

Homological characterizations of Q -manifolds and l_2 -manifolds

by

Alexandre Karassev and Vesko Valov (North Bay)

Abstract. We investigate to what extent the density of Z_n -maps in the characterization of Q -manifolds and the density of maps $f \in C(\mathbb{N} \times Q, X)$ having discrete images in the l_2 -manifolds characterization can be weakened to the density of homological Z_n -maps and homological Z -maps, respectively. As a result, we obtain homological characterizations of Q -manifolds and l_2 -manifolds.

1. Introduction and preliminary results. By a space we always mean a complete separable metric space without isolated points.

Toruńczyk's well-known fundamental characterizations of manifolds modeled on $Q = [-1, 1]^\infty$ and l_2 state that a locally compact separable ANR-space X is a Q -manifold if and only if X satisfies the disjoint n -disks property for every n ; see [16]. Equivalently, for every n the function space $C(\mathbb{B}^n, X)$ contains a dense set of Z_n -maps. Similarly, a complete separable ANR-space is an l_2 -manifold iff X has the discrete approximation property: $C(\mathbb{N} \times Q, X)$ contains a dense set of maps f such that the family $\{f(\{n\} \times Q) : n \in \mathbb{N}\}$ is discrete in X ; see [15]. Here, both function spaces $C(Q, X)$ and $C(\mathbb{N} \times Q, X)$ are equipped with the limitation topology, and \mathbb{B}^n is the n -dimensional ball. Daverman and Walsh [10] (see also [12]) refine Toruńczyk's characterization of Q -manifolds by combining the disjoint 2-disks property and the disjoint Čech carriers property (the latter property means that Čech homology elements can be made disjoint). Banach and Repovš [2] observed that the disjoint Čech carriers property in Daverman–Walsh's characterization can be replaced by the following one: $C(K, X)$ contains a dense set of homological Z -maps for every compact polyhedron K . Bowers [6] provided an

2020 *Mathematics Subject Classification*: Primary 57N20; Secondary 58B05, 54C10.

Key words and phrases: disjoint n -disks property, homological maps, l_2 -manifolds, Q -manifolds.

Received 26 April 2021; revised 2 March 2022.

Published online 14 June 2022.

l_2 -version of Daverman–Walsh’s result for spaces having nice ANR local compactifications.

In the present paper we investigate to what extent the density of Z_n -maps in the characterization of Q -manifolds and the density of maps $f \in C(\mathbb{N} \times Q, X)$ having discrete images in the l_2 -manifolds characterization can be weakened to the density of homological Z_n -maps and homological Z -maps, respectively. In Section 2 we establish relations between the density of homological Z_n -maps in $C(\mathbb{B}^n, X)$ and the disjoint n -carriers property. As a corollary we obtain a proof of the Banach–Repovš result mentioned above, and obtain a characterization of spaces having a nice ANR local compactification that is a Q -manifold. We also discuss the question whether for any locally compact ANR-space X we know that $X \times \mathbb{B}^1$ is a Q -manifold if and only if for any n the space $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps. This can be compared to Daverman–Walsh’s result [10] that a locally compact ANR-space has the disjoint Čech carriers property if and only if $X \times \mathbb{B}^2$ is a Q -manifold.

In Section 3 we provide homological characterizations of l_2 -manifolds. Two cases are considered, the case of the boundary set setting (when the spaces under consideration have “nice” ANR local compactifications) and the general case of ANR-spaces. Among ANR-spaces l_2 -manifolds are exactly the spaces X having the discrete 2-cells property such that $C(\mathbb{N} \times Q, X)$ contains a dense G_δ -set of homological Z_∞ -maps (see Corollary 3.6). A combination of the disjoint disks property and the existence of a dense G_δ -set in $C(\mathbb{N} \times \mathbb{B}^n, X)$ of homological Z_n -maps for every n characterizes l_2 -manifolds in the boundary set setting (Corollary 3.2).

The singular and Čech homology groups are denoted, respectively, by H_k and \check{H}_k . If $V \subset U$ are open subsets of X and $z \in \check{H}_q(U, V)$ for some integer $q \geq 0$, a compact pair $(C, \partial C) \subset (U, V)$ is said to be a *Čech carrier* [10] for z provided $z \in i_* (\check{H}_q(C, \partial C))$, where $i_* : \check{H}_q(C, \partial C) \rightarrow \check{H}_q(U, V)$ is the inclusion-induced homomorphism. A *singular carrier* of an element $z \in H_q(U, V)$ is defined in a similar way. Following [10], we say that a space X has the *disjoint n -carriers property* (br., DC^nP) provided for any open in X sets $V_i \subset U_i$, $i = 1, 2$, and any elements $z_i \in \check{H}_{q(i)}(U_i, V_i)$ with $0 \leq q(i) \leq n$ there are Čech carriers $(C_i, \partial C_i) \subset (U_i, V_i)$ for z_i such that $C_1 \cap C_2 = \emptyset$. A space X has the *disjoint Čech carriers property* (br., DCP) provided $X \in DC^nP$ for every n . The disjoint n -carriers property is a homological analogue of the well-known *disjoint n -disks property* (br., DD^nP): any two maps $\mathbb{B}^n \rightarrow X$ can be approximated by maps with disjoint images. Recall that a closed set $F \subset X$ is said to be a Z_n -set if the set $C(\mathbb{B}^n, X \setminus F)$ is dense in $C(\mathbb{B}^n, X)$. Note that if X is an LC^{n-1} -space, then a closed set $F \subset X$ is a Z_n -set iff for each at most n -dimensional metric compactum Y the set $\{f \in C(Y, X) :$

$f(Y) \cap F = \emptyset$ is dense in $C(Y, X)$; see [3]. We also say that a map $f : K \rightarrow X$, where K is a compactum, is a Z_n -map provided $f(K)$ is a Z_n -set in X . According to [16], $X \in \text{DD}^n\text{P}$ if and only if the set of all Z_n -maps $\mathbb{B}^n \rightarrow X$ is dense and G_δ in $C(\mathbb{B}^n, X)$.

Replacing \mathbb{B}^n in the above definition of Z_n -sets with Q , we obtain the definition of Z_∞ -sets and Z_∞ -maps. It is well known that a set is a Z -set iff it is a Z_n -set for every n . A closed set $A \subset X$ is a *strong Z -set* [4] if for every open cover \mathcal{U} of X and a sequence of maps $\{f_i\} \subset C(Q, X)$, there is a sequence $\{g_i\} \subset C(Q, X)$ such that each g_i is \mathcal{U} -close to f_i and $\bigcup_{i \geq 1} g_i(Q) \cap A = \emptyset$.

