Topology and its Applications 281 (2020) 107199

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On quasi κ-metrizable spaces

Vesko Valov [1]

Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, P.O. Box 5002, North Bay, ON, P1B 8L7, Canada

A R T I C L E I N F O

Article history:
Received 18 February 2019
Received in revised form 1 July 2019
Accepted 5 July 2019
Available online 8 April 2020

MSC:
primary 54C10
secondary 54F65

Keywords:
Compact spaces
I-favorable spaces
κ-metrizable spaces
Skeletal maps
Skeletally generated spaces

A B S T R A C T

The aim of this paper is to investigate the class of quasi κ-metrizable spaces. This class is invariant with respect to arbitrary products and contains Shchepin's [8] κ-metrizable spaces as a proper subclass.

1. Introduction

Recall that a *κ-metric* [8] on a space X is a non-negative function $\rho(x, C)$ of two variables, a point $x \in X$ and a regularly closed set $C \subset X$, satisfying the following conditions:

K1) $\rho(x, C) = 0$ iff $x \in C$;
K2) If $C \subset C'$, then $\rho(x, C') \le \rho(x, C)$ for every $x \in X$;
K3) $\rho(x, C)$ is continuous function of x for every C;
K4) $\rho(x, \overline{\bigcup C_\alpha}) = \inf_\alpha \rho(x, C_\alpha)$ for every increasing transfinite family $\{C_\alpha\}$ of regularly closed sets in X.

A κ-metric on X is said to be *regular* if it satisfies also next condition

---

E-mail address: veskov@nipissingu.ca.

[1] Research supported in part by NSERC Grant 261914-19.

https://doi.org/10.1016/j.topol.2020.107199

K5) $\rho(x, C) \leq \rho(x, C') + \overline{\rho}(C', C)$ for any $x \in X$ and any two regularly closed sets C, C' in X, where $\overline{\rho}(C', C) = \sup\{\rho(y, C) : y \in C'\}$.

We say that a function $\rho(x, C)$ is an *quasi κ-metric* (resp., a regular quasi κ-metric) on X if it satisfies the axioms $K2) - K4)$ (resp., $K2) - K5)$) and the following one:

K1)* For any C there is a dense open subset V of $X \setminus C$ such that $\rho(x, C) > 0$ iff $x \in V$.

Obviously, we can assume that $\rho(x, C) \leq 1$ for all x and C, in such a case we say that ρ is a normed quasi κ-metric.

Quasi κ-metrizable spaces were introduced in [9]. Our interest of this class was originated by Theorem 1.4 from [9] stating that a compact space is quasi κ-metrizable if and only if it is skeletally generated. Unfortunately, the presented there proof of the implication that any skeletally generated compactum is quasi κ-metrizable is not correct. Despite of this incorrectness, the class of quasi κ-metrizable is very interesting. It is closed with respect to arbitrary products and contains as a proper subclass the κ-metrizable spaces. The aim of this paper is to investigate the class of quasi κ-metrizable spaces, and to provide a correct characterization of skeletally generated spaces.

The class of skeletally generated spaces was introduced in [10]. According to [9, Theorem 1.1], a space is skeletally generated iff it is I-favorable in the sense of [2]. Recall that a map $f : X \to Y$ is skeletal if $\mathrm{Int}\overline{f(U)} \neq \varnothing$) for every open $U \subset X$. A space X is skeletally generated [10] if there is an inverse system $S = \{X_\alpha, p_\alpha^\beta, A\}$ of separable metric spaces X_α and skeletal surjective bounding maps p_α^β satisfying the following conditions: (1) the index set A is σ-complete (every countable chain in A has a supremum in A); (2) for every countable chain $\{\alpha_n\}_{n\geq 1} \subset A$ with $\beta = \sup\{\alpha_n\}_{n\geq 1}$ the space X_β is a (dense) subset of $\varprojlim\{X_{\alpha_n}, p_{\alpha_n}^{\alpha_{n+1}}\}$; (3) X is embedded in $\varprojlim S$ and $p_\alpha(X) = X_\alpha$ for each α, where $p_\alpha \colon \varprojlim S \to X_\alpha$ is the α-th limit projection. If in the above definition all bounding maps p_α^β are open, we say that X is openly generated.

All topological spaces are Tychonoff and the single-valued maps are continuous. The paper is organized as follows: Section 2 contains the proof that any product of quasi κ-metrizable spaces is also quasi κ-metrizable. In Section 3 we provide some additional properties of quasi κ-metrizable spaces. For example, it is shown that this property is preserved by open and perfect surjections, and that the Čech-Stone compactification of any pseudocompact quasi κ-metrizable space is quasi κ-metrizable. The results from Section 3 imply that there exist quasi κ-metrizable spaces which are not κ-metrizable. In Section 4 we introduce a similar wider class of spaces, the weakly κ-metrizable spaces, and proved that a compact space is skeletally generated if and only if it is weakly κ-metrizable. Hence, every skeletally generated space is also weakly κ-metrizable. The converse implication is interesting for spaces with a countable cellularity only, but it is still unknown, see Questions 4.3 - 4.4.

2. Products of quasi κ-metrizable spaces

Let $\mathcal{B}$ be a base for a space X consisting of regularly open sets. A real-valued function $\xi : X \times \mathcal{B} \to [0, 1]$ will be called a π-capacity if it satisfies the following conditions:

E1) $\xi(x, U) = 0$ for $x \notin U$, and $0 \leq \xi(x, U) \leq 1$ for $x \in U$.
E2) For any $U \in \mathcal{B}$ the set $\{x \in U : \xi(x, U) > 0\}$ is dense in U.
E3) For any U the function $\xi(x, U)$ is lower semi-continuous, i.e. if $\xi(x_0, U) > a$ for some $x_0 \in X$ and $a \in \mathbb{R}$, then there is a neighborhood O_{x_0} with $\xi(x, U) > a$ for all $x \in O_{x_0}$.

