_{\Pi_\lambda}$, where $\Pi_\lambda = \prod_{\delta<\lambda} \Pi_{A_\delta}$. Let $\pi\colon \Pi \to \Pi_\lambda$ denote the diagonal product of all homomorphisms π_{A_δ}, and for every open $U \subset G$ let $\theta(U) = \pi^{-1}(\mathrm{e}(U))$. It is easily seen that $\theta\colon \mathscr{T}_Q \to \mathscr{T}_\Pi$ is regular.

(3) $\Rightarrow$ (1). This implication follows from Corollary 1 and Theorem 2.

The theorem is proved.

Corollary 2. *Let $S_G = \{G_\gamma, p_\gamma^\delta, \gamma \leqslant \delta < \lambda\}$ be a transfinite almost continuous spectrum of near-openly generated groups G_γ and nearly open homomorphisms p_γ^δ such that $G = \mathrm{a}-\varprojlim S_G$. Then G is a near-openly generated group.*

Proof. By Theorem 3, each G_γ is regularly embedded in a product H_γ of second-countable groups. So, according to Proposition 4, G is regularly embedded in the product $\prod_{\gamma<\delta} H_\gamma$. Finally, Theorem 3 completes the proof.

The next theorem provides a local characterization of near-openly generated groups.

Theorem 4. *An ω-narrow group G is near-openly generated if and only if G has an* I*-favourable neighbourhood of the identity.*

Proof. If G is near-openly generated, then it is I-favourable and, by Proposition 2, every open subset of G is I-favourable.

Suppose G is ω-narrow and let V be an I-favourable neighbourhood of the identity.

First, we show that G is $\mathbb{R}$-factorizable. As an ω-narrow group, G is isomorphically embedded in a product $\Pi = \prod_{\alpha\in A} G_\alpha$ of second-countable groups. Taking an open subset of V if necessary, we can assume that $V = G \cap O$ where O is a co-zero set in Π. By Theorem 1, V is π-regularly embedded in O and, according to Proposition 3, V is z-embedded in O. Therefore, for every co-zero set U in V there is a co-zero set $W \subset O$ with $U = V \cap W$. Thus, $U = G \cap W$. Since G is ω-narrow, there is a sequence $\{g_i\} \subset G$ such that $\{g_i V : i \in \mathbb{N}\}$ covers G. Each set $g_i V$ is I-favourable, being homeomorphic to V. Moreover, $g_i O \cap G = g_i V$ and each $g_i V$ is z-embedded in $g_i O$. As above, we can show that for every co-zero set $U_i \subset g_i V$ there is a co-zero set $W_i \subset g_i O$ with $U_i = G \cap W_i$, $i \geqslant 1$. This yields that G is z-embedded in Π. Indeed, if U is a co-zero set in G, then each $U_i = U \cap g_i V$ is a co-zero set in $g_i V$. Hence, for every i there is a co-zero set $W_i \subset g_i O$ such that $U_i = G \cap W_i$. Finally, the set $W = \bigcup_{i=1}^{\infty} W_i$ is a co-zero set in Π because so is every W_i, and $W \cap G = U$. Thus, G is z-embedded in Π. Hence G is an $\mathbb{R}$-factorizable group (see [10], Theorem 8.2.6).

We can complete the proof. Since V is I-favourable, there exists a σ-lattice L of skeletal maps on V (see Proposition 1). On the other hand, since G is $\mathbb{R}$-factorizable, all continuous homomorphisms on G having second-countable images form a factorizing almost continuous σ-spectrum S_G with $G = \mathrm{a} - \varprojlim S_G$. Then, by Proposition 13, the restriction $S_G|_V$ is a factorizing almost continuous σ-spectrum and $V = \mathrm{a} - \varprojlim S_G|_V$. So, according to Proposition 10, there exists a cofinal almost continuous σ-subspectrum S of $S_G|_V$ consisting of skeletal maps. Observe that S is the restriction of an almost continuous σ-spectrum $\widetilde{S} = \{G_\alpha, p_\alpha^\beta, A\}$, a subspectrum of S_G. Since S is cofinal in $S_G|_V$, $\widetilde{S}$ is also cofinal in S_G. Hence $\widetilde{S}$ is factorizing (because so is S_G) and $G = \mathrm{a} - \varprojlim \widetilde{S}$. Moreover, because every locally skeletal homomorphism is skeletal, all the projections $p_\alpha : G \to G_\alpha$ are nearly open homomorphisms (see [10], Lemma 4.3.29). Therefore, G is a near-openly generated group.