Recall that a space X is LC^n if for every $x \in X$ and its neighborhood U in X there is another neighborhood V of x such that $V \xrightarrow{m} U$ for all $m \leq n$ (here $V \xrightarrow{m} U$ means that $V \subset U$ and every map from the m -dimensional sphere S^m into V can be extended to a map of \mathbb{B}^{m+1} to U). We also say that a set $A \subset X$ is k -LCC in X if for every point $x \in A$ and its neighborhood U in X there exists another neighborhood V of x with $V \subset U$ and $V \setminus A \xrightarrow{k} U \setminus A$. If A is k -LCC in X for all $k \leq n$, then A is said to be LCC^n in X .

There are homological analogues of Z_n -sets and Z_n -maps. A closed set $F \subset X$ is called a *homological Z_n -set* in X if the singular homology groups $H_k(U, U \setminus F)$ are trivial for all open sets $U \subset X$ and all $k \leq n$; see [1]. It can be shown that every homological Z_n -set in X is nowhere dense. The homological Z_n -property is finitely additive and hereditary with respect to closed subsets [1]. A map $f : K \rightarrow X$ is a *homological Z_n -map* provided the image $f(K)$ is a homological Z_n -set in X . Homological Z_∞ -sets were considered in [10] under the name of sets of infinite codimension.

Combining [17, Corollary 3.3] and [1, Theorem 2.1], we have the following:

PROPOSITION 1.1. *Let X be an LC^n -space with $n \geq 2$. Then a closed subset A of X is a Z_n -set in X provided A is an LCC^1 homological Z_n -set in X . Equivalently, A is a Z_n -set iff it is a Z_2 -set and a homological Z_n -set. In particular, Z_n -sets in LC^n -spaces are homological Z_n -sets.*

2. Homological Z_n -maps and Q -manifolds. It is well known that for every pair (X, A) with $A \subset X$ there is a natural homomorphism $T_{X,A} : H_*(X, A) \rightarrow \check{H}_*(X, A)$. Recall that a space X is called *homologically locally connected up to dimension n* if for any point x in X and any neighborhood U of x there exists a neighborhood V of x such that $V \subset U$ and the inclusion-induced homomorphism $H_k(V) \rightarrow H_k(U)$ is trivial for all $k \leq n$. If both X and A are homologically locally connected with respect to the singular homology up to dimension n , then $T_{X,A} : H_k(X, A) \rightarrow \check{H}_k(X, A)$ is an isomorphism for all $k \leq n$; see [13]. In particular, this is true if X and A are LC^n -spaces. We say that an element $z \in H_k(U, V)$ has a *homological Čech*

Z_n -carrier provided z has a Čech carrier $(C_z, \partial C_z) \subset (U, V)$ such that C_z is a homological Z_n -set in X .

The following result is well known.

LEMMA 2.1. *Let X be an LC^n -space and $V \subset U$ be open in X . Then $H_k(U, V)$ is countable for all $k \leq n$.*

LEMMA 2.2. *Let X be an LC^n -space and (U, V) a pair of open sets in X . If $(C_z, \partial C_z) \subset (U, V)$ is a singular carrier for some $z \in H_k(U, V)$ with $k \leq n$, then $(C_z, \partial C_z)$ is also a Čech carrier for z .*

Proof. Since X is LC^n , the homomorphism $T_{U,V} : H_k(U, V) \rightarrow \check{H}_k(U, V)$ is an isomorphism. Then the commutative diagram

$$\begin{CD} H_k(C_z, \partial C_z) @>i_*>> H_k(U, V) \\ @V T_{C_z, \partial C_z} VV @VV T_{U,V} V \\ \check{H}_k(C_z, \partial C_z) @>i_*>> \check{H}_k(U, V) \end{CD}$$

implies that $(C_z, \partial C_z)$ is a Čech carrier for z . ■

The next lemma is an analogue of [10, Lemma 3.1].

LEMMA 2.3. *A closed set $A \subset X$ is a homological Z_n -set in X if and only if each $z \in H_m(U, V)$, where $m \leq n$ and U, V are open sets in X with $V \subset U$, has a singular carrier $(C_z, \partial C_z) \subset (U, V)$ such that $C_z \cap A = \emptyset$.*

Proof. Suppose A is a homological Z_n -set and let $z \in H_m(U, V)$ for some $m \leq n$ and open sets $V \subset U$. Then we have the commutative diagram

$$\begin{CD} H_m(U \setminus A) @>>> H_m(U \setminus A, V \setminus A) @>>> H_{m-1}(V \setminus A) @>>> H_{m-1}(U \setminus A) \\ @V i VV @V j VV @V p VV @V l VV \\ H_m(U) @>>> H_m(U, V) @>>> H_{m-1}(V) @>>> H_{m-1}(U) \end{CD}$$

where the rows are exact sequences of the pairs $(U \setminus A, V \setminus A)$ and (U, V) . Since A is a homological Z_n -set, i is an epimorphism, while p and l are isomorphisms. Hence, by the Four-Lemma, j is an epimorphism. This means that there is $z' \in H_m(U \setminus A, V \setminus A)$ with $j(z') = z$. So, z has a singular carrier $(C_z, \partial C_z) \subset (U \setminus A, V \setminus A)$.

The other implication of Lemma 2.3 is obvious. ■

Let \mathcal{B} be a finitely additive base for X and $\mathcal{H}_n = \bigcup \{H_k(U, V) : k \leq n \text{ and } U, V \in \mathcal{B}\}$. Then $\mathcal{H}_n = \bigcup \{H_k(U, V) : k \leq n \text{ and } U, V \text{ open in } X\}$.

COROLLARY 2.4. *Every closed subset of X contained in $X \setminus \bigcup \{C_z : z \in \mathcal{H}_n\}$ is a homological Z_n -set in X .*

PROPOSITION 2.5. Consider the following conditions for an LC^n -space:

- (1) $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps;
- (2) each $C(\mathbb{B}^k, X)$, $k \leq n$, contains a dense set of homological Z_n -maps;
- (3) every $z \in H_k(U, V)$, $k \leq n$, has a homological Čech Z_n -carrier $(C_z, \partial C_z) \subset (U, V)$.
- (4) $X \in DC^n P$.

Then (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).