E4) For any two mappings $U : T \to \mathcal{B}$ and $\lambda : T \to X$, where T is a set with an ultrafilter $\mathcal{F}$, such that the limit $\widetilde{\lambda} = \lim\limits_{\mathcal{F}} \lambda(t)$ exists and $\lim\limits_{\mathcal{F}} \xi(\lambda(t), U(t)) > a > 0$, then there exists $\widetilde{U} \in \mathcal{B}$ such that $\widetilde{U} \subset \overline{\lim}_{\mathcal{F}} U(t)$ and $\xi(\widetilde{\lambda}, \widetilde{U}) > a$. Here, $\overline{\lim}_{\mathcal{F}} U(t) = \bigcap_{F\in\mathcal{F}} \overline{\bigcup_{t\in F} U(t)}$.

A capacity is called regular if it satisfies also the following condition:

E5) If $\xi(x, U) > a > 0$, there exists $U_a \in \mathcal{B}$ such that $\xi(x, U_a) \geq a$ and $\xi(y, U) \geq \xi(x, U) - a$ for all $y \in U_a$.

Our definition of a π-capacity is almost the same as the Shchepin's definition [7] of capacity, the only difference is that Shchepin requires $\xi(x, U) > 0$ for all $x \in U$.

Lemma 2.1. *Suppose $\xi : X \times \mathcal{B} \to [0, 1]$ is a (regular) π-capacity on X. Then the function $\rho_\xi(x, C)$, $\rho_\xi(x, C) = 0$ if $x \in C$ and $\rho_\xi(x, C) = \sup\{\xi(x, U) : U \cap C = \varnothing\}$ otherwise, is a (regular) quasi κ-metric on X.*

Proof. Suppose ξ is a π-capacity on X. Clearly, ρ_ξ satisfies condition $K2$). According to the proof of [8, Proposition 6, chapter 3] ρ_ξ also satisfies conditions $K3) - K4$). To check condition $K1)^*$, let C be a proper regularly closed subset of X. Then there is a subfamily $\{U_\alpha : \alpha \in A\}$ of $\mathcal{B}$ covering $X \setminus C$. For every α the set $V_\alpha = \{x \in U_\alpha : \xi(x, U_\alpha) > 0\}$ is dense in U_α. So $V = \bigcup_{\alpha\in A} V_\alpha$ is dense in $X \setminus C$ and $\rho_\xi(x, C) > 0$ for all $x \in V$.

Let show that if ξ is a regular π-capacity, then ρ_ξ satisfies condition $K5$). Suppose C, C' are two regularly closed subsets of X and $x \in X$. Obviously, $\rho_\xi(x, C) \leq \rho_\xi(x, C')$ implies $\rho_\xi(x, C) \leq \rho_\xi(x, C') + \overline{\rho_\xi}(C', C)$. So, let $\rho_\xi(x, C) > \rho_\xi(x, C')$, and choose an integer m such that $\rho_\xi(x, C) > \rho_\xi(x, C') + 1/n$ for all $n \geq m$. Hence, there is $U \in \mathcal{B}$ such that $U \cap C = \varnothing$ and $\xi(x, U) > a_n = \rho_\xi(x, C') + 1/n$. So, according to condition $E5$), there is $U_{a_n} \in \mathcal{B}$ such that $\xi(x, U_{a_n}) \geq a_n$ and $\xi(x, U) \leq \xi(y, U) + a_n$ for all $y \in U_{a_n}$. Since $\xi(x, U_{a_n}) \geq a_n$, $U_{a_n} \cap C' \neq \varnothing$ (otherwise we would have $\rho_\xi(x, C') \geq \rho_\xi(x, C') + 1/n$). Hence, $\xi(x, U) \leq \xi(z, U) + a_n$ for every $z \in U_{a_n} \cap C'$, which yields $\xi(x, U) \leq \overline{\rho_\xi}(C', C) + \rho_\xi(x, C') + 1/n$ for all $n \geq m$ and $U \in \mathcal{B}$ with $U \cap C = \varnothing$. Therefore, $\rho_\xi(x, C) \leq \rho_\xi(x, C') + \overline{\rho_\xi}(C', C)$. □

Lemma 2.2. *Let ρ be a (regular) normed quasi κ-metric on X and $\mathcal{B}$ be the family of all regularly open subsets of X. Then the formula $\xi(x, U) = \sup\{\rho(x, C) : C \cup U = X\}$ defines a (regular) π-capacity on X.*

Proof. It is easy to show that ξ satisfies conditions $E1$) and $E3$). Condition $E4$) was established in [7, Lemma 3] in the case ρ is a κ-metric, but the same proof works for quasi κ-metrics as well. Let show condition $E2$). We fix $U \in \mathcal{B}$ and consider the family $\mathcal{B}_U = \{G \in \mathcal{B} : G \subsetneq U\}$. For every $G \in \mathcal{B}_U$ the set $V_G = \{x \in G : \rho(x, C_G) > 0\}$ is open and non-empty, where $C_G = X \setminus G$. Hence, $V = \bigcup_{G\in\mathcal{B}_U} V_G$ is dense in U. Moreover, $\xi(x, U) \geq \rho(x, C_G)$ for every $G \in \mathcal{B}_U$ and $x \in V_G$ because $U \cup C_G = X$. So, $\xi(x, U) > 0$ for all $x \in V$.