The theorem is proved.

3.2. Examples and questions.

Example 1. Any compact group is near-openly generated and has a σ-lattice of open homomorphisms onto metrizable compact groups.

Example 2. The function space $C_p(X)$ equipped with the pointwise convergence topology is near-openly generated as a dense subgroup of the product $\mathbb{R}^X$.

Example 3. Any topological group G homeomorphic to an I-favourable space with a σ-lattice of open maps is near-openly generated and has a σ-lattice of open homomorphisms.

Indeed, let Ψ_{op} be a σ-lattice on G of open maps. Then, by Theorem 2, G is near-openly generated and has a σ-lattice Ψ_{no} of nearly open homomorphisms. According to Proposition 11, the family $\Psi = \Psi_{\mathrm{no}} \cap \Psi_{\mathrm{op}}$ is also a σ-lattice on G and consists of open homomorphisms.

Example 4. Let a group G be homeomorphic to a product $\Pi = \Pi\{G_t : t \in T\}$ of near-openly generated groups such that every G_t has a σ-lattice of open homomorphisms. Then G is near-openly generated and has a σ-lattice of open homomorphisms.

Let Ψ_t be a σ-lattice for G_t, $t \in T$, consisting of open homomorphisms. We can assume that no Ψ_t contains a constant homomorphism. For every countable set $A \subset T$ we consider the family $\mathscr{M}_A$ of all maps on the subproduct $\Pi_A = \prod_{t\in A} G_t$ that have the form $\prod_{t\in A} f_t$ with $f_t \in \Psi_t$ for each $t \in A$. Denote by $\mathscr{L}_A$ the family $\{f \circ \pi_A : f \in \mathscr{M}_A\}$, where π_A is the projection onto Π_A, and let Ψ_G consist of all families $\mathscr{L}_A$, A being a countable subset of T. One can show that if $f \circ \pi_A \prec g \circ \pi_B$ for some $f = \prod_{t\in A} f_t \in \mathscr{L}_A$ and $g = \prod_{t\in B} g_t \in \mathscr{L}_B$, then $B \subset A$ and $f_t \prec g_t$ for all $t \in B$. This easily implies that Ψ_G satisfies condition (L1) in the definition of a σ-lattice. It is also true that for any two maps $f \circ \pi_A \in \mathscr{L}_A$ and $g \circ \pi_B \in \mathscr{L}_B$ there is $h \circ \pi_{A\cup B} \in \mathscr{L}_{A\cup B}$ such that $h \circ \pi_{A\cup B} \prec f \circ \pi_A$ and $h \circ \pi_{A\cup B} \prec g \circ \pi_B$. So, Ψ_G is directed with respect to the relation $\prec$. Moreover, Ψ_G consists of open homomorphisms and generates the topology of Π. Therefore, Ψ_G generates an almost continuous σ-spectrum S_G with $\Pi = \mathrm{a} - \varprojlim S_G$ (see Proposition 9). Since Π is I-favourable (as a product of I-favourable spaces), Π is $\mathbb{R}$-factorizable by Proposition 5. Finally, Theorem 5 yields that S_G is a factorizable spectrum. Consequently, Ψ_G is a σ-lattice of open homomorphisms, and we apply Example 3 to conclude that G is near-openly generated and has a σ-lattice of open homomorphisms.

Example 5 (almost connected pro-Lie groups). Pro-Lie groups were introduced in [1] as groups that are projective limits of Lie groups. Any pro-Lie group G such that the quotient group G/G_0 is compact, where G_0 is the connected component of G, is called an *almost connected pro-Lie group* (see [2]). By Corollary 8.9 in [2] any almost connected pro-Lie group is homeomorphic to a product of a compact topological group and a power of $\mathbb{R}$. Thus, G is near-openly generated and has a σ-lattice of open homomorphisms (see Example 4).