Proof. (1) \Rightarrow (2): Suppose $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps. For every $k < n$ we embed \mathbb{B}^k in \mathbb{B}^n and consider the restriction map $\pi_k^n : C(\mathbb{B}^n, X) \rightarrow C(\mathbb{B}^k, X)$. The maps π_k^n are open and continuous. Moreover, since \mathbb{B}^k is a retract of \mathbb{B}^n , π_k^n are also surjective. Therefore, if $M_n \subset C(\mathbb{B}^n, X)$ is a dense subset consisting of homological Z_n -maps, the sets $M_k = \pi_k^n(M_n)$ are also dense in $C(\mathbb{B}^k, X)$ and consist of homological Z_n -maps.

(2) \Rightarrow (3): If $z \in H_k(U, V)$ for some open sets U, V then there is a singular chain $c_z = \sum_{i=1}^p m_i f_i$ representing z with $f_i(\mathbb{S}^{k-1}) \subset V$ for all i . Using the density of M_k in $C(\mathbb{B}^k, X)$ we approximate each f_i with a map $g_i \in M_k$ such that $g_i(\mathbb{B}^k) \subset U$ and $g_i(\mathbb{S}^{k-1}) \subset V$. Because $X \in LC^n$, we can suppose that each g_i is homotopic to f_i in U . This means that $\sum_{i=1}^p m_i g_i$ is another representation of z and $C_z = \bigcup_{i=1}^p g_i(\mathbb{B}^k)$ is a singular carrier for z . Because each $g_i(\mathbb{B}^k)$ is a homological Z_n -set in X , so is C_z ; see [1]. It remains to show that C_z is also a Čech carrier for z . And this follows from Lemma 2.2.

(3) \Rightarrow (4): Let $z_j \in H_{k(j)}(U_j, V_j)$, $j = 1, 2$, and $(C_1, \partial C_1) \subset (U_1, V_1)$ be a Čech carrier for z_1 such that C_1 is a homological Z_n -set. Then, by Lemma 2.3, z_2 has a singular carrier $(C_2, \partial C_2) \subset (U_2, V_2)$ with $C_1 \cap C_2 = \emptyset$. Lemma 2.2 implies that $(C_2, \partial C_2)$ is also a Čech carrier for z_2 . ■

LEMMA 2.6. Let X be an LC^n -space. Then the set Λ_n of all homological Z_n -maps $f : \mathbb{B}^n \rightarrow X$ is G_δ in $C(\mathbb{B}^n, X)$.

Proof. Let \mathcal{B} be a countable finitely additive base for X . Let \mathcal{C}_k be a countable dense set of k -chains in X , where $k = 0, 1, \dots, n$ (density here is with respect to the compact-open topology for the respective maps), and $\mathcal{C} = \bigcup_{k=0}^n \mathcal{C}_k$. For a set $A \subset X$ and $\varepsilon > 0$ let $B(A, \varepsilon)$ denote the closed ε -neighborhood of A in X . For any $U \in \mathcal{B}$, $c \in \mathcal{C}$ with $c \subset U$, and $m = 1, 2, \dots$ let $G_{U,c}^m$ be the set of all maps $f : \mathbb{B}^n \rightarrow X$ such that if $\partial c \subset U \setminus B(f(\mathbb{B}^n), 1/m)$ then there exists a $(k+1)$ -chain $c' \subset U$ such that $\partial c' - c \subset U \setminus f(\mathbb{B}^n)$.

First we will show that $\Lambda_n = \bigcap \{G_{U,c}^m : U \in \mathcal{B}, c \in \mathcal{C}, m \in \mathbb{N}\}$. Denote the latter intersection by \mathcal{G} and suppose that $f \in \mathcal{G}$. According to the results of [1] it is sufficient to show that $H_k(U, U \setminus f(\mathbb{B}^n)) = 0$ for all $k = 0, 1, \dots, n$ and all $U \in \mathcal{B}$. Consider a chain $c \in H_k(U, U \setminus f(\mathbb{B}^n))$. We may assume that

$c \in \mathcal{C}_k$. Because the carrier of c and $f(\mathbb{B}^n)$ are compact, there exists m such that $\partial c \subset U \setminus B(f(\mathbb{B}^n), 1/m)$. Since $f \in G_{U,c}^m$, there exists a $(k + 1)$ -chain $c' \subset U$ such that $\partial c' - c \subset U \setminus f(\mathbb{B}^n)$. This implies that c is homologous to 0 in $H_k(U, U \setminus f(\mathbb{B}^n))$. Thus $f \in A_n$. Now consider any $f \in A_n$ and a set $G_{U,c}^m$ with $c \in \mathcal{C}_k$. Suppose $\partial c \subset U \setminus B(f(\mathbb{B}^n), 1/m)$. Then, in particular, $c \in H_k(U, U \setminus f(\mathbb{B}^n)) = 0$. Therefore there exists a $(k + 1)$ -chain $c' \subset U$ such that $\partial c' - c \subset U \setminus f(\mathbb{B}^n)$. This implies that f is contained in each set $G_{U,c}^m$, and hence $f \in \mathcal{G}$.

Next, we will show that each $G_{U,c}^m$ is open. For this, consider $f \in G_{U,c}^m$. If $\partial c \not\subset U \setminus B(f(\mathbb{B}^n), 1/m)$, the same is true for all maps g that are sufficiently close to f . Suppose that $\partial c \subset U \setminus B(f(\mathbb{B}^n), 1/m)$. Then there exists a $(k + 1)$ -chain c' such that $\partial c' - c \subset U \setminus f(\mathbb{B}^n)$. This implies that $\partial c' - c \subset U \setminus g(\mathbb{B}^n)$ for all maps g sufficiently close to f . ■

We say that a space X has the *property* $DD^{\{n,m\}}P$ if any two maps $f : \mathbb{B}^n \rightarrow X$ and $g : \mathbb{B}^m \rightarrow X$ can be approximated by maps $f' : \mathbb{B}^n \rightarrow X$ and $g' : \mathbb{B}^m \rightarrow X$ with $f'(\mathbb{B}^n) \cap g'(\mathbb{B}^m) = \emptyset$. The property $DD^{\{1,2\}}P$ (resp., $DD^{\{1,1\}}P$) is called the *disjoint arc-disk* (resp., *disjoint arcs*) *property*.

PROPOSITION 2.7. *Suppose $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_1 -maps. Then*

- (1) $X \in DD^{\{1,n\}}P$;
- (2) $C(\mathbb{B}^1, X)$ contains a dense G_δ -subset consisting of Z_n -maps;
- (3) $C(\mathbb{B}^n, X)$ contains a dense G_δ -subset consisting of Z_1 -maps.

