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On Q-manifolds bundles

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We prove a homological characterization of Q-manifolds bundles over C-spaces. This provides a partial answer to Question QM22 from [25].

1. Introduction

The second author raised the question [25, QM22] whether there exists a Čech-cohomology version of the fibred general position theory of Toruńczyk-West [24] along the lines of Daverman-Walsh results [6] that detects the Q-manifold bundles among the ANR-fibrations with compact fibers. In the present paper we present a homological version of Toruńczyk-West's [24] characterization of Q-manifold bundles over C-spaces. Dranishnikov's results from [7] and [8], as well as Torunczyk-West example [24, Theorem 4.2], show that the C-space condition is essential in Theorems 3.5, 4.1 and 4.4 below.

Following [24], a *trivial ANR-fibration (resp., AR-fibration)* is a map $p : E \to B$ of paracompact Hausdorff spaces such that for some separable metric ANR-space (resp., AR-space) X there is a closed embedding $i : E \hookrightarrow X \times B$ and a retraction $r : X \times B \to i(E)$ making the diagram commute

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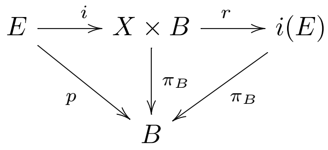


and a fiber preserving homotopy $H : X \times B \times \mathbb{I} \to X \times B$ from r to $r_b \times \mathrm{id_B}$ for some $b \in B$, where $r_b : X \to i(p^{-1}(b))$ is defined by $r_b(x) = r(x, b)$. The space X is called *an associated space* to the trivial fibration p. A map $p : E \to B$ of paracompact Hausdorff spaces is *an ANR-fibration* if there is an open cover $\mathcal{U}$ of B such that for each $U \in \mathcal{U}$ the map $p : p^{-1}(U) \to U$ is a trivial ANR-fibration with an associated space X_U. We call the cover $\mathcal{U}$ a *local trivialization of* p. Note that every $U \in \mathcal{U}$ is also paracompact. If the maps r and H may be taken proper, p is called a *proper ANR-fibration*. If each X_U is locally compact, compact, or a complete separable metric ANR-space, we say that $p : E \to B$ is a *locally compact, compact, or complete ANR-fibration*. Every complete trivial ANR-fibration $p : E \to B$ has the following property: For every paracompact space Z, a closed subset $A \subset Z$ and maps $f : A \to E$ and $g : Z \to B$ such that $p \circ f = g|A$ there is a map $\widetilde{g} : V \to B$, where V is a neighborhood of A in Z, such that $p \circ \widetilde{g} = g|V$ and $\widetilde{g}$ extends f. A map $p : E \to B$ with this property is said to be a *locally soft map*. A map p is *soft* if in the above definition V is the whole space Z [18]. Let us note that every locally soft map is open.

Let $p : E \to B$ be an ANR-fibration and $A \subset B$. We say that $p^{-1}(A)$ has the *fibred disjoint n-disks property* if each pair of fiber preserving maps $f, g : \mathbb{I}^n \times A \to p^{-1}(A)$ can be approximated arbitrary closely by fiber preserving maps $f', g' : \mathbb{I}^n \times A \to p^{-1}(A)$ with disjoint images, see [24]. Here, a map $f : \mathbb{I}^n \times A \to p^{-1}(A)$ is said to be *fiber preserving* provided $p(f(x, y)) = y$ for all $(x, y) \in \mathbb{I}^n \times A$. The fibred disjoint 2-disks property is simply called *fibred disjoint disks property*. If there is an open cover $\mathcal{U}$ of B such that $p^{-1}(U)$ has the fibred disjoint n-disks property for all n and $U \in \mathcal{U}$, then p satisfies the *fibred general position* (FGP)-property [24].

Recall that a closed set $F \subset X$ is said to be a *Z_n-set* in X if the set $C(\mathbb{I}^n, X \setminus F)$ is dense in $C(\mathbb{I}^n, X)$. Note that if X is a metric LC^{n-1}-space, then a closed set $F \subset X$ is Z_n-set iff for each at most n-dimensional metric compactum Y the set $\{f \in C(Y, X) : f(Y) \cap F = \varnothing\}$ is dense in $C(Y, X)$, see [4]. We also say that a map $f : K \to X$ is a *Z_n-map* provided $f(K)$ is a Z_n-set in X. For consistency with [22], instead of using the term Z_∞ to mean Z_n for all n as is often done, we call Z_∞-sets and Z_∞-maps, respectively, Z-sets and Z-maps.

There are homological analogues of Z_n-sets and Z_n-maps. A closed set $F \subset X$ is called a *homological Z_n-set in X* if the singular homology groups $H_k(U, U \setminus F)$ are trivial for all open sets $U \subset X$ and all $k \leq n$, see [2]. It can be shown that if X does not have isolated points then every homological Z_n-set in X is nowhere dense. The homological Z_n-property is finitely additive and hereditary with respect to closed subsets [2]. A map $f : K \to X$ is a *homological Z_n-map* provided the image $f(K)$ is a homological Z_n-set in X. Homological Z_∞-sets were considered in [6] under the name sets of infinite codimension.

Combining [23, Corollary 3.3] and [2, Theorem 2.1], we have the following:

Proposition 1.1. *Let X be an LC^n-space with $n \geq 2$. Then, a closed subset A of X is a Z_n-set in X provided A is a Z_2-set and a homological Z_n-set in X.*

All function spaces in this paper, if not explicitly stated otherwise, are equipped with the limitation topology, see [5] and [21]. Recall that a set $U \subset C(X, Y)$ is open with respect to the limitation topology if for every $f \in U$ there is $\mathcal{V} \in \mathrm{cov(Y)}$ of Y such that U contains the set $B(f, \mathcal{V}) = \{g \in C(X, Y) : g$ is $\mathcal{V}-$close to $f\}$. Here, $\mathrm{cov(Y)}$ is the family of all open covers of Y. Let $p : E \to B$ be an ANR-fibration, $K \subset B$ and Z an arbitrary space. We denote by $C_P(Z \times K, p^{-1}(K))$ the set of all continuous fiber preserving maps from $Z \times K$ into $p^{-1}(K)$. One can show that each $f \in C_P(Z \times K, p^{-1}(K))$ is a perfect map provided Z is compact.

2. Preliminary results

Lemma 2.1. *Let $p : E \to B$ be a locally compact trivial ANR-fibration and $A \subset B$ be a closed set. Then for every space K such that $K \times B$ is paracompact the restriction map $q_A : C_P(K \times B, E) \to C_P(K \times A, p^{-1}(A))$, $q_A(f) = f|(K \times A)$, is open.*

Proof. Let $W \subset C_P(K \times B, E)$ be an open set and $f \in W$. Then there exists an open cover $\mathcal{U}$ of E such that $B(f, \mathcal{U}) \subset W$. By [24, Proposition (A10)], there is $\mathcal{V} \in \mathrm{cov(E)}$ such that any map $h \in C_P(K \times A, p^{-1}(A))$, which is $\mathcal{V}$-close to $f|(K \times A)$, is extended to a map $\widetilde{h} \in B(f, \mathcal{U})$. This means that the set $B(q_A(f), \mathcal{V})$ is contained in $q_A(W)$. Since the interior of $B(q_A(f), \mathcal{V})$ in $C_P(K \times A, p^{-1}(A))$ contains $q_A(f)$ (see [5]), $q_A(W)$ is open in $C_P(K \times A, p^{-1}(A))$. □

In case $p : E \to B$ is a trivial locally compact fibration, we need another description of the limitation topology on $C_P(K \times B, E)$. If (X, d) is the associated with p separable metric ANR-space, we consider the pseudo-metric d_E on E defined by $d_E(x, y) = d(\pi_X(x), \pi_X(y))$, where $\pi_X : X \times B \to X$ is the projection.