It remains to show that ξ is regular, i.e. it satisfies $E5$), provided ρ is regular. Let $U \in \mathcal{B}$ and $\xi(x, U) > a > 0$ for some $x \in U$. Since $C_U = X \setminus U$ is the smallest regularly closed subset C of X with $U \cup C = X$, we have $\xi(z, U) = \rho(z, C_U)$ for all $z \in X$. The set $W = \{y \in X : \rho(y, C_U) > \rho(x, C_U) - a\}$ is open and non-empty because $\rho(\cdot, C)$ is continuous and $x \in W$. Let $U_a \in \mathcal{B}$ be the set $U_a = \mathrm{Int}\overline{W}$ and $C_{U_a} = X \setminus U_a$. Observe that $W \subset U$, which implies $U_a \subset U$ and $C_U \subset C_{U_a}$. Then, by $K5$), $\rho(x, C_U) \leq \rho(x, C_{U_a}) + \overline{\rho}(C_{U_a}, C_U)$. Consequently, $\rho(x, C_{U_a}) \geq \rho(x, C_U) - \overline{\rho}(C_{U_a}, C_U)$. On the other hand, $\overline{\rho}(C_{U_a}, C_U) = \sup_{y\in C_{U_a}} \rho(y, C_U) = \sup_{y\in C_{U_a}\setminus C_U} \rho(y, C_U)$ because $\rho(y, C_U) = 0$ for all $y \in C_U$. Observe also that $\rho(y, C_U) \leq \rho(x, C_U) - a$ for all $y \in C_{U_a} \setminus C_U$. So, $\overline{\rho}(C_{U_a}, C_U) \leq \rho(x, C_U) - a$, and thus $\xi(x, U_a) = \rho(x, C_{U_a}) \geq a$. Finally, $\xi(y, U) = \rho(y, C_U) \geq \rho(x, C_U) - a = \xi(x, U) - a$ for all $y \in U_a$ because $U_a \subset \overline{W}$. Hence, ξ satisfies $E5$). □

Let consider the following condition, where $\rho(x, C)$ is a non-negative function with C being a regularly closed subset of X:

K1)** For any regularly closed $C \subsetneqq X$ there is $y \notin C$ with $\rho(y, C) > 0$ and $\rho(x, C) = 0$ for all $x \in C$.

Remark 2.3. Observe that in the previous lemma we actually proved the following more general statement: Suppose ρ satisfies conditions $K1)^{**}$ and $K2) - K4)$, and $\rho(x, C) \leq 1$ for all $x \in X$ and all regularly closed sets $C \subset X$. Then $\xi(x, U) = \sup\{\rho(x, C) : C \cup U = X\}$ defines a π-capacity on X. Moreover, ξ is regular if ρ satisfies also $K5)$.

Corollary 2.4. *Suppose there is a function ρ on X satisfying conditions $K1)^{**}$ and $K2) - K4)$. Then there is a quasi κ-metric d on X. Moreover, d is regular if ρ satisfies also condition $K5)$.*

Proof. We can suppose that ρ is normed. Then, by Lemma 2.2, there is a π-capacity ξ on X. Finally, Lemma 2.1, implies the existence of a quasi κ-metric d on X. Moreover, if ρ satisfies condition $K5)$, then ξ is regular, so is d. □

Theorem 2.5. *Any product of (regularly) quasi κ-metrizable spaces is (regularly) quasi κ-metrizable.*

Proof. Suppose $X = \prod_{\alpha \in A} X_\alpha$ and for every α there is a normed (regular) quasi κ-metric ρ_α on X. Following the proof of [7, Theorem 2], for every α let $\mathcal{T}_\alpha$ be the family of all regularly open subsets of X_α and let $\mathcal{B}$ be the standard base for X consisting of sets of the form $U = \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_i)$ with $U_i \in \mathcal{T}_{\alpha_i}$ and $U_i \neq X_{\alpha_i}$, where $\pi_\alpha : X \to X_\alpha$ is the projection. Denote by $v(U)$ the collection $\{\alpha_1, ..., \alpha_n\}$. According to Lemma 2.2, for every α there exists a (regular) π-capacity ξ_α on X_α. Consider the function $\xi : X \times \mathcal{B} \to \mathbb{R}$ defined by $\xi(x, U) = \inf\limits_{\alpha \in v(U)} \xi_\alpha(\pi_\alpha(x), \pi_\alpha(U))/|v(U)|$. Obviously, condition $E1)$ is satisfied. Moreover, since for each α_i the set $W_i = \{z \in X_{\alpha_i} : \xi_i(z, U_i) > 0\}$ is open and dense in U_i, the set $W = \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(W_i)$ is open and dense in U. So, ξ satisfies condition $E2)$. Shchepin has shown that the function ξ is a (regular) capacity provide each ξ_α is so, see the proof of [8, Theorem 15] and [7, Theorem 2]. The same arguments show that ξ also satisfies conditions $E3) - E4)$, and condition $E5)$ in case each ξ_α is regular. Therefore, ξ is a (regular) π-capacity. Finally, by Lemma 2.1, there exists a (regular) quasi κ-metric on X. □

3. Some more properties of quasi κ-metrizable spaces

Proposition 3.1. *Let X be a quasi κ-metrizable space and $Y \subset X$. The Y is also quasi κ-metrizable in each of the following cases: (i) Y is dense in X; (ii) Y is regularly closed in X; (iii) Y is open in X.*

Proof. If ρ is a quasi κ-metric on X and $Y \subset X$ is dense, the equality $d(y, \overline{U}^Y) = \rho(y, \overline{U}^X)$, where $U \subset Y$ is open, defines a quasi κ-metric on Y. The second case follows from the observation that every regularly closed subset of Y is also regularly closed in X. The third case is a consequence of the first two because every open subset of X is dense in its closure. □

Let consider the following condition.