Example 6. Let K be a compact invariant subgroup of G. If the quotient group $H = G/K$ is an I-favourable space with a σ-lattice of open maps, then G is a near-openly generated group with a σ-lattice of open homomorphisms.

We use a construction from [6], Theorem 3.11. By Example 3, H has a σ-lattice Ψ_H of open homomorphisms. Let $\mathscr{A}_H = \{\mathrm{Ker}(\varphi) : \varphi \in \Psi_H\}$. Consider also the family $\mathscr{A}_G$ of all invariant subgroups N of G such that the quotient group G/N is a second-countable space and $\pi(N) \in \mathscr{A}_H$, where $\pi : G \to H$ is the quotient map. It follows from Lemma 3.3 in [6] and the proof of Theorem 3.11 in [6] that for G consisting of open homomorphisms the family Ψ_G of all quotient maps correspondent to $\mathscr{A}_G$ is a σ-lattice. Note that the requirement in [6], Theorem 3.11, to have a strong σ-lattice on H is not necessary in our case.

Corollary 3. *Let K be a compact invariant subgroup of G. If the quotient group $H = G/K$ is homeomorphic to an almost connected pro-Lie group, then G is a near-openly generated group with a σ-lattice of open homomorphisms.*

Example 7 (Lindelöf almost metrizable groups). A group is almost metrizable [7] if it contains a compact subset of countable character (almost metrizable groups coincide with *feathered groups* [10]). In the case that an almost metrizable group G is Lindelöf, it has a compact invariant subgroup K of countable character such that the quotient group G/K is second-countable. So, by Example 6, G is a near-openly generated group with a σ-lattice of open homomorphisms.

Example 8. Any near-openly generated group has countable cellularity. Since there are Lindelöf groups of uncountable cellularity and any Lindelöf group is $\mathbb{R}$-factorizable, there is an $\mathbb{R}$-factorizable group which is not near-openly generated.

Question 1. Let K be a compact invariant subgroup of a topological group G such that the quotient group $H = G/K$ is near-openly generated. Is it true that G is near-openly generated?

Question 2. Let G be an almost limit of an almost σ-continuous spectrum S_G (see the definition in § 4, just before Proposition 10) of near-openly generated groups and nearly open homomorphisms. Is G a near-openly generated group?

§ 4. Appendix

4.1. Some properties of spectra. In this section we provide results used in the previous sections.

By a *cofinal subspectrum* of a spectrum $S = \{X_\alpha, p_\alpha^\beta, A\}$ we mean a spectrum $S' = \{X_\alpha, p_\alpha^\beta, A'\}$ such that the index set A' is cofinal in A, that is, for every $\alpha \in A$ there is $\gamma \in A'$ with $\alpha < \gamma$. Clearly, every cofinal subspectrum of a factorizing spectrum is also factorizing. Below, if Ψ is a family of maps on a space X, we say that Ψ is *directed* if it is directed with respect to the order on Ψ defined by $f < g$ if and only if $g \prec f$. For any two maps $f, g \in \Psi$ with $f < g$ there is a unique map $p_f^g\colon g(X) \to f(X)$ such that $f = p_f^g \circ g$. The proof of the next proposition needs only a comparison of the corresponding definitions.

Proposition 9. *Let Ψ be a directed family of maps on X onto second-countable spaces such that Ψ generates the topology of X and satisfies condition (L1). Then $S_\Psi = \{f(X), p_f^g, \Psi\}$ is an almost continuous σ-spectrum with $X = \mathrm{a} - \varprojlim S_\Psi$. If, in addition, Ψ is a σ-lattice, S_Ψ is a factorizing. Moreover, every σ-sublattice of Ψ generates a cofinal almost continuous σ-subspectrum S' of S_Ψ with $X = \mathrm{a} - \varprojlim S'$.*

Besides almost continuous σ-spectra, we also consider spectra $S = \{X_\alpha, p_\alpha^\beta, A\}$ such that all bonding maps p_α^β are surjective, the directed set A is σ-complete and for any countable chain $\{\alpha_n\}_{n \geqslant 1} \subset A$ with $\beta = \sup\{\alpha_n\}_{n \geqslant 1}$ the space X_β is a (dense) subset of $\varprojlim\{X_{\alpha_n}, p_{\alpha_n}^{\alpha_{n+1}}, n \geqslant 1\}$. Any such spectrum is said to be *almost σ-continuous*. In other words, almost σ-continuous spectra satisfy the conditions in the definition of an almost continuous σ-spectrum but the spaces X_α are not required to be second-countable.