Lemma 2.2. *Let $p : E \to B$ be a trivial ANR-fibration and (X, d) be a locally compact separable metric space associated with p. Then for any space K a set $U \subset C_P(K \times B, E)$ is open with respect to the limitation topology iff for every $f \in U$ there is a continuous function $\varepsilon : E \to (0, 1]$ such that U contains the set*

$$B_{d_E}(f, \varepsilon) = \{g \in C_P(K \times B, E) : d_E(f(y), g(y)) < \varepsilon(f(y)) \forall y \in K \times B\}.$$

Proof. Suppose $U \subset C_P(K \times B, E)$ is open with respect to the limitation topology and $f \in U$. Then there is $\mathcal{V} \in \mathrm{cov(E)}$ such that U contains the set $B(f, \mathcal{V})$. Since $X \times B$ is paracompact, so is E. Let $r : X \times B \to E$ be a fiber-preserving retraction and $\mathcal{W}$ be a locally finite open cover of $X \times B$ which is star-refinement of the cover $\widetilde{\mathcal{V}} = r^{-1}(\mathcal{V})$. We fix a countable open cover $\{X_i\}_{i \geq 1}$ of X such that each X_i has a compact closure and $\overline{X}_i \subset X_{i+1}$. Since $\overline{X}_i$ is compact, every $b \in B$ has a neighborhood O^i_b in B such that for every $x \in \overline{X}_i$ there is $W_x \in \mathcal{W}$ and a neighborhood O_x of x in X with $O_x \times O^i_b \subset W_x$. Choose a bounded continuous pseudo-metric $\sigma_i \leq 1$ on B such that the family of all open balls $\{B_{\sigma_i}(b, 1) : b \in B\}$ refines the cover $\{O^i_b : b \in B\}$. Let ρ be the pseudo-metric on $X \times B$ defined by $\rho((x, b), (x', b')) = d(x, x') + \sigma(b, b')$, where $\sigma(b, b') = \sum_{i=1}^{\infty} \sigma_i(b, b')/2^i$. Define a function ε on E by

$$\varepsilon(z) = \rho(z, E \backslash \mathrm{St(z, \mathcal{W})}).$$

We can assume that $\varepsilon \leq 1$. Clearly ε is continuous and bounded, and it is positive. Indeed, suppose $\varepsilon(z) = 0$ for some $z = (x, b) \in E$, where $x \in X_j$. Then there is a sequence $\{z_n\} \subset E \backslash \mathrm{St(z, \mathcal{W})}$, $z_n = (x_n, b_n)$, such that $\rho(z, z_n) < 1/n$ for all n. Then $d(x, x_n) < 1/n$ and, since d is a metric on X, the sequence $\{x_n\}$ converges to x. We can assume that each $x_n \in X_j$. On the other hand, $\sigma(b, b_n) < 1/n$ implies that $\sigma_j(b, b_n) < 2^j/n$ for all n. Hence, there exists n_0 with $2^j/n < 1$ for every $n \geq n_0$. Consequently, $b, b_n \in O^j_{b^*}$ for any $n \geq n_0$ and some $b^* \in B$. According to the construction of the neighborhoods O^j_b, we have $O_x \times O^j_{b^*} \subset W_x$ for some $W_x \in \mathcal{W}$ and a neighborhood O_x of x. Now, take $m > n_0$ with $x_m \in O_x$, and observe that $z, z_m \in O_x \times O^j_{b^*} \subset W_x$. Hence, $z_m \in \mathrm{St(z, \mathcal{W})}$, a contradiction. Moreover, for any $z \in E$ the inequality $\rho(z, z') < \varepsilon(z)$ implies that $z, z' \in W$ for some $W \in \mathcal{W}$. Hence, $B_\rho(f, \varepsilon) \subset B(f, \mathrm{St}\mathcal{W}) \subset \mathrm{B(f, \mathcal{V})}$. Because $B_{d_E}(f, \varepsilon) = B_\rho(f, \varepsilon)$, we finally have $B_{d_E}(f, \varepsilon) \subset U$.

To show the other implication of Lemma 2.2, let $B_{d_E}(f, \varepsilon) \subset U$ for some positive continuous function ε. For every $z \in E$ let

$$W_z = \{e \in E : d_E(e, z) < \varepsilon(z)/4 \text{ and } \varepsilon(z)/2 < \varepsilon(e) < 3\varepsilon(z)/2\}.$$

Then $\mathcal{V} = \{W_z : z \in E\} \in \mathrm{cov(E)}$ and $B(f, \mathcal{V}) \subset B_{d_E}(f, \varepsilon)$. □

According to [21, Lemma 1.1], $C(X,Y)$ with the limitation topology has the Baire property provided Y is a complete metric space (by the Baire property we mean the well known property of a given space that the intersection of countably many open dense subsets is also dense). Next proposition provides a similar result for the space of fiber preserving maps.

Proposition 2.3. *Let $p : E \to B$ be a trivial locally compact ANR-fibration and K a space. Then for any space K the space $C_P(K \times B, E)$ has the Baire property.*

Proof. We follow the arguments from the proof of [21, Lemma 1.1] given in [5]. Let X be a locally compact separable metric space associated with p and d a complete metric on X. Let also $r : X \times B \to E$ be a fiber preserving retraction. Then the map $r_* : C_P(K \times B, X \times B) \to C_P(K \times B, E)$, $r_*(f) = r \circ f$, is a well defined continuous retraction. If $\mathcal{V} \in \mathrm{cov(X \times B)}$ and $f \in C_P(K \times B, E)$ we denote by $B_*(f, \mathcal{V})$ the set of all $g \in C_P(K \times B, X \times B)$ such that $B(g, \mathcal{W}) \subset B(f, \mathcal{V})$ for some $\mathcal{W} \in \mathrm{cov(X \times B)}$. The proof of [5, Lemma 2.1] implies $B_*(f, \mathcal{U})$ is open in $C_P(K \times B, X \times B)$, and obviously contains f. We say that a cover $\mathcal{V} \in cov(X \times B)$ has a d-mesh less than ε if diameter of $\pi_X(V)$ is less than ε for every $V \in \mathcal{V}$, where $\pi_X : X \times B \to X$ is the projection.

Suppose $\{U_n\}$ is a sequence of open dense subsets of $C_P(K \times B, E)$. To prove that $\bigcap_{n=1}^{\infty} U_n$ is dense in $C_P(K \times B, E)$ it suffices to show that if $f \in C_P(K \times B, E)$ and $\mathcal{U} \in \mathrm{cov(E)}$, then there is a map $g \in \bigcap_{n=1}^{\infty} U_n$ that is $\mathcal{U}$-close to f. We define inductively a sequence of triples $\{(f_k, \mathcal{A}_k, \mathcal{B}_k)\}$ with $f_k \in C_P(K \times B, E)$ and $\mathcal{A}_k, \mathcal{B}_k \in \mathrm{cov(X \times B)}$ such that:

(i) $f_1 \in U_1 \cap B_*(f, \mathcal{B}_1)$, where $\mathcal{A}_1 = \mathcal{B}_1 = r^{-1}(\mathcal{U})$;
(ii) $f_k \in U_k \cap B_*(f_{k-1}, \mathcal{B}_k)$, where $\mathcal{B}_k$ double-star refines $\mathcal{A}_k$ and has d-mesh less than 2^{-k}, $k \geq 2$;
(iii) $\mathcal{A}_{k+1}$ refines $\mathcal{B}_k$ and $B(f_k, \mathcal{A}_{k+1}) \subset r_*^{-1}(U_k) \cap B_*(f_{k-1}, \mathcal{B}_k)$, where $f_0 = f$, $k \geq 1$.