K4)* $\rho(x, \overline{\bigcup C_n}) = \inf_n \rho(x, C_n)$ for every increasing sequence $\{C_n\}$ of regularly closed sets in X.

Lemma 3.2. *Suppose X is a space admitting a non-negative function $\rho(x, C)$ satisfying conditions $K1)^*$, $K2)$, $K3)$ and $K4)^*$. Then X is quasi κ-metrizable provided X has countable cellularity. In particular, every compact space admitting such a function ρ is quasi κ-metrizable.*

Proof. It suffices to show that ρ satisfies condition $K4)$ in case X is of countable cellularity, and this follows from [3, Proposition 2.1]. For reader's convenience we provide a proof. Let $\{C_\alpha\}_\alpha$ be an increasing transfinite family of regularly closed sets in X. Then $\overline{\bigcup_\alpha C_\alpha} = \overline{\bigcup_\alpha U_\alpha}$ and $\{U_\alpha\}_\alpha$ is also increasing, where U_α is the interior of C_α. Since X has countable cellularity, there are countably many α_i such that $\bigcup_{i\geq 1} U_{\alpha_i}$ is dense in $\bigcup_\alpha U_\alpha$. We can assume that the sequence $\{\alpha_i\}$ is increasing, so is the sequence $\{U_{\alpha_i}\}$. Because ρ satisfies condition $K4)^*$, we have $\rho(x, \overline{\bigcup C_{\alpha_i}}) = \inf_i \rho(x, C_{\alpha_i})$. This implies that $\rho(x, \overline{\bigcup C_\alpha}) = \inf_\alpha \rho(x, C_\alpha)$. Indeed, since $\overline{\bigcup C_{\alpha_i}} = \overline{\bigcup C_\alpha}$, $\inf_\alpha \rho(x, C_\alpha) < \inf_i \rho(x, C_{\alpha_i})$ for some $x \in X$ would imply the existence of α_0 with $\rho(x, C_{\alpha_0}) < \inf_i \rho(x, C_{\alpha_i})$ for all i. Because any two elements of the family $\{C_\alpha\}_\alpha$ are comparable with respect to inclusion, the last inequality means that C_{α_0} contains all C_{α_i}. Hence, $C_{\alpha_0} = \overline{\bigcup_\alpha C_\alpha}$ and $\rho(x, C_{\alpha_0})$ would be equal to $\inf_i \rho(x, C_{\alpha_i})$, a contradiction.

It was shown in [9, Theorem 1.4] that every compact space X admitting a non-negative function $\rho(x, C)$ satisfying conditions $K1)^*$, $K2)$, $K3)$ and $K4)^*$ is skeletally generated, and hence X has countable cellularity. Therefore, any such compactum is quasi κ-metrizable. □

It was shown by Chigogidze [1] that the Čech-Stone compactification of every pseudocompact κ-metrizable space is κ-metrizable. We have a similar result for quasi κ-metrizable spaces.

Theorem 3.3. *If X is a pseudocompact (regularly) quasi κ-metrizable space, then βX is (regularly) quasi κ-metrizable.*

Proof. Suppose $\rho(x, C)$ is a quasi κ-metric on X. We can assume that $\rho(x, \overline{U}^X) \leq 1$ for all $x \in X$ and all open $U \subset X$ ($\overline{U}^X$ denotes the closure of U in X). For every open $W \subset \beta X$ consider the function f_W on X defined by $f_W(x) = \rho(x, \overline{W \cap X}^X)$. Let $\widetilde{f}_W : \beta X \to \mathbb{R}$ be the continuous extension of f_W, and define $d(y, \overline{W}) = \widetilde{f}_W(y)$, $y \in \beta X$. Obviously, $d(y, \overline{W}) = 0$ if $y \in W \cap X$. Since $W \cap X$ is dense in $\overline{W}$, $d(y, \overline{W}) = 0$ for all $y \in \overline{W}$. Moreover, if $\overline{W} \neq \beta X$, then $\overline{W \cap X}^X \neq X$. So, there is an open dense subset V of $X \setminus \overline{W}$ with $\rho(x, \overline{W \cap X}^X) > 0$ for all $x \in V$. Since f_W is continuous, the set $\widetilde{V} = \{y \in \beta X : f_W(y) > 0\}$ is open in βX and disjoint from $\overline{W}$. Finally, because $V \subset \widetilde{V}$ and V is dense in $X \setminus \overline{W}$, $\widetilde{V}$ is dense in $\beta X \setminus \overline{W}$. So, d satisfies condition $K1)^*$. Conditions $K2)$ and $K3)$ also hold. Hence, by Lemma 3.2, it suffices to show that d satisfies $K4)^*$. To this end, let $\{\overline{W}_n\}$ be an increasing sequence of regularly closed subsets of βX and $W = \bigcup_{n\geq 1} W_n$. We have $d(y, \overline{W}) \leq \inf_n d(y, \overline{W}_n)$ for all $y \in \beta X$. Moreover, since ρ satisfies $K4)$, $d(y, \overline{W}) = \inf_n d(y, \overline{W}_n)$ if $y \in X$. Suppose there is $y_0 \in \beta X \setminus X$ with $d(y_0, \overline{W}) < \inf_n d(y_0, \overline{W}_n)$. Consequently, for every n there exists a neighborhood V_n of y_0 in βX such that $\delta < d(y, \overline{W}_n)$ for all $y \in V_n$, where $d(y_0, \overline{W}) < \delta < \inf_n d(y_0, \overline{W}_n)$. We also choose a neighborhood V_0 of y_0 with $d(y, \overline{W}) < \delta$ for all $y \in V_0$. This implies that $d(y, \overline{W}) < \delta \leq \inf_n d(y, \overline{W}_n)$ provided $y \in V = \bigcap_{n\geq 1} V_0 \cap V_n$. But $V \cap X \neq \varnothing$ because X is pseudocompact. Thus, $d(y, \overline{W}) < \inf_n d(y, \overline{W}_n)$ for any $y \in V \cap X$, a contradiction.