Proof. The following statement was actually established in [2, proof of Proposition 6]: If A is a homological Z_1 -set in a space X , then every map $g : \mathbb{B}^1 \rightarrow X$ can be approximated by maps $g' : \mathbb{B}^1 \rightarrow X \setminus A$. This statement implies that $X \in DD^{\{1,n\}}P$ provided $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_1 -maps.

To prove the second item, choose a countable base $\{U_i\}$ for $C(\mathbb{B}^n, X)$. For every i let G_i be the set of all $f \in C(\mathbb{B}^1, X)$ such that $f(\mathbb{B}^1) \cap g(\mathbb{B}^n) = \emptyset$ for some $g \in U_i$. Since $X \in DD^{\{1,n\}}P$, one can show that each G_i is open and dense in $C(\mathbb{B}^1, X)$. So, $G = \bigcap G_i$ is dense and G_δ in $C(\mathbb{B}^1, X)$. Observe that for every $f \in G$ and every i there is $g_i \in U_i$ with $f(\mathbb{B}^1) \cap g_i(\mathbb{B}^n) = \emptyset$. Since $\{g_i\}$ is a dense set in $C(\mathbb{B}^n, X)$, each $f(\mathbb{B}^1)$, $f \in G$, is a Z_n -set in X . The proof of item (3) is similar. ■

PROPOSITION 2.8. *Let X be a locally compact LC^n -space with $n \geq 2$. Then the following are equivalent:*

- (1) $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps;
- (2) $X \in DC^nP$ and $C(\mathbb{B}^2, X)$ contains a dense set of homological Z_n -maps;
- (3) every $C(\mathbb{B}^k, X)$, $k \leq n$, contains a dense G_δ -set of homological Z_n -maps.

Proof. (1) \Rightarrow (2): This implication follows from Proposition 2.5.

(2) \Rightarrow (3): Suppose X satisfies condition (2) and choose a dense sequence $\{g_j\}$ in $C(\mathbb{B}^2, X)$ of homological Z_n -maps. Let $\mathcal{H}_n = \{z_j \in H_{k(j)}(U_j, V_j) : k(j) \leq n \text{ and } U_j, V_j \in \mathcal{B}\}$, where \mathcal{B} is a countable additive base for X . Then the arguments from [10, proof of Lemma 3.2] imply that every $z_j \in H_{k(j)}(U_j, V_j)$ has a homological Čech Z_n -carrier $(C_j, \partial C_j) \subset (U_j, V_j)$. According to Corollary 2.4, every compact subset of $X \setminus \bigcup C_j$ is a homological Z_n -set in X . Therefore, we have a sequence $\{D_j = C_j \cup g_j(\mathbb{B}^2)\}$ of homological Z_n -sets such that every compact subset of $X \setminus \bigcup D_j$ is also a homological Z_n -set. Moreover, the density of $\{g_j\}$ in $C(\mathbb{B}^2, X)$ implies that all compact subsets of $X \setminus \bigcup D_j$ are Z_2 -sets. So, by Proposition 1.1, every compact subset of $X \setminus \bigcup D_j$ is a Z_n -set. Following the arguments of [10, Lemma 3.8], one can show that there is another sequence $\{A_i\} \subset X \setminus \bigcup D_j$ of compact sets with each compact subset of $X \setminus \bigcup A_i$ being a homological Z_n -set. Because A_i are Z_n -sets (as subsets of $X \setminus \bigcup D_j$), for every $k \leq n$ the maps $g \in C(\mathbb{B}^k, X)$ with $g(\mathbb{B}^k) \cap \bigcup A_i = \emptyset$ form a dense G_δ -subset W_k of $C(\mathbb{B}^k, X)$. Finally, observe that $g(\mathbb{B}^k)$ is a homological Z_n -set for every $g \in W_k$.

(3) \Rightarrow (1): This implication is obvious. ■

THEOREM 2.9. *The following conditions are equivalent for any LCⁿ-space X :*

- (1) $X \in \text{DD}^2\text{P}$ and $C(\mathbb{B}^n, X)$ contains a dense set A_n of homological Z_n -maps;
- (2) X has the disjoint n -disks property.

Proof. Suppose X satisfies condition (1). Since $X \in \text{DD}^2\text{P}$, there is a dense G_δ -set $G' \subset C(\mathbb{B}^2, X)$ of Z_2 -maps. Observe that there is a dense sequence $\{g_i\} \subset C(\mathbb{B}^2, X)$ such that each $g_i(\mathbb{B}^2)$ is a Z_2 -set and a homological Z_n -set. Indeed, because the restriction map $\pi_2^n : C(\mathbb{B}^n, X) \rightarrow C(\mathbb{B}^2, X)$ is surjective and open, the set $(\pi_2^n)^{-1}(G')$ is dense and G_δ in $C(\mathbb{B}^n, X)$. Lemma 2.6 implies that A_n is G_δ in $C(\mathbb{B}^n, X)$. So, $(\pi_2^n)^{-1}(G') \cap A_n$ is also dense in $C(\mathbb{B}^n, X)$ and it contains a dense sequence $\{f_i\}$. Obviously, the sequence $\{g_i = \pi_2^n(f_i)\}$ has the required property. Thus, by Proposition 1.1, all $g_i(\mathbb{B}^2)$ are Z_n -sets. Consequently, the set $\Gamma_n = \{f \in C(\mathbb{B}^n, X) : f(\mathbb{B}^n) \cap \bigcup g_i(\mathbb{B}^2) = \emptyset\}$ is dense and G_δ in $C(\mathbb{B}^n, X)$. Moreover, the density of $\{g_i\}$ in $C(\mathbb{B}^2, X)$ implies that $f(\mathbb{B}^n)$ is a Z_2 -set in X for all $f \in \Gamma_n$. Then $\Gamma_n \cap A_n$ is a dense subset of $C(\mathbb{B}^n, X)$ and consists of maps f such that $f(\mathbb{B}^n)$ is both a homological Z_n -set and a Z_2 -set in X . Therefore, each $f \in \Gamma_n \cap A_n$ is a Z_n -map, which shows that X has the disjoint n -disks property.

The implication (2) \Rightarrow (1) is obvious. ■

Following Bowers [6], we say that a space X has a *nice ANR local compactification* if there is a locally compact ANR-space Y containing X such

that $X = Y \setminus F$ for some σZ -set F (i.e., a countable union of Z -sets) in Y . Any such X is complete ANR; see [17]. Toruńczyk’s [16] characterization theorem of Q -manifolds yields the following proposition (the special case when $F = \emptyset$ was established in [2]).