The step (i) is possible because U_1 is dense in $C_P(K \times B, E)$ and $B_*(f, \mathcal{B}_1)$ is open in $C_P(K \times B, X \times B)$ meeting $C_P(K \times B, E)$. For the inductive step, suppose $\{(f_k, \mathcal{A}_k, \mathcal{B}_k)\}$ have been constructed for $1 \leq k \leq i$. Since $f_i \in U_i \cap B_*(f_{i-1}, \mathcal{B}_i) \subset r_*^{-1}(U_i) \cap B_*(f_{i-1}, \mathcal{B}_i)$ and $r_*^{-1}(U_i) \cap B_*(f_{i-1}, \mathcal{B}_i)$ is open in $C_P(K \times B, X \times B)$, there exists a cover $\mathcal{A}_{i+1} \in \mathrm{cov(X \times B)}$ refining $\mathcal{B}_i$ such that $B(f_i, \mathcal{A}_{i+1}) \subset r_*^{-1}(U_i) \cap B_*(f_{i-1}, \mathcal{B}_i)$. Next, let $\mathcal{B}_{i+1} \in \mathrm{cov(X \times B)}$ be a double-star refinement of $\mathcal{A}_{i+1}$ having d-mesh less than 2^{-i-1}. Since U_{i+1} is dense in $C_P(K \times B, E)$ and $B_*(f_i, \mathcal{B}_{i+1})$ is an open set in $C_P(K \times B, X \times B)$ meeting $C_P(K \times B, E)$, we may choose $f_{i+1} \in U_{i+1} \cap B_*(f_i, \mathcal{B}_{i+1})$. This completes the induction.

Observe that each map f_k is of the form $f_k(y, b) = (h_k(y, b), b)$ for each $(y, b) \in K \times B$, where $h_k \in C(K \times B, X)$. Since $f_k \in B_*(f_{k-1}, \mathcal{B}_k) \subset B(f_{k-1}, \mathcal{B}_k)$ and the d-mesh of $\mathcal{B}_k$ is less than 2^{-k}, we have $d(h_k(y, b), h_{k-1}(y, b)) < 2^{-k}$ for all $(y, b) \in K \times B$. Because d is a complete metric on X, the sequence $\{h_k\}$ converges to a map $h_0 \in C(K \times B, X)$. Then $h(y, b) = (h_0(y, b), b)$ defines a map $h \in C_P(K \times B, X \times B)$, so $g = r \circ h \in C_P(K \times B, E)$. It remains to show that g is $\mathcal{U}$-close to f and $g \in U_k$ for all k. Condition (ii) implies that f_i is $\mathrm{st}\mathcal{B}_{k+1}$-close to f_k for all $i \geq k$. Consequently, h is $\mathrm{st}^2\mathcal{B}_{k+1}$-close to f_k and hence $\mathcal{A}_{k+1}$-close to f_k. So, by (iii), $h \in r_*^{-1}(U_k)$ which yields $g \in U_k$. Finally, since h is $\mathcal{A}_2$-close to f_1, (iii) implies $h \in B_*(f_0, \mathcal{B}_1) \subset B(f, \mathcal{B}_1) = B(f, r^{-1}(\mathcal{U}))$. Hence, g is $\mathcal{U}$-close to f. □

For a space Y let $\mathcal{F}(Y)$ denote the family of all non-empty closed subsets of Y. If Y is a convex subspace of a linear space, then $\mathcal{F}_c(Y)$ stands for the closed convex subsets of Y. A set-valued map $\varphi : X \to \mathcal{F}(Y)$ is said to be lower semi-continuous, or l.s.c., if $\varphi^{-1}(U) = \{x \in X : \varphi(x) \cap U \neq \varnothing\}$ is open in X for every open $U \subset Y$.

We need to consider another topology on function spaces $C(X, Y)$, the so called *source limitation topology* [3]. The local base in that topology at given $h : X \to Y$ consists of all sets

$$\Lambda(h, \eta) = \{h' \in C(X, Y) : \rho(h(x), h'(x)) < \eta(x) \ \forall x \in X\},$$

where ρ is a continuous pseudo-metric on Y and η is a continuous and positive function on X. This topology, called sometimes the fine topology, was initially introduced for metrizable spaces Y (in this case ρ is a fixed compatible metric on Y), see [14], [17]. Clearly, the source limitation topology is stronger than the limitation one. This topology has the Baire property provided X is paracompact and Y is completely metrizable, see [12].

Proposition 2.4. *Let $p : E \to B$ be a locally compact ANR-fibration. Then the fibration $p' : E \times Q \to B$ has the* FGP-property*.*

Proof. It suffices to show that there exists a local trivialization $\mathcal{U} \in \mathrm{cov}(\mathrm{B})$ of p such that $p^{-1}(U) \times Q$ has the fibred disjoint n-disks property for all n and $U \in \mathcal{U}$. Therefore, without loss of generality, we may assume that p is a trivial fibration. We need to show that every fiber preserving map $f : Q \times \{1, 2\} \times B \to E \times Q$ can be approximated in the limitation topology by maps $g \in C_P(Q \times \{1, 2\} \times B, E \times Q)$ such that $g(Q \times \{1\} \times B) \cap g(Q \times \{2\} \times B) = \varnothing$. Here $C_P(Q \times \{1, 2\} \times B, E \times Q)$ is the set of all p'-preserving maps from $C(Q \times \{1, 2\} \times B, E \times Q)$. To this end, we fix $f \in C_P(Q \times \{1, 2\} \times B, E \times Q)$ and an open cover $\mathcal{U}$ of $E \times Q$. Let f_E, f_Q be the maps $\pi_E \circ f$ and $\pi_Q \circ f$, where π_E and π_Q are the projections of $E \times Q$ onto E and Q, respectively. Since E is paracompact, so is $E \times Q$. Using that Q is compact, every $e \in E$ has a neighborhood O_e such that for every $x \in Q$ there is $V \in \mathcal{U}$ with $O_e \times \{x\} \subset V$. As in the proof of Lemma 2.2, we can find a bounded continuous pseudo-metric σ on E and a continuous bounded positive function ε on $E \times Q$ such that any $g \in C_P(Q \times \{1, 2\} \times B, E \times Q)$ is $\mathcal{U}$-close to f provided $\rho(f(y), g(y) < \varepsilon(f(y))$ for all $y \in Q \times \{1, 2\} \times B$, where ρ be the pseudo-metric on $E \times Q$ defined by $\rho((e, x), (e', x')) = d(x, x') + \sigma(e, e')$ (here d is the ordinary convex metric on Q).

We are going to show there is a function $g_Q : Q \times \{1, 2\} \times B \to Q$ such that $g_Q(Q \times \{1\} \times B) \cap g_Q(Q \times \{2\} \times B) = \varnothing$ and $d(f_Q(y), g_Q(y)) < \varepsilon(f(y))$ for all $y \in Q \times \{1, 2\} \times B$. Since ε is bounded, there is a continuous extension $\widetilde{\varepsilon} : \beta(E \times Q) \to [0, \infty)$, where $\beta(E \times Q)$ is the Čech-Stone extension of $E \times Q$. Let also $\widetilde{f} : \beta(Q \times B) \times \{1\} \oplus \beta(Q \times B) \times \{2\} \to \beta(E \times Q)$ and $\widetilde{\pi}_Q : \beta(E \times Q) \to Q$ be the extensions of f and π_Q. (We use $\oplus$ to denote the disjoint union.) Then $\widetilde{E} = \widetilde{\varepsilon}^{-1}(0, \infty)$ and $Y = \widetilde{f}^{-1}(\widetilde{E})$ are locally compact and σ-compact spaces. We represent Y as the union of the sets $K_i = Y_i \times \{1\} \oplus Y_i \times \{2\}$, where Y_i is an increasing sequence of compact subspaces of $\beta(Q \times B)$. Let $\delta = \widetilde{\varepsilon} \circ \widetilde{f} : Y \to (0, \infty)$ and $\widetilde{f}_Q = \widetilde{\pi}_Q \circ \widetilde{f} : Y \to Q$. We consider the space $C(Y, Q)$ with the source limitation topology. Recall that the local base in that topology at given $h : Y \to Q$ consists of all sets

$$\Lambda(h, \eta) = \{h' \in C(Y, Q) : d(h(y), h'(y)) < \eta(y) \ \forall y \in Y\},$$

where η is a continuous and positive function on Y. As we already noted, this topology has the Baire property.