It follows from the definition of d that it satisfies condition $K5)$ provided ρ is regular. □

Corollary 3.4. *Every pseudocompact quasi κ-metrizable space X is skeletally generated.*

Proof. We already noted that every quasi κ-metrizable compactum is skeletally generated, see [9]. So, by Theorem 3.3, βX is skeletally generated. Finally, by [2] and [9], every dense subset of a skeletally generated space is also skeletally generated. □

Proposition 3.5. *Suppose $f : X \to Y$ is a perfect open surjection and X is (regularly) quasi κ-metrizable. Then Y is also (regularly) quasi κ-metrizable.*

Proof. Let ρ be a quasi κ-metric on X. Since f is open, $f^{-1}(\overline{U}) = \overline{f^{-1}(U)}$ for any open $U \subset Y$. So, $f^{-1}(\overline{U})$ is regularly closed set in X and we define

$$d(y,\overline{U})=\sup\{\rho(x,f^{-1}(\overline{U})):x\in f^{-1}(y)\}.$$

One can check that d satisfies conditions $K2$) and $K4$), and condition $K5$) in case ρ is regular. Moreover, $\overline{U}\neq Y$ implies $f^{-1}(\overline{U})\neq X$. So, there is a dense open subset $V\subset X\setminus f^{-1}(\overline{U})$ such that $\rho(x,f^{-1}(\overline{U}))>0$ iff $x\in V$. Then $f(V)$ is a dense and open subset of $Y\setminus\overline{U}$ such that $f^{-1}(y)\cap V\neq\varnothing$ for all $y\in f(V)$. Hence, $d(y,\overline{U})>0$ if $y\in f(V)$. If $y\notin f(V)$, then $f^{-1}(y)\cap V=\varnothing$. Thus, $d(y,\overline{U})>0$ iff $y\in f(V)$. Finally, let check continuity of the functions $d(\cdot,\overline{U})$. Suppose $d(y_0,\overline{U})<\varepsilon$ for some y_0 and U. Then $\rho(x,f^{-1}(\overline{U}))<\varepsilon$ for all $x\in f^{-1}(y_0)$. Consequently, there is a neighborhood W of $f^{-1}(y_0)$ with $\rho(x,f^{-1}(\overline{U}))<\varepsilon$ for all $x\in W$. Since, f is closed, y_0 has a neighborhood G such that $f^{-1}(G)\subset W$. This implies that $d(y,\overline{U})<\varepsilon$ for all $y\in G$. Now, let $d(y_0,\overline{U})>\delta$ for some $\delta\in\mathbb{R}$. So, there exists $x_0\in f^{-1}(y_0)$ with $\rho(x_0,f^{-1}(\overline{U}))>\delta$. Choose a neighborhood O of x_0 such that $\rho(x,f^{-1}(\overline{U}))>\delta$ for all $x\in O$. Then, $f(O)$ is a neighborhood of y_0 and $d(y,\overline{U})>\delta$ for any $y\in f(O)$. Therefore, each $d(.,\overline{U})$ is continuous. □

Recall that a surjective map $f:X\to Y$ is *irreducible* provided there is no a proper closed subset F of X with $f(F)=Y$.

Proposition 3.6. *Let $f:X\to Y$ be a perfect irreducible surjection, and Y is (regularly) quasi κ-metrizable. Then X is also (regularly) quasi κ-metrizable.*

Proof. Suppose ρ is a quasi κ-metric on Y. For every regularly closed $C\subset X$ define $d(x,C)=\rho(f(x),f(C))$. This definition is correct because $f(C)$ is regularly closed in Y. Indeed, let $C=\overline{U}$ with U open in X. Since f perfect and irreducible, we have $f(C)=\overline{U_\sharp}$, where $U_\sharp=\{y\in Y:f^{-1}(y)\subset U\}=Y\setminus f(X\setminus U)$ is open in Y. It is easily seen that d satisfies conditions $K2$) and $K3$). Condition $K4$) follows from the equality $f(\overline{\bigcup_\alpha C_\alpha})=\overline{\bigcup_\alpha f(C_\alpha)}$ for any family of regularly closed sets in X. To see that d satisfies also condition $K1)^*$, we observe that for every regularly closed $C\subsetneqq X$ there is a dense open subset $V\subset Y\setminus f(C)$ such that $\rho(y,f(C))>0$ iff $y\in V$. Then $W=f^{-1}(V)$ is open in X and disjoint from C. Moreover, $d(x,C)>0$ iff $x\in W$. It remains to show that W is dense in $X\setminus C$. And that is really true because for every open $O\subset X\setminus C$ the set $O_\sharp$ is a non-empty open subset of $Y\setminus f(C)$. So, $O_\sharp\cap V\neq\varnothing$, which implies $W\cap O\neq\varnothing$.