PROPOSITION 2.10. *Let \overline{X} be a nice ANR local compactification of a space X . Then \overline{X} is a Q -manifold if and only if X has the disjoint disks property and for every n the space $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps.*

Proof. Let $X = \overline{X} \setminus F$, where F is a σZ -set in \overline{X} (i.e., F is the union of countably many Z -sets). Suppose X has the disjoint disks property and every $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps. To show that \overline{X} is a Q -manifold, according to Theorem 2.9 and Toruńczyk’s [16] characterization of Q -manifolds, it suffices to prove \overline{X} satisfies the following two conditions: (i) \overline{X} has the disjoint disks property and (ii) every $C(\mathbb{B}^n, \overline{X})$ contains a dense set of homological Z_n -maps. Let $f, g : \mathbb{B}^2 \rightarrow \overline{X}$ be two maps. Since F is a σZ -set in \overline{X} , we can approximate f, g , respectively, by maps $f', g' : \mathbb{B}^2 \rightarrow X$. Then, using the fact that $X \in \text{DD}^2\text{P}$, we approximate f', g' by maps $f'', g'' : \mathbb{B}^2 \rightarrow X$ with $f''(\mathbb{B}^2) \cap g''(\mathbb{B}^2) = \emptyset$. To show condition (ii), let $f \in C(\mathbb{B}^n, \overline{X})$. Since $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps, and recalling again that F is a σZ -set in \overline{X} , we can suppose that $f(\mathbb{B}^n)$ is a homological Z_n -set in X . It remains to show that $f(\mathbb{B}^n)$ is a homological Z_n -set in \overline{X} . To this end, by Lemma 2.3, it suffices to show that any $z \in H_m(U, V)$, where $m \leq n$ and U, V are open sets in \overline{X} , has a singular carrier disjoint from $f(\mathbb{B}^n)$. If $\sum_{i=1}^k m_i h_i$ is a singular representation of z with $h_i \in C(\mathbb{B}^n, \overline{X})$, we approximate each h_i by a map $h'_i \in C(\mathbb{B}^n, X)$ such that $h'(\mathbb{B}^n, \mathbb{S}^{n-1}) \subset (U \cap X, V \cap X)$ and h' is homotopic to h in \overline{X} . Therefore, we may assume that $h_i \in C(\mathbb{B}^n, X)$ for all i and $z \in H_m(U \cap X, V \cap X)$. Since $f(\mathbb{B}^n)$ is a homological Z_n -set in X , by Lemma 2.3, there exists a singular carrier $(C_z, \partial C_z) \subset (U \cap X, V \cap X)$ of z with $C_z \cap f(\mathbb{B}^n) = \emptyset$.

If \overline{X} is a Q -manifold, then it has the disjoint n -disks property for every n ; see [16]. Since F is a σZ -set in \overline{X} , this implies that X also has the disjoint n -disks property for every n . Equivalently, each function space $C(\mathbb{B}^n, X)$ contains a dense set of Z_n -maps. Because every Z_n -map is a homological Z_n -map, the proof is complete. ■

THEOREM 2.11. *Let X be a locally compact LC^n -space such that $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps. Then $X \times Y$ has the disjoint n -disks property for every non-trivial locally compact ANR-space Y .*

Proof. According to Proposition 2.7, X has the disjoint arc-disk property. Then, following [8, proof of Proposition 2.10], one can show that $X \times Y$ has the disjoint disks property. Since $X \times Y$ is LC^n , by Theorem 2.9, it suffices

to show that $C(\mathbb{B}^n, X \times Y)$ contains a dense set of homological Z_n -maps. To this end, let $f = (f_1, f_2) \in C(\mathbb{B}^n, X \times Y)$, where $f_1 \in C(\mathbb{B}^n, X)$ and $f_2 \in C(\mathbb{B}^n, Y)$. We can approximate f_1 by maps $g \in C(\mathbb{B}^n, X)$ such that $g(\mathbb{B}^n)$ are homological Z_n -sets in X . Then, for any such g , consider the map $h_g = (g, f_2) : \mathbb{B}^n \rightarrow X \times Y$. Obviously, the maps h_g approximate f and $h_g(\mathbb{B}^n) \subset g(\mathbb{B}^n) \times Y$. It remains to show that $g(\mathbb{B}^n) \times Y$ is a homological Z_n -set in $X \times Y$. Indeed, let \mathcal{B}_X and \mathcal{B}_Y be bases for X, Y , respectively. Then, by [1, Proposition 3.6], it suffices to show that for any $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ we have $H_k(U \times V, (U \times V) \setminus (g(\mathbb{B}^n) \times Y)) = 0$ for all $k \leq n$. This is really true because by the Künneth formula the group $H_k(U \times V, (U \times V) \setminus (g(\mathbb{B}^n) \times Y))$ is isomorphic to the direct sum of $\sum_{i+j \leq k} H_i(U, U \setminus g(\mathbb{B}^n)) \otimes H_j(V)$ and $\sum_{i+j \leq k-1} H_i(U, U \setminus g(\mathbb{B}^n)) * H_j(V)$, where $H_i(U, U \setminus g(\mathbb{B}^n)) \otimes H_j(V)$ and $H_i(U, U \setminus g(\mathbb{B}^n)) * H_j(V)$ stand for the tensor and the torsion products of $H_i(U, U \setminus g(\mathbb{B}^n))$ and $H_j(V)$, respectively. ■

Another application of Toruńczyk’s [16] characterization of Q -manifolds provides the next corollary.

COROLLARY 2.12. *Let X be a locally compact ANR such that for every n the space $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps. Then $X \times Y$ is a Q -manifold for every non-trivial locally compact ANR-space Y .*

A similar statement was established in [2, Theorem 14].

Corollary 2.12 implies that $X \times \mathbb{B}^1$ is a Q -manifold provided X is a locally compact ANR such that for every n the space $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps. We say that a space X is a *fake Q -manifold* if $X \times \mathbb{B}^1$ is a Q -manifold but X is not. According to [10], any fake Q -manifold has the disjoint Čech carrier property but not the disjoint disks property. All existing examples (see [2], [10] and [14]) of fake Q -manifolds X have the property that for any n the space $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps. So, the following question is very natural.

QUESTION 2.13. *Let X be a locally compact ANR. Is it true that $X \times \mathbb{B}^1$ is a Q -manifold if and only if for every n the space $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps?*

According to the next proposition, Question 2.13 has a positive solution if we can show that for every $n \geq 2$ the space $C(\mathbb{B}^2, X)$ contains a dense set of homological Z_n -maps provided $X \times \mathbb{B}^1$ is a Q -manifold. In particular, that would be true if $C(\mathbb{B}^2, X)$ contains a dense subset of maps with finite-dimensional images.