Proposition 10. *Let f be a homeomorphism of X onto Y, $S_X = \{X_\alpha, p_\beta^\alpha, A_X\}$ be a factorizing almost continuous σ-spectrum with $X = a - \varprojlim S_X$ and Ψ_Y be a σ-lattice on Y. Then f is induced by an isomorphism of cofinal subspectra S_X' of S_X and S_Y' of S_Y, where S_X' is an almost continuous σ-subspectrum of S_X and S_Y is the factorizing almost continuous σ-spectrum generated by Ψ_Y.*

Proof. Without loss of generality we can assume that $X = Y$ and f is the identity. So, $\Psi = \Psi_Y$ is a σ-lattice on X. Denote the factorizing almost continuous σ-spectrum generated by Ψ (see Proposition 9) by S_Ψ. Hence our proof is reduced to showing that there is a cofinal almost continuous σ-subspectrum of S_X which is a cofinal subspectrum of S_Ψ. To this end, let $\mathscr{P} = \{p_\alpha\colon \alpha \in A_X\}$. We define a partial order on $\mathscr{P}$ by setting $p_\alpha <_{\mathscr{P}} p_\beta$ if and only if $\alpha < \beta$. Obviously, $p_\alpha <_{\mathscr{P}} p_\beta$

implies that $p_\beta \prec p_\alpha$. Setting $\varphi_1 <_\Psi \varphi_2$ if and only if $\varphi_2 \prec \varphi_1$ also defines a partial order on the set Ψ.

Claim 2. *$\Psi \cap \mathscr{P}$ satisfies condition (L2) and it is cofinal in both Ψ and $\mathscr{P}$.*

Indeed, let h be a continuous function on X. Using that Ψ satisfies (L2) and S_X is factorizing, we can construct by induction two sequences $\{p_{\alpha_n}\} \subset \mathscr{P}$ and $\{\varphi_n\} \subset \Psi$ such that $\varphi_{n+1} \prec p_{\alpha_n} \prec \varphi_n \prec h$ and $\alpha_n < \alpha_{n+1}$ for all n. Then, because Ψ is a σ-lattice and S_X is almost σ-continuous, we have $\varphi = \triangle_{n\geqslant 1}\varphi_n \in \Psi$ and $p_\alpha \in \mathscr{P}$, where $\alpha = \sup \alpha_n \in A_X$. Moreover, our construction yields that $\varphi(X)$ is homeomorphic to X_α. So, $\varphi = p_\alpha$ belongs to $\Psi \cap \mathscr{P}$ and $\varphi \prec h$. Similarly, taking $h \in \Psi$ ($h \in \mathscr{P}$) we can find $\varphi \in \Psi \cap \mathscr{P}$ such that $h <_\Psi \varphi$ ($h <_{\mathscr{P}} \varphi$, respectively). This completes the proof of the claim.