Claim 2.5. All restriction maps $\theta_i : C(Y, Q) \to C(K_i, Q)$, $\theta_i(h) = h|K_i$, are surjective and open in the source limitation topology.

Indeed, if $W \subset C(Y, Q)$ is open let $h' \in \theta_i(W)$. Then $h' = \theta_i(h)$ for some $h \in W$ and there is $\eta \in C(Y, (0, \infty))$ with $\Lambda(h, \eta) \subset W$. Consider the open in $C(K_i, Q)$ set

$$\Lambda_i(h', \eta/2) = \{g \in C(K_i, Q) : d(h'y), g(y)) < \eta(y)/2 \ \forall y \in K_i\}.$$

If $g \in \Lambda_i(h', \eta/2)$, we define the lower semi-continuous set valued map $\Phi : Y \to \mathcal{F}_c(Q)$, $\Phi(y) = \{z \in Q : d(z, h(y) \leq \eta(y)/2\}$ if $y \notin K_i$ and $\Phi(y) = g(y)$ if $y \in K_i$. Then by Michael's selection theorem [15], there is a continuous function $\widetilde{g} \in C(Y, Q)$ with $\widetilde{g}(y) \in \Phi(y)$ for all $y \in Y$. Obviously, $\widetilde{g} \in \Lambda(h, \eta) \subset W$ and $\theta_i(\widetilde{g}) = g$.

This means that $\Lambda_i(h', \eta/2) \subset \theta_i(W)$. Hence, each θ_i is open. Surjectivity of θ_i is obvious because Q is an absolute retract for the normal spaces.

Now, for every i let G_i be the set of all maps $h \in C(K_i, Q)$ such that $h(Y_i \times \{1\}) \cap h(Y_i \times \{2\}) = \varnothing$. Using that K_i are compact, one can show that each G_i is open and dense in $C(K_i, Q)$ (note that the source limitation topology on $C(K_i, Q)$ coincides with the compact open topology). Hence, $G = \bigcap_{i=1}^{\infty} \theta_i^{-1}(G_i)$ is dense in $C(Y, Q)$ with respect to the source limitation topology. Since $\Lambda(\widetilde{f}_Q, \delta)$ is open in that topology, there is $\widetilde{g}_Q \in G \cap \Lambda(\widetilde{f}_Q, \delta)$. Let $g_Q = \widetilde{g}_Q|(Q \times \{1,2\} \times B)$ and define $g : Q \times \{1,2\} \times B \to E \times Q$ by $g(y) = (f_E(y), g_Q(y))$. Clearly, g is p'-preserving and $\rho(f(y), g(y)) = d(f_Q(y), g_Q(y)) < \varepsilon(f(y))$ for all $y \in Q \times \{1,2\} \times B$. This means that g is $\mathcal{U}$-close to f. Moreover, $\widetilde{g}_Q \in G$ implies $g(Q \times \{1\} \times B) \cap g(Q \times \{2\} \times B) = \varnothing$. □

3. Homological characterization of Q-manifold bundles over C-spaces

In this section we prove a homological characterization of Q-manifold bundles over C-spaces. This provides a partial answer to Question QM22 from [25].

The C-space property was originally defined by Haver [11] for compact metric spaces. Addis and Gresham [1] reformulated Haver's definition for arbitrary spaces: A space X has property C if for every sequence $\{\mathcal{U}_n\}$ of open covers of X there exists a sequence $\{\mathcal{V}_n\}$ of open disjoint families in X such that each $\mathcal{V}_n$ refines $\mathcal{U}_n$ and $\bigcup_{n\geq 1} \mathcal{V}_n$ is a cover of X. Every finite-dimensional paracompact space, as well as every countable-dimensional (a countable union of finite-dimensional sets) metric space, is a C-space [1], but there is a compact metric C-space which is not countable-dimensional [19]. On the other hand, normal C-spaces are weakly infinite-dimensional in the sense of Alexandroff [1]. We say that X is an *hereditary C-space* if every open subset of X is a C-space. For example, every paracompact C-space whose open sets are F_σ is an hereditary C-space.

Although not formulated in this form, Lemma 3.1 below was actually established in [10, Proposition 3.1].

Lemma 3.1. *[10] Let Y be a closed convex subset of a Banach space and $V \subset Y$ be open. Suppose X is a paracompact C-space and $\varphi : X \rightsquigarrow \mathcal{F}_c(Y)$ is l.s.c. such that $\varphi(x) \subset V$ for all $x \in X$. Then for every set-valued map $\psi : X \to \mathcal{F}(V)$ with a closed graph (in $X \times V$) there is a map $f : X \to V$ with $f(x) \in \varphi(x) \backslash \psi(x)$ for all $x \in X$ provided each $\varphi(x) \cap \psi(x)$ is a Z-set in $\varphi(x)$.*

Proposition 3.2. *Let $p : E \to B$ be a locally compact ANR-fibration with compact Q-manifold fibers such that B is a locally compact C-space. Then every $b \in B$ has a basis of neighborhoods U with compact closures such that the fibrations $p|p^{-1}(\overline{U}) : p^{-1}(\overline{U}) \to \overline{U}$ have the* FGP-property.

Proof. By Proposition 2.4, the fibration $E \times Q \to B$ has the FGP-property. Hence, according to [24, Theorem 2.3], $p' : E \times Q \to E \to B$ is a Q-manifold bundle. Consequently, $p'' : E \times Q \times [0,1) \to E \to B$ is also a Q-manifold bundle. Let $\mathcal{U}$ be a simultaneous local trivialization of p and p', and hence of p''. It is clear that the proof is reduced to show that for every compact set $K \subset B$ which is contained in some $U \in \mathcal{U}$ the fibration $p|p^{-1}(K) : p^{-1}(K) \to K$ has the FGP-property. To this end, we can assume that $p : E \to B$ is a trivial ANR-fibration over a compact space B such that both $p' : E \times Q \to E \to B$ and $p'' : E \times Q \times [0,1) \to E \to B$ are trivial Q-manifold bundles. Therefore, we need to show that for every n every fiber-preserving map $f : \mathbb{I}^n \times \{1,2\} \times B \to E$ can be approximated by fiber-preserving maps $f' : \mathbb{I}^n \times \{1,2\} \times B \to E$ such that $f'(\mathbb{I}^n \times \{1\}) \cap f'(\mathbb{I}^n \times \{2\}) = \varnothing$. To do this, take a fiber-preserving homeomorphism

$$h : E \times Q \times [0,1) \to B \times M,$$

where M is a Q-manifold homeomorphic to $p^{-1}(b) \times Q \times [0,1)$ for all $b \in B$. Since M is a product of the Q-manifolds $p^{-1}(b) \times Q$ and $[0,1)$, it can be embedded as an open subset of a copy Q_M of the Hilbert cube [16, Corollary 7.4.4(2)]. Since the domains of all function spaces in the present proof are compact spaces, the limitation topology coincides with the compact open topology. Therefore, all function spaces will be considered with the compact open topology. We also consider the projection $\pi_E : E \times Q \times [0,1) \to E$. Then π_E generates the continuous map

$$\pi_E^* : C(\mathbb{I}^n \times \{1,2\}, E \times Q \times [0,1)) \to C(\mathbb{I}^n \times \{1,2\}, E),$$

$\pi_E^*(w) = \pi_E \circ w$. Since, $C(\mathbb{I}^n \times \{1,2\}, E \times Q \times [0,1))$ is homeomorphic to the product $C(\mathbb{I}^n \times \{1,2\}, E) \times C(\mathbb{I}^n \times \{1,2\}, Q \times [0,1))$, π_E^* is the projection onto $C(\mathbb{I}^n \times \{1,2\}, E)$. For every $b \in B$ let $C(b) = C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b) \times Q \times [0,1))$ and $C = \bigcup\{C(b) : b \in B\}$ considered as a subspace of $C(\mathbb{I}^n \times \{1,2\}, E \times Q \times [0,1))$. Let also $L(b)$ be the subspace of $C(b)$ consisting of all maps g such that $\pi_E^*(g)(\mathbb{I}^n \times \{1\}) \cap \pi_E^*(g)(\mathbb{I}^n \times \{2\}) \neq \varnothing$. Actually, $L(b)$ is the product $L_1(b) \times C(\mathbb{I}^n \times \{1,2\}, Q \times [0,1))$, where $L_1(b)$ is the set of all $g' \in C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b))$ with $g'(\mathbb{I}^n \times \{1\}) \cap g'(\mathbb{I}^n \times \{2\}) \neq \varnothing$.