PROPOSITION 2.14. *Let X be a locally compact ANR such that for every n the space $C(\mathbb{B}^2, X)$ contains a dense set of homological Z_n -maps. Then $X \times \mathbb{B}^1$ is a Q -manifold if and only if each $C(\mathbb{B}^n, X)$, $n \geq 2$, contains a dense set of homological Z_n -maps.*

Proof. If $X \times \mathbb{B}^1$ is a Q -manifold, then so is $X \times \mathbb{B}^2$. Hence, by [10, Corollary 6.2], $X \in \text{DC}^n\text{P}$. This, according to Proposition 2.8, implies that every $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps. The other implication follows from Theorem 2.11. ■

According to [9], the so-called *disjoint path concordance property* characterizes locally compact ANRs $X \in \text{DD}^{\{1,1\}}\text{P}$ such that $X \times \mathbb{R}$, or equivalently $X \times \mathbb{B}^1$, has the disjoint disks property. The disjoint path concordance property is quite different from the property that for every n the space $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps, but the results from [10] yield the following description of the locally compact ANR-spaces X such that $X \times \mathbb{B}^1$ is a Q -manifold.

PROPOSITION 2.15. *Let X be a locally compact ANR-space. Then $X \times \mathbb{B}^1$ is a Q -manifold if and only if X has both the disjoint path concordance property and the disjoint Čech carrier property.*

3. Homological Z_n -maps and l_2 -manifolds. In this section the function spaces $C(Y, X)$ are equipped with the *limitation topology*; see [5, 15]. Let $\text{cov}(X)$ denote the collection of all open covers of X . For a map $f \in C(Y, X)$ and $\mathcal{U} \in \text{cov}(X)$ let $B(f, \mathcal{U})$ be the set of maps $g \in C(Y, X)$ that are \mathcal{U} -close to f . A set $U \subset C(Y, X)$ is open in the limitation topology if for every $f \in U$ there exists $\mathcal{U} \in \text{cov}(X)$ such that $B(f, \mathcal{U}) \subset U$. According to [5, 15], $C(Y, X)$ with the limitation topology is a Baire space.

We say X satisfies the *discrete n -cells property*, where $n \leq \infty$, if for each map $f : \bigoplus_{i=1}^{\infty} \mathbb{B}_i^n \rightarrow X$ of the countable free union of n -cells (∞ -cells are Hilbert cubes Q_i) into X and each open cover \mathcal{U} of X there exists a map $g : \bigoplus_{i=1}^{\infty} \mathbb{B}_i^n \rightarrow X$ such that g is \mathcal{U} -close to f and $\{g(\mathbb{B}_i^n)\}_{i=1}^{\infty}$ is a discrete family in X . The discrete ∞ -cells property is usually called the *discrete approximation property*.

Our first result in this section provides a homological characterization of l_2 -manifolds in the boundary set setting.

THEOREM 3.1. *Suppose X has a nice ANR local compactification. Then X is an l_2 -manifold if and only if X satisfies the following conditions:*

- (1) X has the disjoint disks property;
- (2) for every $n \geq 2$ the space $C(\bigoplus_{i=1}^{\infty} \mathbb{B}_i^n, X)$, equipped with the limitation topology, contains a dense G_δ -set of maps f such that the set $f(\bigoplus_{i=1}^{\infty} \mathbb{B}_i^n) \subset X$ is closed and each $f(\mathbb{B}_i^n)$ is a homological Z_n -set in X .

Proof. Let \overline{X} be a locally compact ANR-compactification of X such that $X = \overline{X} \setminus F$, where F is a σZ -set in \overline{X} . Suppose X satisfies (1) and (2). Observe that, by (2), each $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps. Therefore, by Proposition 2.10, \overline{X} is a Q -manifold.

CLAIM 1. Every $C(\mathbb{B}^n, X)$ contains a dense G_δ -set of Z_n -maps.

According to [16], a given space Y has the disjoint n -disks property iff all Z_n -maps in $C(\mathbb{B}^n, Y)$ form a dense and G_δ -subset. Since \overline{X} is a Q -manifold, it has the disjoint n -disks property for every n (see [16]), or equivalently, every $C(\mathbb{B}^n, \overline{X})$ contains a dense set of Z_n -maps. Let $\{g_i\}_{i \geq 1}$ be a dense in $C(\mathbb{B}^n, \overline{X})$ sequence of Z_n -maps and $D = \bigcup_{i \geq 1} g_i(\mathbb{B}^n) \cup F$. Then D is a σZ_n -subset of \overline{X} . Hence, every compactum in $X \setminus D$ is a Z_n -set in \overline{X} and every map $f \in C(\mathbb{B}^n, \overline{X})$ can be approximated by maps into $X \setminus D$. This implies that X has the disjoint n -disks property and $C(\mathbb{B}^n, X)$ contains a dense G_δ -set of Z_n -maps.

For every n we fix a dense G_δ -subset $A^n \subset C(\bigoplus_{i=1}^\infty \mathbb{B}_i^n, X)$ consisting of maps f satisfying the condition

$$(*)_n \quad f(\bigoplus_{i=1}^\infty \mathbb{B}_i^n) \subset X \text{ is closed and each } f(\mathbb{B}_i^n) \text{ is a homological } Z_n\text{-set in } X.$$

Then, for $k \geq 2$, the space $C(\bigoplus_{i=k}^\infty \mathbb{B}_i^n, X)$ contains a dense G_δ -set A_k^n of maps f such that $f(\bigoplus_{i=k}^\infty \mathbb{B}_i^n) \subset X$ is closed and all $f(\mathbb{B}_i^n)$, $i \geq k$, are homological Z_n -sets in X . Since each restriction map $p_k^n : C(\bigoplus_{i=1}^\infty \mathbb{B}_i^n, X) \rightarrow C(\bigoplus_{i=k}^\infty \mathbb{B}_i^n, X)$ is open (see [15]) and surjective, all sets $(p_k^n)^{-1}(A_k^n)$, $k \geq 2$, are dense and G_δ in $C(\bigoplus_{i=1}^\infty \mathbb{B}_i^n, X)$. So is the set $\tilde{A}^n = \bigcap_{k \geq 2} A^n \cap (p_k^n)^{-1}(A_k^n)$.