One can also see that d is regular provided so is ρ. □

It is well known that for every space X there is a unique extremally disconnected space $\widetilde{X}$ and a perfect irreducible map $f:\widetilde{X}\to X$. The space $\widetilde{X}$ is said to be the *absolute of* X. A space Y is called *co-absolute to* X if their absolutes are homeomorphic.

Corollary 3.7. *The absolute of any (regularly) quasi κ-metrizable space is (regularly) quasi κ-metrizable.*

Remark 3.8. The last corollary shows that the class of κ-metrizable spaces is a proper subclass of the quasi κ-metrizable spaces. Indeed, let X be a κ-metrizable compact infinite space. Then its absolute aX is quasi κ-metrizable. On the other hand, aX being extremally disconnected can not be κ-metrizable (otherwise, it should be discrete by [8, Theorem 11]).

Corollary 3.9. *Every compact space co-absolute to a quasi κ-metrizable space is skeletally generated.*

Proof. Let X and Y be compact spaces having the same absolute Z. So, there are perfect irreducible surjections $g:Z\to Y$ and $f:Z\to X$. If Y is quasi κ-metrizable, then so is Z, see Proposition 3.6. Hence, Z is skeletally generated, and by [4, Lemma 1], X is also skeletally generated. □

Recall that the hyperspace expX consists of all compact non-empty subsets F of X such that the sets of the form

$$[U_1,..,U_k]=\{H\in \mathrm{expX}:\mathrm{H}\subset\bigcup_{\mathrm{i}=1}^{\mathrm{k}}\mathrm{U_i}\text{ and }\mathrm{H}\cap\mathrm{U_i}\neq\varnothing\text{ for all i}\}$$

form a base $\mathcal{B}_{\exp}$ for expX, where each U_i belongs to a base $\mathcal{B}$ for X, see [6].

Proposition 3.10. *If X is (regularly) quasi κ-metrizable, so is* expX.

Proof. Let $\mathcal{B}$ be a base for X and ρ be a (regular) quasi κ-metric on X. Then ρ generates a (regular) π-capacity $\xi_\rho : X\times\mathcal{B}\to\mathbb{R}$ on X. Following the proof of [7, Theorem 3], we define a function $\xi:\mathrm{expX}\times\mathcal{B}_{\exp}\to\mathbb{R}$ by

$$\xi(F,[U_1,..,U_k])=\frac{1}{k}\min\{\inf_{x\in F}\max_i\xi_\rho(x,U_i),\min_i\sup_{x\in F}\xi_\rho(x,U_i)\}.$$

It was shown in [7] that ξ satisfies conditions $E1$), $E3$) and $E4$), and that ξ is regular provided ξ_ρ is regular. Let show that ξ satisfies condition $E2$). Since ξ is satisfies $E3$), it suffices to prove that for every $[U_1,..,U_k]$ there is a dense subset $V_{\exp}\subset[U_1,..,U_k]$ with $\xi(F,[U_1,..,U_k])>0$ for all $F\in V_{\exp}$. To this end, for each i fix an open dense subset V_i of U_i such that $\xi_\rho(x,U_i)>0$ if $x\in V_i$. Let $V_{\exp}$ consist of all finite sets $F\subset X$ such that $F\subset\bigcup_{i=1}^n V_i$ and $F\cap V_i\neq\varnothing$ for all i. Then $V_{\exp}$ is dense in $[U_1,..,U_k]$ and $\xi(F,[U_1,..,U_k])>0$ for all $F\in V_{\exp}$. Hence, by Lemma 2.1, expX is (regularly) quasi κ-metrizable. □

Shchepin [7, Theorem 3a] has shown that if expX is κ-metrizable, then so is X. We don't know if a similar result is true for quasi κ-metrizable spaces.

4. Skeletally generated spaces

In this section we provide a characterization of skeletally generated compact spaces in terms of functions similar to quasi κ-metrics. We say that a non-negative function $d:X\times\mathcal{C}\to\mathbb{R}$ is a *weak κ-metric*, where $\mathcal{C}$ is the family of all regularly closed subsets of X, if it satisfies conditions $K1)^*$, $K2)-K3)$ and the following one:

$K4)_0$ For every increasing transfinite family $\{C_\alpha\}_\alpha\subset\mathcal{C}$ the function $f(x)=\inf_\alpha d(x,C_\alpha)$ is continuous.

Theorem 4.1. *A compact space is skeletally generated if and only if it is weakly κ-metrizable.*

Proof. First, let show that every skeletally generated compactum X is weakly κ-metrizable. We embed X as a subset of $\mathbb{R}^\tau$ for some cardinal τ. Then, according to [9, Theorem 1.1], there is a function $\mathrm{e}:\mathcal{T}_{\mathrm{X}}\to\mathcal{T}_{\mathbb{R}^\tau}$ between the topologies of X and $\mathbb{R}^\tau$ such that: (i) $\mathrm{e(U)}\cap\mathrm{e(V)}=\varnothing$ provided U and V are disjoint; (ii) $\mathrm{e(U)}\cap\mathrm{X}$ is dense in U. We define a new function $\mathrm{e}_1:\mathcal{T}_X\to\mathcal{T}_{\mathbb{R}^\tau}$,

$$\mathrm{e}_1(U)=\bigcup\{\mathrm{e}(V):V\in\mathcal{T}_X\text{ and }\overline{V}\subset U\}.$$