Below we identify every element $\phi \in \Psi \cap \mathscr{P}$ with a couple (φ, p_α) such that $\varphi \in \Psi$, $\alpha \in A_X$ and the spaces $\varphi(X)$ and $X_\alpha = p_\alpha(X)$ are homeomorphic. We introduce a partial order on $\Psi \cap \mathscr{P}$: $(\varphi_1, p_{\alpha_1}) <* (\varphi_2, p_{\alpha_2})$ if and only if $\alpha_1 < \alpha_2$. Obviously, if $\phi_1 <* \phi_2$ for some $\phi_1, \phi_2 \in \Psi \cap \mathscr{P}$, then $\varphi_2 \prec \varphi_1$. Using again that Ψ is a σ-lattice and the spectrum S_X is almost σ-continuous, one can show that the set $\Psi \cap \mathscr{P}$ is σ-complete with respect to the order $<*$, that is, if $\{(\varphi_n, p_{\alpha_n})\}$ is a sequence in $\Psi \cap \mathscr{P}$ such that $(\varphi_n, p_{\alpha_n}) <* (\varphi_{n+1}, p_{\alpha_{n+1}})$ for each n, then $\{(\varphi_n, p_{\alpha_n})\}$ has a supremum in $\Psi \cap \mathscr{P}$ and $\sup\{(\varphi_n, p_{\alpha_n})\} = (\varphi, p_\alpha)$, where $\varphi = \triangle_{n\geqslant 1}\varphi_n$ and $\alpha = \sup \alpha_n \in A_X$. This implies that the subspectrum $S_{\Psi\cap\mathscr{P}}$ of S_X is almost σ-continuous. Finally, by Claim 2, $S_{\Psi\cap\mathscr{P}}$ is also factorizing and it is a cofinal subspectrum of S_X and S_Ψ.

The proposition is proved.

The next proposition is an analogue of Shchepin's theorem [17] on intersections of lattices. It follows from Proposition 10.

Proposition 11. *If Ψ_1 and Ψ_2 are two σ-lattices on X, then $\Psi_1 \cap \Psi_2$ is a σ-lattice on X.*

Proposition 12. *If $X = \mathrm{a} - \varprojlim S$, where S is a factorizing almost continuous σ-spectrum, then $\varprojlim S$ is the realcompactification of X.*

Proof. Because S is factorizing, $\varprojlim S$ is a realcompact space in which X is C-embedded. Hence, $\upsilon X = \varprojlim S$. The proposition is proved.

Proposition 13. *Let $S_X = \{X_\alpha, p^\beta_\alpha, A\}$ be a factorizing almost σ-continuous spectrum and $X = \mathrm{a} - \varprojlim S$. If $Y \subset X$ is a co-zero set, then $S_Y = \{Y_\alpha = p_\alpha(Y), p^\beta_\alpha|_{Y_\beta}, A\}$ is a factorizing almost σ-continuous spectrum.*

Proof. Obviously, S_Y is almost σ-continuous, so we need to show that it is also factorizing. Suppose $Y = g^{-1}((0,1])$ for some continuous function g on X. Since S_X is factorizing, there exists $\gamma \in A$ and a continuous function $g_\gamma\colon X_\gamma \to \mathbb{R}$ with $g = g_\gamma \circ p_\gamma$.

Let $f\colon Y \to \mathbb{R}$ be a continuous function. Put

$$f_n(x) = \begin{cases} f'_n(x)g_n(x), & x \in Y, \\ 0, & x \notin Y, \end{cases}$$

where $g_n(x) = \min\{n \cdot g(x), 1\}$, $x \in X$, and

$$f'_n(y) = \begin{cases} \min\{f(y), n\}, & f(y) \geqslant 0, \\ \max\{f(y), -n\}, & f(y) < 0. \end{cases}$$

The sequence $\{f_n\}$ consists of continuous functions on X such that $\lim_{n\to\infty} f_n(y) = f(y)$ for any $y \in Y$. Because the spectrum S_X is factorizing, we can construct by induction a sequence $\{\alpha_n\} \subset A$ and functions $h_n \colon X_{\alpha_n} \to \mathbb{R}$ such that $\alpha_{n+1} \geqslant \alpha_n > \gamma$ and $f_n = h_n \circ p_{\alpha_n}$. Let $\beta = \sup\{\alpha_n \colon n \in \mathbb{N}\}$. Then for any points $y, y' \in Y$ such that $p_\beta(y) = p_\beta(y')$ we have

$$\lim_{n\to\infty} f_n(y) = \lim_{n\to\infty} h_n \circ p_{\alpha_n}(y) = \lim_{n\to\infty} h_n \circ p_{\alpha_n}(y') = \lim_{n\to\infty} f_n(y'),$$