For every $k \neq l$ let A_{kl}^n be the set of all maps $f \in C(\bigoplus_{i=1}^\infty \mathbb{B}_i^n, X)$ such that $f(\mathbb{B}_k^n) \cap f(\mathbb{B}_l^n) = \emptyset$. Obviously, each A_{kl}^n is open in $C(\bigoplus_{i=1}^\infty \mathbb{B}_i^n, X)$.

CLAIM 2. Every A_{kl}^n is dense in $C(\bigoplus_{i=1}^\infty \mathbb{B}_i^n, X)$.

Indeed, let $f \in C(\bigoplus_{i=1}^\infty \mathbb{B}_i^n, X)$. Then the maps $f_k = f|_{\mathbb{B}_k^n}$ and $f_l = f|_{\mathbb{B}_l^n}$ can be approximated, respectively, by maps $f'_k \in C(\mathbb{B}_k^n, X)$ and $f'_l \in C(\mathbb{B}_l^n, X)$ such that $f'_k(\mathbb{B}_k^n) \cap f'_l(\mathbb{B}_l^n) = \emptyset$. This can be done because $C(\mathbb{B}_k^n, X)$ and $C(\mathbb{B}_l^n, X)$ contain dense sets of Z_n -maps. Define a map $g \in C(\bigoplus_{i=1}^\infty \mathbb{B}_i^n, X)$ by $g|_{\mathbb{B}_k^n} = f'_k$, $g|_{\mathbb{B}_l^n} = f'_l$ and $g|_{\mathbb{B}_i^n} = f|_{\mathbb{B}_i^n}$ for all $i \notin \{k, l\}$. Then g is an approximation of f and $g \in A_{kl}^n$.

Therefore, the set $\Gamma^n = \bigcap_{k \neq l} A_{kl}^n$ is dense and G_δ in $C(\bigoplus_{i=1}^\infty \mathbb{B}_i^n, X)$. Consequently, so is the set $\Gamma^n \cap \tilde{A}^n$. Observe that $\Gamma^n \cap \tilde{A}^n$ consists of maps f satisfying the following conditions:

- $f(\mathbb{B}_k^n) \cap f(\mathbb{B}_l^n) = \emptyset$ for all $k \neq l$;
- $f(\bigoplus_{i=k}^\infty \mathbb{B}_i^n)$ is a closed set in X for all $k \geq 1$.

The last two conditions imply that the family $\{f(\mathbb{B}_i^n)\}_{i=1}^\infty$ is discrete in X for all $f \in \Gamma^n \cap \tilde{A}^n$. Hence, X has the discrete n -cells property for every n , and by [7], X has the discrete approximation property. Finally, we apply Toruńczyk's [15] characterization of l_2 -manifolds.

If X is an l_2 -manifold, then for every n there is a dense G_δ -set in $C(\bigoplus_{i=1}^\infty \mathbb{B}_i^n, X)$ consisting of closed embeddings; see [15]. Because every compact subset of an l_2 -manifold is a Z -set, X satisfies conditions (1) and (2) from Theorem 3.1. ■

Since every closed subset of a homological Z_n -set is also a homological Z_n -set, we have the following corollary.

COROLLARY 3.2. *A space X having a nice ANR local compactification is an l_2 -manifold if and only if X has the disjoint disks property and for every n the space $C(\bigoplus_{i=1}^\infty \mathbb{B}_i^n, X)$, equipped with the limitation topology, contains a dense G_δ -set of homological Z_n -maps.*

One can show that condition (2) in Theorem 3.1 can be replaced by the following two conditions:

- (2') for every n the space $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps;
- (2'') for every n the space $C(\bigoplus_{i=1}^\infty \mathbb{B}_i^n, X)$ contains a dense G_δ -set of maps with closed images.

COROLLARY 3.3. *A space X having a nice ANR local compactification is an l_2 -manifold if and only if X has the disjoint disks property and satisfies conditions (2') and (2'').*

Following [11] we say that a subset $A \subset X$ is *almost strongly negligible* if for every open cover \mathcal{U} of X there is a homeomorphism h from X onto $X \setminus A$ that is \mathcal{U} -close to the identity of X . We provide another version of Theorem 3.1.

THEOREM 3.4. *Suppose X has a nice ANR local compactification \overline{X} . Then X is an l_2 -manifold if and only if X satisfies the following conditions:*

- (1) X has the disjoint disks property;
- (2) for every $n \geq 2$ the space $C(\mathbb{B}^n, X)$ contains a dense set of homological Z_n -maps;
- (3) every Z -subset of X is almost strongly negligible.

Proof. It is well known that every l_2 -manifold satisfies the three conditions. For the inverse implication, observe that \overline{X} is a Q -manifold (see the proof of Theorem 3.1). Hence, $C(Q, \overline{X})$ contains a dense set of Z -maps; see [16]. This implies that $C(Q, X)$ contains a dense set \mathcal{Z} of Z -maps because X is the complement of a σZ -set in \overline{X} ; see the proof of Claim 1 in Theorem 3.1. We choose a countable dense subset $\{\varphi_k\}_{k \geq 1}$ of \mathcal{Z} , and consider

the sets

$$\Gamma_k = \left\{ f \in C\left(\bigoplus_{i=1}^{\infty} Q_i, X\right) : \right. \\ \left. \left(f\left(\bigoplus_{i=1}^k Q_i\right) \cup \varphi_k(Q) \right) \cap \overline{f\left(\bigoplus_{i=k+1}^{\infty} Q_i\right)} = \emptyset \right\}, \quad k \geq 1.$$

CLAIM 3. Each Γ_k , $k \geq 1$, is open and dense in $C(\bigoplus_{i=1}^{\infty} Q_i, X)$.

The openness of Γ_k is obvious.

To show the density, choose $f \in C(\bigoplus_{i=1}^{\infty} Q_i, X)$ and an open cover \mathcal{U} of X . We may assume that all sets $f(Q_i)$, $i \leq k$, are Z -sets in X . Since every Z -set in X is a strong Z -set (see [7]), so is the set $f(\bigoplus_{i=1}^k Q_i) \cup \varphi_k(Q)$. Consequently, there is a map $g \in C(\bigoplus_{i=k+1}^{\infty} Q_i, X)$ such that g is \mathcal{U} -close to the restriction map $f|_{\bigoplus_{i=k+1}^{\infty} Q_i}$ and $g(\bigoplus_{i=k+1}^{\infty} Q_i) \cap (f(\bigoplus_{i=1}^k Q_i) \cup \varphi_k(Q)) = \emptyset$. Finally, the map $f' \in C(\bigoplus_{i=1}^{\infty} Q_i, X)$ defined by $f'|_{Q_i} = f|_{Q_i}$ for $i \leq k$ and $f'|_{Q_i} = g|_{Q_i}$ for $i \geq k + 1$ is \mathcal{U} -close to f and $f' \in \Gamma_k$.