Obviously e_1 satisfies conditions (i) and (ii), and it is also monotone, i.e. $U\subset V$ implies $\mathrm{e}_1(U)\subset\mathrm{e}_1(V)$. Moreover, for every increasing transfinite family $\gamma=\{U_\alpha\}$ of open sets in X we have $\mathrm{e}_1(\bigcup_\alpha U_\alpha)=\bigcup_\alpha\mathrm{e}_1(U_\alpha)$. Indeed, if $z\in\mathrm{e}_1(\bigcup_\alpha U_\alpha)$, then there is an open set $V\in\mathcal{T}_X$ with $\overline{V}\subset\bigcup_\alpha U_\alpha$ and $z\in\mathrm{e}(V)$. Since $\overline{V}$ is compact and the family is increasing, $\overline{V}$ is contained in some U_{α_0}. Hence, $z\in\mathrm{e}(V)\subset\mathrm{e}_1(U_{\alpha_0})$. Consequently, $\mathrm{e}_1(\bigcup_\alpha U_\alpha)\subset\bigcup_\alpha\mathrm{e}_1(U_\alpha)$. The other inclusion follows from monotonicity of e_1.

Because $\mathbb{R}^\tau$ is κ-metrizable (see [7]), there is a κ-metric ρ on $\mathbb{R}^\tau$. For every regularly closed $C\subset X$ and $x\in X$ we can define the function $d(x,C)=\rho(x,\overline{\mathrm{e}_1(\mathrm{IntC})})$, where $\overline{\mathrm{e}_1(U)}$ is the closure of $\mathrm{e}_1(U)$ in $\mathbb{R}^\tau$. It is easily seen that d satisfies conditions $K2) - K3)$. Let show that it also satisfies $K4)_0$ and $K1^*)$. Indeed, assume $\{C_\alpha\}$ is an increasing transfinite family of regularly closed sets in X. We put $U_\alpha = \mathrm{Int}C_\alpha$ for every α and $U = \bigcup_\alpha U_\alpha$. Thus, $\mathrm{e}_1(U) = \bigcup_\alpha \mathrm{e}_1(U_\alpha)$. Since $\{\overline{\mathrm{e}_1(U_\alpha)}\}$ is an increasing transfinite family of regularly closed sets in $\mathbb{R}^\tau$, for every $x \in X$ we have

$$\rho(x, \overline{\bigcup_\alpha \mathrm{e}_1(U_\alpha)}) = \inf_\alpha \rho(x, \overline{\mathrm{e}_1(U_\alpha)}) = \inf_\alpha d(x, C_\alpha).$$

Hence, the function $f(x) = \inf_\alpha d(x, C_\alpha)$ is continuous on X because so is $\rho(., \overline{\bigcup_\alpha \mathrm{e}_1(U_\alpha)})$. To show that $K1^*)$ also holds, observe that $d(x, C) = 0$ if and only if $x \in X \cap \overline{\mathrm{e}_1(\mathrm{IntC})}$. Because $\mathrm{e}_1(\mathrm{IntC}) \cap \mathrm{X}$ is dense in C, $C \subset \overline{\mathrm{e}_1(\mathrm{IntC})}$. Hence, $V = X \setminus \overline{\mathrm{e}_1(\mathrm{IntC})}$ is contained in $X \setminus C$ and $d(x, C) > 0$ iff $x \in V$. To prove V is dense in $X \setminus C$, let $x \in X \setminus C$ and $W_x \subset X \setminus C$ be an open neighborhood of x. Then $W \cap \mathrm{IntC} = \varnothing$, so $\mathrm{e}_1(W) \cap \mathrm{e}_1(\mathrm{IntC}) = \varnothing$. This yields $\mathrm{e}_1(W) \cap X \subset V$. On the other hand, $\mathrm{e}_1(W) \cap X$ is a non-empty subset of W, hence $W \cap V \neq \varnothing$. Therefore, d is a weak κ-metric on X.

The other implication was actually established in the proof of Theorem 1.4] from [9], and we sketch the proof here. Suppose d is a weak κ-metric on X and embed X in a Tychonoff cube $\mathbb{I}^A$ with uncountable A, where $\mathbb{I} = [0, 1]$. For any countable set $B \subset A$ let $\mathcal{A}_B$ be a countable base for $X_B = \pi_B(X)$ consisting of all open sets in X_B of the form $X_B \cap \prod_{\alpha \in B} V_\alpha$, where each V_α is an open subinterval of $\mathbb{I}$ with rational end-points and $V_\alpha \neq \mathbb{I}$ for finitely many α. Here $\pi_B : \mathbb{I}^A \to \mathbb{I}^B$ denotes the projection, and let $p_B = \pi_B|X$. For any open $U \subset X$ denote by f_U the function $d(\cdot, \overline{U})$. We also write $p_B \prec g$, where g is a map defined on X, if there is a map $h : p_B(X) \to g(X)$ such that $g = h \circ p_B$. Since X is compact this is equivalent to the following: if $p_B(x_1) = p_B(x_2)$ for some $x_1, x_2 \in X$, then $g(x_1) = g(x_2)$. We say that a countable set $B \subset A$ is *d-admissible* if $p_B \prec f_{p_B^{-1}(V)}$ for every $V \in \mathcal{A}_B^{<\omega}$. Here $\mathcal{A}_B^{<\omega}$ is the family of all finite unions of elements from $\mathcal{A}_B$. Denote by $\mathcal{D}$ the family of all d-admissible subsets of A. We are going to show that all maps $p_B : X \to X_B$, $B \in \mathcal{D}$, are skeletal and the inverse system $S = \{X_B : p^B_D : D \subset B, D, B \in \mathcal{D}\}$ is σ-continuous. Since $X = \varprojlim S$, this would imply that X is skeletally generated, see [9] and [10].