Claim 3.3. $L(b)$ is a Z-set in $C(b)$ for each $b \in B$.

Since $L(b)$ is homeomorphic to the product $L_1(b) \times C(\mathbb{I}^n \times \{1,2\}, Q \times [0,1))$ and $C(b)$ is homeomorphic to $C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b)) \times C(\mathbb{I}^n \times \{1,2\}, Q \times [0,1))$, it suffices to show that $L_1(b)$ is a Z-set in $C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b))$. Obviously, $L_1(b)$ is closed in $C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b))$. So, we need to show that every map $u : Q \to C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b))$ is approximated by maps $u' : Q \to C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b))$ with $u'(Q) \cap L_1(b) = \varnothing$. Since Q is compact, the exponential map

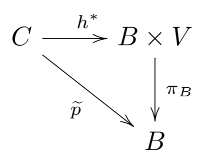
$$\Lambda_b : C(Q \times \mathbb{I}^n \times \{1,2\}, p^{-1}(b)) \to C(Q, C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b))),$$

defined by $\{[\Lambda_b(v)](q)\}(x) = v(q,x)$, $q \in Q$ and $x \in \mathbb{I}^n \times \{1,2\}$, is a homeomorphism, see [9, Theorem 3.4.3]. So, $v = \Lambda_b^{-1}(u) \in C(Q \times \mathbb{I}^n \times \{1,2\}, p^{-1}(b))$ for every $u \in C(Q, C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b)))$. Because $p^{-1}(b)$ is a Q-manifold, v can be approximated by maps $v' \in C(Q \times \mathbb{I}^n \times \{1,2\}, p^{-1}(b))$ such that $v'(Q \times \mathbb{I}^n \times \{1\}) \cap v'(Q \times \mathbb{I}^n \times \{2\}) = \varnothing$. Therefore, $u'(Q) \cap L_1(b) = \varnothing$, where $u' = \Lambda_b(v')$. This completes the proof of the claim.

Next, we consider the set $L = \bigcup\{L(b) : b \in B\} \subset C$. It is easily seen that L is closed in C and that $\widetilde{p} : C \to B$, $\widetilde{p}(g) = b$ for all $g \in C(b)$, defines a continuous map. The fiber-preserving homeomorphism h provides a homeomorphism between the sets $p^{-1}(b) \times Q \times [0,1)$ and $\{b\} \times M$ for every $b \in B$. Consequently, h generates a homeomorphism

$$h^* : C(\mathbb{I}^n \times \{1,2\}, E \times Q \times [0,1)) \to C(\mathbb{I}^n \times \{1,2\}, B \times M)$$

such that $h^*(g) \in C(\mathbb{I}^n \times \{1,2\}, \{b\} \times M)$ for all $g \in C(b)$. On the other hand, there is a natural homeomorphism between $C(\mathbb{I}^n \times \{1,2\}, B \times M)$ and the product $C(\mathbb{I}^n \times \{1,2\}, B) \times C(\mathbb{I}^n \times \{1,2\}, M)$. Therefore, we have the commutative diagram



with $h^*(C(b)) = \{b\} \times V$, where $V = C(\mathbb{I}^n \times \{1,2\}, M)$. Let us show that any fiber-preserving map $f \in C_P(\mathbb{I}^n \times \{1,2\} \times B, E)$ can be approximated by fiber-preserving maps $g : \mathbb{I}^n \times \{1,2\} \times B \to E$ such that $g(\mathbb{I}^n \times \{1\} \times B) \cap g(\mathbb{I}^n \times \{2\} \times B) = \varnothing$. To this end, fix a map $f \in C_P(\mathbb{I}^n \times \{1,2\} \times B, E)$ and consider the exponential homeomorphism

$$\Lambda : C(\mathbb{I}^n \times \{1,2\} \times B, E) \to C(B, C(\mathbb{I}^n \times \{1,2\}, E)).$$

Then $\Lambda(f)(b) \in C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b))$ for all $b \in B$. Therefore, Λ is a homeomorphism between $C_P(\mathbb{I}^n \times \{1,2\} \times B, E)$ and the set Ω consisting of all $\{l \in C(B, C(\mathbb{I}^n \times \{1,2\}, E))$ with $l(b) \in C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b))$ for every $b \in B$. We identify E with $E \times \{\overline{0}\} \times 0$ and each $p^{-1}(b)$ with $p^{-1}(b) \times \{\overline{0}\} \times 0$, where $\overline{0} = (0,0,...) \in Q$. We also consider the retraction $r_E : E \times Q \times [0,1) \to E \times \{\overline{0}\} \times \{0\}$ defined by $r_E(z_1, z_2, z_3) = (z_1, \overline{0}, 0)$. This retraction generates a continuous map

$$r_E^* : C(\mathbb{I}^n \times \{1,2\}, E \times Q \times [0,1)) \to C(\mathbb{I}^n \times \{1,2\}, E \times \{\overline{0}\} \times \{0\}).$$

Therefore, we identify $C(\mathbb{I}^n \times \{1,2\}, E)$ with $C(\mathbb{I}^n \times \{1,2\}, E \times \{\overline{0}\} \times \{0\})$ and every $C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b))$ with $C(\mathbb{I}^n \times \{1,2\}, p^{-1}(b) \times \{\overline{0}\} \times \{0\})$. So, every $\Lambda(f)(b)$ can be considered as an element of $C(b)$. This means that $\Lambda(f)$ is a map from B to C. Moreover, one can show that $\Lambda(f) : B \to C$ is an embedding with $\Lambda(f)(b) \in \widetilde{p}^{-1}(b) = C(b)$ for all $b \in B$. So is the composition $u = h^* \circ \Lambda(f) : B \to B \times V$ such that $u(b) \in \pi_B^{-1}(b)$, $b \in B$.

Claim 3.4. f can be approximated by maps $g \in C_P(\mathbb{I}^n \times \{1,2\} \times B, E)$ such that $g(\mathbb{I}^n \times \{1\} \times B) \cap g(\mathbb{I}^n \times \{2\} \times B) = \varnothing$.

Let $O_{\Lambda(f)} = \bigcap_{i=1}^k < P_i, W_i >$ be a neighborhood of $\Lambda(f)$ in the space $C(B, C(\mathbb{I}^n \times \{1,2\}, E \times \{\overline{0}\} \times \{0\}))$ with respect to the compact-open topology. Here, $P_i \subset B$ are compact sets, W_i are open in $C(\mathbb{I}^n \times \{1,2\}, E \times \{\overline{0}\} \times \{0\})$ and $< P_i, W_i >$ consists of all maps $l \in C(B, C(\mathbb{I}^n \times \{1,2\}, E \times \{\overline{0}\} \times \{0\}))$ such that $l(P_i) \subset W_i$. The sets $\widetilde{W}_i = (r_E^*)^{-1}(W_i)$ are open in $C(\mathbb{I}^n \times \{1,2\}, E \times Q \times [0,1))$. So are the sets $h^*(\widetilde{W}_i)$ in $C(\mathbb{I}^n \times \{1,2\}, B \times M)$. Hence, each $G_i = h^*(\widetilde{W}_i) \cap (B \times V)$ is open in $B \times V$. Since $\Lambda(f)(P_i) \subset W_i$, $u(P_i) \subset G_i$ for all i. Because M is open in Q_M and $Y = C(\mathbb{I}^n \times \{1,2\}, Q_M)$ is homeomorphic to a closed convex subset of the Banach space $\big(C(\mathbb{I}^n \times \{1,2\}, l_2), ||.||\big)$, $V = C(\mathbb{I}^n \times \{1,2\}, M)$ is homeomorphic to an open sunset of Y. Therefore, the continuous function $\alpha : B \to \mathbb{R}$, $\alpha(b) = \mathrm{dist}(\pi_V(u(b)), Y\backslash V)$ is positive, where $\pi_V : B \times V \to V$ is the projection. Using that $\{P_i : i \leq k\}$ is a finite family of compact sets in B with $u(P_i) \subset G_i$ and each $\pi_B|u(P_i) : u(P_i) \to P_i$ being a homeomorphism, one can show that there exists m such that the sets $T_b = \{(b,v) \in B \times V : ||v - \pi_V(u(b))|| < \dfrac{\alpha(b)}{m}\}$ are contained in G_i for every $b \in P_i$ and $i \leq k$. Now, we consider the set-valued map