Since $C(\bigoplus_{i=1}^{\infty} Q_i, X)$ with the limitation topology is a Baire space, the set $\Gamma = \bigcap_{k \geq 1} \Gamma_k$ is dense in $C(\bigoplus_{i=1}^{\infty} Q_i, X)$. Observe that $f \in \Gamma$ implies that $f(Q_i) \cap f(Q_j) = \emptyset$ for all $i \neq j$ and $A_f = \overline{f(\bigoplus_{i=1}^{\infty} Q_i)} \setminus f(\bigoplus_{i=1}^{\infty} Q_i)$ is a closed set in X disjoint from each $\varphi_k(Q)$, $k \geq 1$. Because $\{\varphi_k\}_{k \geq 1}$ is dense in $C(Q, X)$, A_f are Z -sets of X . We can complete the proof by showing that X has the discrete approximation property. To this end, let \mathcal{U}, \mathcal{V} be open covers of X such that \mathcal{V} is a star-refinement of \mathcal{U} and $f \in C(\bigoplus_{i=1}^{\infty} Q_i, X)$. We first take $g \in \Gamma$ that is \mathcal{V} -close to f . As A_g is a Z -set in X , there is a homeomorphism h from X onto $X \setminus A_g$ that is \mathcal{V} -close to the identity of X . Then $\tilde{f} = h \circ g$ is \mathcal{U} -close to f and $\{\tilde{f}(Q_i)\}_{i \geq 1}$ is a discrete family in X . ■

The next theorem is a homological version of Toruńczyk’s [15] characterization of l_2 -manifolds.

THEOREM 3.5. An ANR-space X is an l_2 -manifold if and only if X has the discrete 2-disks property and $C(\bigoplus_{i=1}^{\infty} Q_i, X)$, equipped with the limitation topology, contains a dense G_δ -set of maps f satisfying the condition

$$(*) \quad f(\bigoplus_{i=1}^{\infty} Q_i) \text{ is closed and each } f(Q_i) \text{ is a homological } Z_\infty\text{-set in } X.$$

Proof. Toruńczyk’s characterization of l_2 -manifolds guarantees that every l_2 -manifold satisfies the hypothesis of Corollary 3.3. For the other implication, we first observe that every compact subset of X is a Z_2 -set because X has the discrete 2-disks property. Thus, every compact homological Z_∞ -subset of X is a Z -set in X . So, condition (*) implies that each $C(Q_i, X)$ contains a dense G_δ -set of Z -maps. Then, proceeding as in the proof of Theorem 3.1, we can show that $C(\bigoplus_{i=1}^{\infty} Q_i, X)$ contains a dense subset Γ such

that $\{f(Q_i)\}_{i \geq 1}$ is a discrete family in X for all $f \in \Gamma$. Therefore, X has the discrete approximation property. ■

COROLLARY 3.6. *An ANR-space X is an l_2 -manifold if and only if X has the discrete 2-cells property and $C(\bigoplus_{i=1}^{\infty} Q_i, X)$ contains a dense G_δ -set of homological Z_∞ -maps.*

Acknowledgements. The authors would like to express their gratitude to J. West for his careful reading and helpful comments.

The first author was partially supported by NSERC Grant 257231-14.

The second author was partially supported by NSERC Grant 261914-19.

References

- [1] T. Banakh, R. Cauty and A. Karassev, *On homotopical and homological Z_n -sets*, Topology Proc. 38 (2011), 29–82.
- [2] T. Banakh and D. Repovš, *Division and k -th root theorems for Q -manifolds*, Sci. China Ser. A 50 (2007), 313–324.
- [3] M. Bestvina, *Characterizing k -dimensional universal Menger compacta*, Mem. Amer. Math. Soc. 71 (1988), no. 380.
- [4] M. Bestvina, P. Bowers, J. Mogilski and J. Walsh, *Characterization of Hilbert space manifolds revisited*, Topology Appl. 24 (1986), 53–69.
- [5] P. Bowers, *Limitation topologies on function spaces*, Trans. Amer. Math. Soc. 314 (1989), 421–431.
- [6] P. Bowers, *Homological characterization of boundary set complements*, Compos. Math. 62 (1987), 63–94.
- [7] P. Bowers, *Nonshrinkable “cell-like” decompositions of s* , Pacific J. Math. 124 (1986), 257–273.
- [8] R. Daverman, *Detecting the disjoint disks property*, Pacific J. Math. 93 (1981), 277–298.
- [9] R. Daverman and D. Halverson, *Path concordances as detectors of codimension-one manifold factors*, in: Exotic Homology Manifolds—Oberwolfach 2003, Geom. Topol. Monogr. 9, Geom. Topol. Publ., Coventry, 2006, 7–15.
- [10] R. Daverman and J. Walsh, *Čech homology characterizations of infinite-dimensional manifolds*, Amer. J. Math. 103 (1981), 411–435.
- [11] J. Dijkstra, *Strong negligibility of σ -compacta does not characterize Hilbert space*, Pacific J. Math. 127 (1987), 19–30.
- [12] T. Lay and J. Walsh, *Characterizing Hilbert cube manifolds by their homological structure*, Topology Appl. 15 (1983), 197–203.
- [13] S. Mardešić, *Comparison of singular and Čech homology in locally connected spaces*, Michigan Math. J. 6 (1959), 151–166.
- [14] S. Singh, *Exotic ANR’s via null decompositions of Hilbert cube manifolds*, Fund. Math. 125 (1985), 175–183.
- [15] H. Toruńczyk, *Characterizing Hilbert space topology*, Fund. Math. 111 (1981), 247–262.
- [16] H. Toruńczyk, *On CE-images of the Hilbert cube and characterization of Q -manifolds*, Fund. Math. 106 (1980), 31–40.

- [17] H. Toruńczyk, *Concerning locally homotopy negligible sets and characterization of l_2 -manifolds*, *Fund. Math.* 101 (1978), 93–110.

Alexandre Karassev, Vesko Valov
Department of Computer Science and Mathematics
Nipissing University
100 College Drive, P.O. Box 5002
North Bay, ON, P1B 8L7, Canada
E-mail: alexandk@nipissingu.ca
veskov@nipissingu.ca