which implies $f(y) = f(y')$. Therefore, there is a map $h_\beta \colon Y_\beta \to \mathbb{R}$ such that $f = h_\beta \circ (p_\beta|_Y)$. It remains to show h_β is continuous. Suppose $p_\beta(y) = z$ for some $z \in Y_\beta$ and $y \in Y$. Then there is k such that $g_n(y) = 1$ and $f_n(y) = f'_n(y) = f(y) \in (-k, k)$ for all $n \geqslant k$. Fix $m > k$ and $\varepsilon > 0$, and let $y_m = p_{\alpha_m}(y)$. So, $f_m(y) = h_m(p_{\alpha_m}(y)) \in (-k, k)$ and $y \in g^{-1}((1/m, 1])$. Since $\alpha_m > \gamma$, there is a map $p^{\alpha_m}_\gamma \colon X_{\alpha_m} \to X_\gamma$ with $p_\gamma = p^{\alpha_m}_\gamma \circ p_{\alpha_m}$. Then $g_\gamma\big(p^{\alpha_m}_\gamma(y_m)\big) = g(y) \in (1/m, 1]$. Consequently, there is a neighbourhood $W \subset X_{\alpha_m}$ of y_m such that $W \subset (g_\gamma \circ p^{\alpha_m}_\gamma)^{-1}((1/m, 1]) \cap h^{-1}_m((-k, k))$ and $|h_m(t) - h_m(y_m)| < \varepsilon$ for every $t \in W$. Since $g_m(x) = 1$ and $f_m(x) \in (-k, k)$ for all $x \in p^{-1}_{\alpha_m}(W)$, it follows that $p^{-1}_{\alpha_m}(W) \subset f^{-1}((-m, m))$. Therefore, if $t = p_{\alpha_m}(x)$, where $x \in p^{-1}_{\alpha_m}(W)$, then

$$\begin{aligned} \varepsilon &> |h_m(t) - h_m \circ p_{\alpha_m}(y)| = |h_m \circ p_{\alpha_m}(x) - h_m \circ p_{\alpha_m}(y)| \\ &= |f_m(x) - f_m(y)| = |f(x) - f(y)|. \end{aligned}$$

The last inequality implies that

$$|h_\beta(z') - h_\beta(z)| = |h_\beta \circ p_\beta(x) - h_\beta \circ p_\beta(y)| = |f(x) - f(y)| < \varepsilon$$

for all $z' = p_\beta(x) \in (p^\beta_{\alpha_m})^{-1}(W)$. Thus, h_β is continuous.

The proposition is proved.

4.2. Spectral representations of $\mathbb{R}$-factorizable groups. If G is an $\mathbb{R}$-factorizable group, then the family of all continuous homomorphisms from G into second-countable groups is a σ-lattice. So, Proposition 10 yields that if G is an $\mathbb{R}$-factorizable group, then every factorizing almost continuous σ-spectrum S_G with $G = \mathrm{a} - \varprojlim S_G$ has a cofinal almost continuous σ-subspectrum consisting of groups and homomorphisms.

The next theorem implies that if G is an $\mathbb{R}$-factorizable group, then any almost σ-continuous spectrum S of groups and homomorphisms with $G = \mathrm{a} - \varprojlim S$ is factorizing.

Theorem 5. *Let G be an $\mathbb{R}$-factorizable group and let $G = \mathrm{a} - \varprojlim S_G$, where S_G is an almost σ-continuous spectrum consisting of topological groups and homomorphisms. Then S_G is factorizing.*

Proof. Since G is $\mathbb{R}$-factorizable, it has the property ω-U [24]: for any continuous function $f\colon G \to \mathbb{R}$ and any $\varepsilon > 0$ there exists a countable family $\mathscr{U}(f,\varepsilon) = \{U_n \in N_G(e)\colon n \in \mathbb{N}\}$ such that for every $x \in G$ there exists U_n such that $|f(x) - f(y)| < \varepsilon$ whenever $y \in U_n x$. Then $\mathscr{W}(f) = \bigcup_{k\geqslant 1} \mathscr{U}(f, 1/k)$ is also a countable family. Without loss of generality we may assume that each family $\mathscr{W}(f) = \{W_k\colon k \geqslant 1\}$ satisfies the additional condition $W_k = W_k^{-1}$ and $W_{k+1}^2 \subset W_k$.