Following the proof of [9, Theorem 1.4], one can show that for any countable set $B \subset A$ there is $D \in \mathcal{D}$ with $B \subset D$, and the union of any increasing sequence of d-admissible sets is also d-admissible. So, we need to show only that $p_B : X \to X_B$ is a skeletal map for every $B \in \mathcal{D}$. Suppose there is an open set $U \subset X$ such that the interior in X_B of $\overline{p_B(U)}$ is empty. Then $W = X_B \setminus \overline{p_B(U)}$ is dense in X_B. Let $\{W_m\}_{m \geq 1}$ be a countable cover of W with $W_m \in \mathcal{A}_B$ for all m. Since $\mathcal{A}_B^{<\omega}$ is finitely additive, we may assume that $W_m \subset W_{m+1}$, $m \geq 1$. Because B is d-admissible, $p_B \prec f_{p_B^{-1}(W_m)}$ for all m. Hence, there are continuous functions $h_m : X_B \to \mathbb{R}$ with $f_{p_B^{-1}(W_m)} = h_m \circ p_B$, $m \geq 1$. Recall that $f_{p_B^{-1}(W_m)}(x) = d(x, \overline{p_B^{-1}(W_m)})$ and $p_B^{-1}(W) = \bigcup_{m \geq 1} p_B^{-1}(W_m)$. Therefore, $f_{p_B^{-1}(W)}(x) = d(x, \overline{p_B^{-1}(W)}) \leq f(x) = \inf_m f_{p_B^{-1}(W_m)}(x)$ for all $x \in X$. Moreover, f is continuous and $f_{p_B^{-1}(W_{m+1})}(x) \leq f_{p_B^{-1}(W_m)}(x)$ because $W_m \subset W_{m+1}$. The last inequalities together with $p_B \prec f_{p_B^{-1}(W_m)}$ yields that $p_B \prec f$. So, there exists a continuous function h on X_B with $f(x) = h(p_B(x))$ for all $x \in X$. But $f(x) = 0$ for all $x \in p_B^{-1}(W)$, so $f(\overline{p_B^{-1}(W)}) = 0$. This implies that $h(\overline{W}) = 0$. Since $p_B(\overline{p_B^{-1}(W)}) = \overline{W} = X_B$, we have that h is the constant function zero. Consequently, $f(x) = 0$ for all $x \in X$. Finally, the inequality $d(x, \overline{p_B^{-1}(W)}) \leq f(x)$ yields that $d(x, \overline{p_B^{-1}(W)}) = 0$ for all $x \in X$. On the other hand, $\overline{p_B^{-1}(W)} \cap U = \varnothing$. So, according to $K1^*)$, there is an open subset U' of U with $d(x, \overline{p_B^{-1}(W)}) > 0$ for each $x \in U'$, a contradiction. □

Because any compactification of a skeletally generated space is skeletally generated (see [9]) and the weakly κ-metrizability is a hereditary property with respect to dense subsets, we have the following

Corollary 4.2. *Every skeletally generated space is weakly κ-metrizable.*

All results in Section 3, except Proposition 3.10, remain valid for weakly κ-metrizable spaces. Theorem 4.1 and a result of Kucharski-Plewik [5, Theorem 6] imply that Proposition 3.10 is also true for weakly κ-metrizable compacta. But the following questions are still open.

Question 4.3. Is any product of weakly κ-metrizable spaces weakly κ-metrizable?

If there exists a counter example to Question 4.3 which, in addition has a countable cellularity, then next question would have also a negative answer.

Question 4.4. Is any weakly κ-metrizable space with a countable cellularity skeletally generated?

Acknowledgements

The author would like to express his gratitude to A. Kucharski for his careful reading and suggesting some improvements of the paper.

References

[1] A. Chigogidze, On κ-metrizable spaces, Usp. Mat. Nauk 37 (2) (1982) 241–242 (in Russian).
[2] P. Daniels, K. Kunen, H. Zhou, On the open-open game, Fundam. Math. 145 (3) (1994) 205–220.
[3] A. Kucharski, S. Turek, A generalization of κ-metrizable spaces, arXiv:1612.08838.
[4] A. Kucharski, S. Plewik, Skeletal maps and I-favorable spaces, Acta Univ. Carol., Math. Phys. 51 (2010) 67–72.
[5] A. Kucharski, S. Plewik, Game approach to universally Kuratowski-Ulam spaces, Topol. Appl. 154 (2) (2007) 421–427.
[6] K. Kuratowski, Topology, vol. I, Academic Press/PWN-Polish Scientific Publishers, New York/Warsaw, 1966.
[7] E. Shchepin, On κ-metrizable spaces, Math. USSR, Izv. 14 (2) (1980) 1–34.
[8] E. Shchepin, Topology of limit spaces of uncountable inverse spectra, Russ. Math. Surv. 315 (1976) 155–191.
[9] V. Valov, I-favorable spaces: revisited, Topol. Proc. 51 (2018) 277–292.
[10] V. Valov, External characterization of I-favorable spaces, Math. Balk. 25 (1–2) (2011) 61–78.