$$\varphi : B \rightsquigarrow \mathcal{F}_c(Y),\ \varphi(b) = \{v \in V : ||v - \pi_V(u(b))|| \leq \frac{\alpha(b)}{N}\},$$

where $N > m$. Obviously, $\{b\} \times \varphi(b) \subset G_i$ for all $b \in P_i$, $i \leq k$. Moreover, φ is lower semi-continuous. Observe also that the set $h^*(L)$ is closed in $B \times V$ and $h^*(L(b))$ is closed in $h^*(C(b)) = \{b\} \times V$ for all $b \in B$. So, $\pi_V(h^*(L(b)))$ is a closed subset of V homeomorphic to $h^*(L(b))$, and by Claim 3.3, it is a Z-set in V. Hence, we have the set-valued map $\psi : B \rightsquigarrow \mathcal{Z}(V)$, $\psi(b) = \pi_V(h^*(L(b)))$, where $\mathcal{Z}(V)$ is the family of all Z-sets in V.

We claim that the graph $Gr(\psi)$ of ψ is closed in $B \times V$. Indeed, suppose $\{(b_\gamma, v_\gamma)\} \in Gr(\psi)$ is a net converging to (b_0, v_0) in $B \times V$. Since h^* is a homeomorphism between C and $B \times V$, there is a net $\{g_\gamma\} \subset C$ converging to some $g_0 \in C$ such that $g_\gamma \in L(b_\gamma)$ and $h^*(g_\gamma) = (b_\gamma, v_\gamma)$. Since L is closed in C, $g_0 \in L$. On the other hand, $\{b_\gamma\}$ converges to b_0 implies that $g_0 \in C(b_0)$. Hence $g_0 \in L \cap C(b_0) = L(b_0)$. Finally, $h^*(g_0) = (b_0, v_0) \in Gr(\psi)$.

Since each set $O_b = \{v \in V : ||v - \pi_V(u(b))|| < \dfrac{\alpha(b)}{N}\}$, $b \in B$, is open in V, $\psi(b) \cap O_b$ is a Z-set in O_b. This implies that $\psi(b) \cap \varphi(b)$ is a Z-set in $\varphi(b)$ for each b. (This follows because radial contraction toward $\pi_V(u(b)$ shows that the boundary of $\phi(b)$ is a Z-set in $\phi(b)$, and as $\phi(b) \cap O_b$ is a countable union of Z-sets of $C(b)$. Thus $\psi(b)$ is a countable union of Z-sets of $\phi(b)$.) Therefore, we may apply Lemma 3.1 to find a map $\phi : B \to V$ with $\phi(b) \in \varphi(b)\backslash\psi(b)$ for all $b \in B$. Then the equality $u'(b) = (b, \phi(b))$ provides a map $u' : B \to B \times V$ such that $u'(P_i) \subset G_i$ for all $i \leq k$ and $u'(B) \cap h^*(L) = \varnothing$. Hence, each $(h^*)^{-1}(u'(b))$ belongs to $C(b)\backslash L$. Consequently, we have a map $g' : B \to C\backslash L$, $g'(b) = (h^*)^{-1}(u'(b))$, with $g'(P_i) \subset \widetilde{W}_i$, $i \leq k$. The composition $g'' = r_E^* \circ g'$ provides a map from B into $C_P(\mathbb{I}^n \times \{1, 2\}, E \times \{\overline{0}\} \times 0))$ such that for every $b \in B$ we have:

(3) $g''(b) \in C(\mathbb{I}^n \times \{1, 2\}, p^{-1}(b) \times \{\overline{0}\} \times 0))$;
(4) $g''(P_i) \subset W_i$, $i \leq k$;
(5) $g''(b)(\mathbb{I}^n \times \{1\}) \cap g''(b)(\mathbb{I}^n \times \{2\}) = \varnothing$.

Condition (4) means that $g'' \in O_{\Lambda(f)}$. Therefore, $g = \Lambda^{-1}(g'') \in C(\mathbb{I}^n \times \{1, 2\} \times B, E)$ is a fiber-preserving approximation of f such that $g(\mathbb{I}^n \times \{1\} \times B) \cap g(\mathbb{I}^n \times \{2\} \times B) = \varnothing$. This provides the proof of Claim 3.4.

Therefore, p has the FGP-property. □

Theorem 3.5. *Let $p : E \to B$ be a locally compact ANR-fibration with compact fibers such that B is a paracompact locally compact C-space. Then p is a Q-manifold bundle if and only if for every $b \in B$ we have:*

(1) *$p^{-1}(b)$ has the disjoint disks property;*
(2) *$C(\mathbb{I}^n, p^{-1}(b))$ contains a dense set of homological Z_n-maps for each $n \geq 2$.*

Proof. Suppose p is a Q-manifold bundle. Then there is a Q-manifold M and an open cover $\mathcal{U}$ of B such that $p^{-1}(U)$ is homeomorphic to $U \times M$ for all $U \in \mathcal{U}$. Because for every $U \in \mathcal{U}$ and every $b \in U$ the fiber $p^{-1}(b)$ is a Q-manifold, conditions (1) and (2) are satisfied.

Suppose now that p satisfies conditions (1) and (2). Then each $p^{-1}(b)$, $b \in B$, is a compact ANR satisfying the disjoint disks property and for every $n \geq 3$ the space $C(\mathbb{I}^n, p^{-1}(b))$ contains a dense set of homological Z_n-maps. Therefore, by [13, Theorem 2.9], each fiber of p is a Q-manifold. We choose a local trivialization $\{U_\alpha : \alpha \in A\}$ of p such that if $K \subset B$ is a compact set contained in some U_α, then the fibration $p|p^{-1}(K) : p^{-1}(K) \to K$ has the FGP-property, see Proposition 3.2. Without loss of generality, we may assume that all U_α are F_σ-subsets of B, hence C-spaces. According to [24, Theorem 2.3], it suffices to show that each $p|p^{-1}(U_\alpha)$ has the FGP-property. Because every U_α is a locally compact C-space, we can assume that p is a trivial fibration such that any fibration $p|p^{-1}(K) : p^{-1}(K) \to K$, where $K \subset B$ is compact, has the FGP-property. Therefore, we need to show that p has the FGP-property. To this end, using the paracompactness of B, for every i choose an open discrete family $\gamma_i = \{V_{s,i} : s \in S\}$ in B such that $\bigcup_{i=1}^{\infty} \gamma_i$ is a cover of B and each $\overline{V}_{s,i}$ is compact, see [9]. Then the family $\overline{\gamma}_i = \{\overline{V}_{s,i} : s \in S\}$ is also discrete in B and $B_i = \bigcup\{\overline{V}_{s,i} : s \in S\}$ is closed in B. Since for every i the fibrations $p|p^{-1}(\overline{V}_{s,i})$, $s \in S$, have the FGP-property, each $p|p^{-1}(B_i)$ also has the FGP-property. Consequently, for every n and i the set $\Theta_{n,i}$ of all $f \in C_P(\mathbb{I}^n \times \{1, 2\} \times B_i, p^{-1}(B_i))$ with $f(\mathbb{I}^n \times \{1\} \times B_i) \cap f(\mathbb{I}^n \times \{2\} \times B_i) = \varnothing\}$ is dense in $C_P(\mathbb{I}^n \times \{1, 2\} \times B_i, p^{-1}(B_i))$.