Suppose $S_G = \{G_\alpha, p_\alpha^\beta, A\}$ and f is a continuous function on G. We construct an increasing sequence $\{\alpha_k\} \subset A$ and neighbourhoods V_{α_k} of the identities e_{α_k} of G_{α_k} respectively, such that $p_{\alpha_k}^{-1}(V_{\alpha_k}) \subset W_k$, $W_k \in \mathscr{W}(f)$. Let $\alpha = \sup\{\alpha_k\}$. Since $p_\alpha^{-1}(\mathrm{e}_\alpha) \subset \bigcap_{k\geqslant 1} W_k$, we have $p_\alpha(x) = p_\alpha(y)$ if and only if $y \in \bigcap\{W_k x\colon k \in \mathbb{N}\}$. Hence $p_\alpha(x) = p_\alpha(y)$ implies that $f(x) = f(y)$. Therefore, we can define a map $f_\alpha\colon G_\alpha \to \mathbb{R}$ such that $f_\alpha \circ p_\alpha = f$.

To finish the proof it remains to show that f_α is continuous. To this end, let $x \in G_\alpha$ and $x' \in p_\alpha^{-1}(x)$. For any $\varepsilon > 0$ there exists $W_k \in \mathscr{W}(f)$ such that $|f(x') - f(y)| < \varepsilon$ for every $y \in W_k x'$. Then $(p_{\alpha_k}^\alpha)^{-1}(V_{\alpha_k})x$ is a neighbourhood of x in G_α and $|f_\alpha(x) - f_\alpha(y)| < \varepsilon$ for every $y \in (p_{\alpha_k}^\alpha)^{-1}(V_{\alpha_k})x$. Hence, f_α is continuous at x.

The theorem is proved.

Corollary 4. *If an $\mathbb{R}$-factorizable group G has a system L of* (*open or nearly open*) *homomorphisms onto second-countable topological groups such that L satisfies condition* (L1) *and generates the topology of G, then L is a σ-lattice of* (*open or nearly open*, *respectively*) *homomorphisms.*

If an $\mathbb{R}$-factorizable group G is a subgroup of a product of second-countable groups, then the images of the restrictions to G of all projections onto countable subproducts form an almost continuous σ-spectrum. Therefore, we have the following corollary.

Corollary 5. *A topological group G is $\mathbb{R}$-factorizable if and only if for any topological isomorphism of G into a product $\Pi = \prod_{\alpha\in A} G_\alpha$ of second-countable groups the family $\{p_B\colon B \subset A$ is countable$\}$, where $p_B\colon G \to \prod_{\alpha\in B} G_\alpha$ denotes the projection, is a σ-lattice on G.*

It is worth noting that any ω-narrow not $\mathbb{R}$-factorizable group is an almost limit of an almost continuous σ-spectrum of second-countable groups and homomorphisms, but no such a spectrum is factorizing. On the other hand $\mathbb{R}$-factorizable groups can be characterized as those groups which are almost limits of almost continuous σ-spectra of second-countable groups and homomorphisms, and any such spectrum is factorizing.

Corollary 6. *If G is an $\mathbb{R}$-factorizable group and $G = \mathrm{a} - \varprojlim S_G$, where S_G is an almost continuous σ-spectrum consisting of groups and homomorphisms*, *then $\varprojlim S_G$ is homeomorphic to the realcompactification of G.*

Proof. By Theorem 5, S_G is a factorizing almost continuous σ-spectrum. The rest follows from Proposition 12.

Corollary 6 improves Theorem 2.22 in [5], which states that if G is an $\mathbb{R}$-factorizable group, then vG is the limit of a factorizing almost continuous σ-spectrum of second-countable groups and homomorphisms.

Acknowledgements. The authors would like to express their gratitude to M. Tkachenko and A. Kucharski for several discussions and to the referee for their attention to this work, comments and a question.

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Received 12/NOV/18 and 7/APR/19

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