Claim 3.6. Each $\Theta_{n,i}$ is open in $C_P(\mathbb{I}^n \times \{1, 2\} \times B_i, p^{-1}(B_i))$.

Indeed, let $f_0 \in \Theta_{n,i}$. Using that $\overline{\gamma}_i$ is discrete in B and the pairs $F^1_{s,i} = f_0(\mathbb{I}^n \times \{1\} \times \overline{V}_{s,i})$, $F^2_{s,i} = f_0(\mathbb{I}^n \times \{2\} \times \overline{V}_{s,i})$ are compact disjoint subsets of $p^{-1}(B_i)$, we can find open subsets $W^j_{s,i}$ and $G^j_{s,i}$ of $p^{-1}(B_i)$, $j = 1, 2$, with the following properties:

(6) $F^j_{s,i} \subset W^j_{s,i} \subset \overline{W}^j_{s,i} \subset G^j_{s,i}$, $j = 1, 2$;
(7) For each $j = 1, 2$ the families $\beta^j_i = \{W^j_{s,i} : s \in S\}$ and $\eta^j_i = \{G^j_{s,i} : s \in S\}$ are discrete in $p^{-1}(B_i)$;
(8) $G^1_{s,i} \cap G^2_{s,i} = \varnothing$ for each s and i.

Now, let $\mathcal{G}_i = \{G^1_{s,i}, G^2_{s,i}, p^{-1}(B_i) \setminus \overline{W} : s \in S\}$, where $\overline{W} = \bigcup\{\overline{W}^1_{s,i} \cup \overline{W}^2_{s,i} : s \in S\}$. Every $\mathcal{G}_i$ is an open cover of $p^{-1}(B_i)$ such that $f \in B(f_0, \mathcal{G}_i)$ implies $f(\mathbb{I}^n \times \{1\} \times \{b\}) \cap f(\mathbb{I}^n \times \{2\} \times \{b\}) = \varnothing$ for every $b \in B_i$. Hence, $B(f_0, \mathcal{G}_i) \subset \Theta_{n,i}$. This completes the proof of the claim because $B(f_0, \mathcal{G}_i)$ is a neighborhood of f_0 with respect to the limitation topology, see [5].

Since each $q_i : C_P(\mathbb{I}^n \times \{1, 2\} \times B, E) \to C_P(\mathbb{I}^n \times \{1, 2\} \times B_i, p^{-1}(B_i))$ is open (see Lemma 2.1), $q_i^{-1}(\Theta_{n,i})$ is open and dense in $C_P(\mathbb{I}^n \times \{1, 2\} \times B, E)$. According to Proposition 2.3, $C_P(\mathbb{I}^n \times \{1, 2\} \times B, E)$ has the Baire property. Therefore, the set $\Theta_n = \bigcap_{i=1}^{\infty} q_i^{-1}(\Theta_{n,i})$ is dense in $C_P(\mathbb{I}^n \times \{1, 2\} \times B, E)$. Finally, observe that $g \in \Theta_n$ implies $g(\mathbb{I}^n \times \{1\} \times \{b\}) \cap g(\mathbb{I}^n \times \{2\} \times \{b\}) = \varnothing$ for every $b \in B$. Therefore, p has the FGP-property. □

4. *ANR*-fibrations over *C*-spaces

The FGP property is necessary and sufficient for a locally compact ANR fibration with compact fibers to be a Q-manifold bundle. In the present section we provide another condition which still guarantees that any locally compact ANR fibration with compact fibers satisfying that condition is a Q-manifold bundle provided the base is a C-space.

A closed set $A \subset E$ is said to be a *fibred* Z_n*-set* if the set of all $f \in C_P(\mathbb{I}^n \times B, E)$ with $f(\mathbb{I}^n \times B) \cap A = \varnothing$ is dense in $C_P(\mathbb{I}^n \times B, E)$. We also define weak versions of fibred Z_n-sets: a closed set $A \subset E$ is a *weak fibred* Z_n*-set in* E (resp., weak fibred homological Z_n-set in E) if $A \cap p^{-1}(b)$ is a Z_n-set (resp., homological Z_n-set) in $p^{-1}(b)$ for all $b \in B$. Fibred Z_∞-sets (resp., weak fibred Z_∞-sets or weak fibred homological Z_∞-sets) will be called *fibred* Z*-sets* (resp., *weak fibred* Z*-sets or weak fibred homological* Z*-sets*). Finally, a map $f \in C_P(K \times B, E)$, where K is a given space, is *fibred* Z*-map, weak fibred* Z*-map or weak fibred homological* Z*-map* if the image $f(K)$ has the corresponding property.

Theorem 4.1. *Let $p : E \to B$ be a locally compact ANR-fibration with compact fibers such that B is a C-space. Then every $b \in B$ has a basis of neighborhoods U such that each weak fibred Z-set in $p^{-1}(U)$ is a fibred Z-set in $p^{-1}(U)$.*

Proof. As in Proposition 3.2, considering a local trivialization $\mathcal{U}$ of p such that each $U \in \mathcal{U}$ is open F_σ subset of B, the proof is reduced to the case when p is a trivial ANR-fibration such that both fibrations $p' : E \times Q \to E \to B$ and $p'' : E \times Q \times [0, 1) \to E \to B$ are trivial Q-manifold bundles. Consequently, there is a fiber-preserving homeomorphism $h : E \times Q \times [0, 1) \to M \times B$, where M is a Q-manifold homeomorphic to $p^{-1}(b_0) \times Q \times [0, 1)$ for some $b_0 \in B$ (actually, M is homeomorphic to $p^{-1}(b) \times Q \times [0, 1)$ for all $b \in B$). We consider M as an open subset of a copy Q_M of Q, and let d be the usual convex metric on Q_M. Suppose $A \subset E$ is a weak fibred Z-set in E. We need to show that for every n, a map $f \in C_P(\mathbb{I}^n \times B, E)$ and an open cover $\mathcal{U}$ of E there is a map $f' \in C_P(\mathbb{I}^n \times B, E)$ such that $f'(\mathbb{I}^n \times B) \cap A = \varnothing$ and f' is $\mathcal{U}$-close to f. To this end, fix such $f \in C_P(\mathbb{I}^n \times B, E)$ and an open cover $\mathcal{U}$ of E. We identify E with $E \times \{\overline{0}\} \times \{0\}$, where $\overline{0} = (0, 0, ...) \in Q$. Consider the retraction $r_E : E \times Q \times [0, 1) \to E \times \{\overline{0}\} \times 0$, $r_E((z_1, z_2, z_3)) = (z_1, \overline{0}, 0)$. Under the above identifications, r_E is a fiber-preserving retraction and $\mathcal{V} = h(r_E^{-1}(\mathcal{U}))$ is an open cover of $M \times B$. Moreover, consider the map $\widetilde{f} : \mathbb{I}^n \times B \to M \times B$, $\widetilde{f}(x, b) = h(f(x, b), \overline{0}, 0))$. Since h is a fiber-preserving homeomorphism, $\widetilde{f}(x, b)$ is of the form $(f_M(x, b), b)$ for each $(x, b) \in \mathbb{I}^n \times B$. Then, by Lemma 2.2, there is a bounded continuous function ε on $M \times B$ such that every map from the set

$$\{g \in C_P(\mathbb{I}^n \times B, M \times B) : d_E(\widetilde{f}(y), g(y)) < \varepsilon(\widetilde{f}(y)) \forall y \in \mathbb{I}^n \times B\}$$

is $\mathcal{V}$-close to $\widetilde{f}$, where d_E is the continuous pseudo-metric on $M \times B$ generated by d.

Since M is open in Q_M, the function $\eta(x,b) = d\big(\pi_M(\widetilde{f}(x,b)), Q_M \backslash M\big)$ is positive and continuous on $\mathbb{I}^n \times B$ ($\pi_M : M \times B \to M$ is the projection. Let $\delta : \mathbb{I}^n \times B \to (0,\infty)$ be the function defined by $\delta(x,b) = \min\{\eta(x,b), \varepsilon(\widetilde{f}(x,b))\}$.

Claim 4.2. There is a bounded continuous function $\alpha : B \to (0,\infty)$ such that $\alpha(b) < \delta(x,b)$ for each $(x,b) \in \mathbb{I}^n \times B$.

Indeed, for every $b \in B$ let $m_b = \min\{\delta(x,b) : x \in \mathbb{I}^n\}$. Then each $m_b > 0$ and consider the set-valued map $\theta : B \rightsquigarrow \mathcal{F}_c((0,\infty))$, $\theta(b) = (0, m_b]$. This map is lower semi-continuous and, by [20, Theorem 6.2, p.116], θ has a continuous selection α. Clearly, α has the required property.

Define the set-valued map $\varphi : \mathbb{I}^n \times B \to \mathcal{F}_c(Q_M)$ by

$$\varphi(x,b) = \{u \in Q_M : d(f_M(x,b), u) \le \alpha(b)\}.$$

Observe that each $\varphi(x,b)$ is a compact convex subset of M because $\alpha(b) < \eta(x,b)$ for all $(x,b) \in \mathbb{I}^n \times B$. Since f_M and α are continuous, φ is lower semi-continuous. Recall that each $A(b) = A \cap p^{-1}(b)$ is a Z-set in $p^{-1}(b)$. So is $A(b) \times Q \times [0,1)$ in the set $p^{-1}(b) \times Q \times [0,1)$. This implies that $H(b) = h(A(b) \times Q \times [0,1))$ is a Z-set in $h(p^{-1}(b) \times Q \times [0,1))$. On the other hand, $h(p^{-1}(b) \times Q \times [0,1)) = M \times \{b\}$. Therefore, $\psi(b) = \pi_M\big(H(b)\big)$ is a Z-set in M for every $b \in B$.

Claim 4.3. The set-valued map $\psi : B \rightsquigarrow \mathcal{F}(M)$ has a closed graph and $\psi(b) \cap \varphi(x,b)$ is a Z-set in $\varphi(x,b)$ for all $(x,b) \in \mathbb{I}^n \times B$.

To show that the graph $Gr(\psi)$ is closed in $B \times M$, take a net $\{(b_\gamma, u_\gamma)\}$ in $Gr(\psi)$ converging to a point $(b^*, u^*) \in B \times M$. Then for each γ we have $u_\gamma \in \psi(b_\gamma)$, so $(u_\gamma, b_\gamma) \in h(A(b) \times Q \times [0,1))$. This means that there exists $z_\gamma \in A(b_\gamma) \times Q \times [0,1)$ with $h(z_\gamma) = (u_\gamma, b_\gamma)$. Since the net $\{(u_\gamma, b_\gamma)\}$ converges to (u^*, b^*) in $M \times B$, $\{z_\gamma\}$ converges to $z^* = h^{-1}(u^*, b^*)$ in $E \times Q \times [0,1)$. Since $A \times Q \times [0,1)$ is closed in $E \times Q \times [0,1)$ and contains all z_γ, z^* belongs to $A \times Q \times [0,1)$. On the other hand, $z^* \in p^{-1}(b^*) \times Q \times [0,1)$ because h is fiber-preserving. Hence, $z^* \in (A \times Q \times [0,1)) \cap (p^{-1}(b^*) \times Q \times [0,1))$. The last intersection is $A(b^*) \times Q \times [0,1)$, so $h(z^*) = (u^*, b^*) \in H(b^*)$. Therefore, $u^* \in \psi(b^*)$, or equivalently, $(b^*, u^*) \in Gr(\psi)$. This shows that $Gr(\psi)$ is closed in $B \times M$.

To prove the second part of Claim 4.3, for each $(x,b) \in \mathbb{I}^n \times B$ consider the open in M set $O(x,b) = \{u \in Q_M : d(f_M(x,b), u) < \alpha(b)\}$. Because $\psi(b)$ is a Z-set in M, $\psi(b) \cap O(x,b)$ is a Z-set in $O(x,b)$. Finally, since $\varphi(x,b)$ is the closure of $O(x,b)$, $\psi(b) \cap \varphi(x,b)$ is a Z-set in $\varphi(x,b)$.

Since ψ has a closed graph, so has the map $\psi' : I^n \times B \rightsquigarrow \mathcal{F}(M)$, $\psi'(x,b) = \psi(b)$. Because $\mathbb{I}^n \times B$ is a paracompact C-space, we can apply Lemma 3.1 to obtain a map $g_M : \mathbb{I}^n \times B \to M$ with $g_M(x,b) \in \varphi(x,b) \backslash \psi'(x,b)$. Then $(g_M(x,b), b)$ defines a map $\widetilde{g} \in C_P(\mathbb{I}^n \times B, M \times B)$ with $\widetilde{g}(x,b) \in M \times \{b\} \backslash H(b)$. So, $g(x,b) \in (p^{-1}(b^*) \backslash A(b)) \times Q \times [0,1)$ for all $(x,b) \in \mathbb{I}^n \times B$, where $g \in C_P(\mathbb{I}^n \times B \to E \times Q \times [0,1))$ is the map defined by $g = h^{-1} \circ \widetilde{g}$. Because $d(f_M(x,b), g_M(x,b)) \le \alpha(b) < \varepsilon(\widetilde{f}(x,b))$, $\widetilde{f}$ and $\widetilde{g}$ are $\mathcal{V}$-close maps. Consequently, $(f(x,b), \overline{0}, 0)$ and $g(x,b)$ are $r_E^{-1}(\mathcal{U})$-close for each (x,b). This implies that f is $\mathcal{U}$-close to the map $f' \in C_P(\mathbb{I}^n \times B \to E)$, $f' = r_E \circ g$. Finally, observe that $g(x,b) \in (p^{-1}(b^*) \backslash A(b)) \times Q \times [0,1)$ yields $f'(x,b) \in p^{-1}(b^*) \backslash A(b)$ for all (x,b). Hence, $f'(\mathbb{I}^n \times B) \cap A = \varnothing$. This completes the proof of Theorem 4.1. □

Theorem 4.4. *Let $p : E \to B$ be a locally compact ANR-fibration with compact fibers. Then p is a Q-manifold bundle provided the following holds:*

(*) *Every point of B has a basis consisting of C-spaces U such that each $C_P(\mathbb{I}^n \times U, p^{-1}(U))$, $n \ge 1$, contains a dense set of weak fibred Z-maps.*

Proof. We take a local trivialization $\mathcal{U}$ of p such that each $U \in \mathcal{U}$ is a C-space and for every n the space $C_P(\mathbb{I}^n \times U, p^{-1}(U))$ contains a dense set of weak fibred Z-maps. Moreover, by Theorem 4.1, we can also assume that each $U \in \mathcal{U}$ satisfies the additional condition:

- Every weak fibred Z-set in $p^{-1}(U)$ is a fibred Z-set in $p^{-1}(U)$.

Let us show that each $p_U = p|p^{-1}(U)$, $U \in \mathcal{U}$, has the FGP-property. It suffices to prove that for every n every weak fibred Z-map $f : \mathbb{I}^n \times U \to p^{-1}(U)$ can be approximated by fiber-preserving maps $f' : \mathbb{I}^n \times U \to p^{-1}(U)$ such that $f'(\mathbb{I}^n \times U) \cap f(\mathbb{I}^n \times U) = \varnothing$. This follows from the fact that the set $A = f(\mathbb{I}^n \times U)$ is a fibred Z-set in $p^{-1}(U)$ because f is weak fibred Z-map. □

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