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SOME CHARACTERIZATIONS OF THE SPACES WITH A LATTICE OF d-OPEN MAPPINGS

V. M. Valov

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Ščepin [5] introduced the class of k-metrizable spaces and proved [4] that for a compact space X the following conditions are equivalent: (i) X is k-metrizable; (ii) X possesses a lattice of open mappings; (iii) X is a limit space of an inverse σ-system with open projections. Shirokov [8] gave an external characterization of k-metrizable compact spaces. In [11] we consider an extension of the class of compact k-metrizable spaces — the class of all completely regular spaces with a lattice of d-open mappings. The present note provides some characterizations of this extended class, similar to those of Ščepin's and Shirokov's mentioned above. We also characterize the class of all completely regular spaces with a lattice of ad-open mappings.

1. Notations. We shall consider only completely regular spaces and continuous mappings.

An embedding j of X in a space Y is called regular [3] if to every open subset U of $j(X)$ we can assign an open subset $e(U)$ of Y such that: (i) $e(U)\cap j(X)=U$; (ii) $e(U_1)\cap e(U_2)\neq\emptyset$ implies $U_1\cap U_2\neq\emptyset$ for each two open subsets U_1 and U_2 of $j(X)$. If in addition $e(\bigcap_{i=1}^{n}U_i)=\bigcap_{i=1}^{n}e(U_i)$ for every finite family $\{U_1,\ldots,U_n\}$ of open subsets of $j(X)$, then j is said to be a d-regular embedding [3].

For two mappings f and g, defined on a given space X, we write $g<f$ if there exists a mapping $h\colon g(X)\to f(X)$ such that $h_0 g=f$.

Let τ be a cardinal number. Following Ščepin [5] we shall state that a family $\mathfrak{L}$ of mappings, defined on X, is a τ-lattice of X if the following conditions are fulfilled:

1. $\{h_s\colon s\in S\}\subset\mathfrak{L}$ and for every finite set $A\subset S$ the diagonal mapping $\mathop{\Delta}\limits_{s\in A} h_s$ belongs to $\mathfrak{L}$, then $\mathop{\Delta}\limits_{s\in S} h_s\in\mathfrak{L}$.

2. For every mapping f, defined on X, there exists a mapping $h\in\mathfrak{L}$ such that $h<f$ and $w(h(X))\leq\tau\,.\,w(f(X))$.

An $\aleph_0$-lattice is called a lattice. Suppose $\mathfrak{L}$ is a τ-lattice of a given space X. $\mathfrak{L}$ is called multiplicative [4] if in the above definition the first condition is replaced by the following one: $\mathop{\Delta}\limits_{h\in\mathfrak{B}} h\in\mathfrak{L}$ for every subfamily $\mathfrak{B}$ of $\mathfrak{L}$. $\mathfrak{L}$ is said to be complete [5] if for every two mappings h and f, defined on X, with $h\in\mathfrak{L}$ and $h<f$ there exists $h'\in\mathfrak{L}$ such that $h<h'<f$ and $w(h'(X))\leq\tau\,.\,w(f(X))$.

A mapping f from X to Y is said to be d-open [9] if $f(U)$ is dense in some open subset of Y for every open set U in X. Let us note that each closed d-open mapping is open [9].

A well ordered inverse system $\{X_\alpha, p_\alpha^\beta\}_{\alpha<\beta<\tau}$ is called almost continuous [10] briefly a. c., if for every limit ordinal $\alpha^*<\tau$ the space X_{α^*} is naturally embedded in $\varprojlim \{X_\alpha, p_\alpha^\beta\}_{\alpha<\beta<\tau^*}$. Next, suppose $S=\{X_\alpha, p_\alpha^\beta\}_{\alpha, \beta \in A}$ is an inverse system and X is dense in $\varprojlim S$. We write $X=a-\varprojlim S$ if for each $\alpha \in A$ we have $p_\alpha(X)=X_\alpha$ where $p_\alpha\colon \varprojlim S \to X_\alpha$ is the natural projection. Now, let $w(X_\alpha)\leq\tau$ for every $\alpha \in A$ and let each increasing transfinite sequence $\{\alpha_\gamma\}_{\gamma\in\Gamma} \subset A$ of cardinality $\leq\tau$ have a supremum α' in A such that $X_{\alpha'}$ is naturally embedded in the space $\varprojlim \{X_{\alpha_\gamma}\colon p_{\alpha_\gamma}^{\alpha_{\gamma'}}\}_{\gamma, \gamma' \in \Gamma}$. Then S is called a τ-system.

2. Spaces with a lattice of *d*-open mappings.

Let $\mathfrak{C}$ be the family of all regular closed subsets of X. By $\lambda(X, \mathfrak{C})$ we denote the superextension of X with respect to $\mathfrak{C}$ (see [8]). If X is perfectly k-normal in the sense of Ščepin [2] the space $\lambda(X, \mathfrak{C})$ is Hausdorff. It is easily seen that $\lambda(X, \mathfrak{C})$ is homeomorphic to $\lambda(X)$ for every normal space X. Here $\lambda(X)$ is the superextension of X [7]. Since X can be embedded in $\lambda(X, \mathfrak{C})$, we always consider X as a subset of $\lambda(X, \mathfrak{C})$.

Let X be a subset of Y. We say that X is τ-embedded in Y if for every mapping f from X to a compact space Z with $w(Z)\leq\tau$ there is an upper semi-continuous mapping $\varphi\colon Y\to Z$ such that $\varphi(x)=f(x)$ for every $x\in X$.

Proposition 2.1. Every C^*-embedded subset of a given space Y is $\aleph_0$-embedded in Y.

Theorem 2.1. For a space X the following conditions are equivalent:

(i) X possesses a (complete) τ-lattice of d-open mappings;

(ii) every τ-embedding of X in any space is regular;

(iii) if X is a τ-embedded subset of Y there exists an upper semi-continuous mapping r from Y to $\lambda(X, \mathfrak{C})$ such that $r(x)=\{x\}$ for every $x\in X$;

(iv) $X=a-\varprojlim S$, where S is a τ-system with d-open projections.

In the case $\tau=\aleph_0$ and X is compact, the implications (i)$\leftrightarrow$(iv) and (i)$\leftrightarrow$(ii) were proved by Ščepin [4] and Shirokov [3] respectively.

In the proof of Theorem 2.1 we use the following propositions

Proposition 2.2. Let $S=\{X_\alpha, p_\alpha^\beta\}_{\alpha<\beta<\tau}$ be an a. c. inverse system with d-open projections and $X=a-\varprojlim S$. Then the family of all open subsets U of X with $|k(U)|<\aleph_0$ is a base for X, where $k(U)=\{\alpha\colon (p_\alpha^{\alpha+1})^{-1}(\overline{p_\alpha(U)})\setminus\overline{p_{\alpha+1}(U)}\neq\emptyset\}$.

Proposition 2.2 is an analogue of Lemma 2 from [5], chapter 3 and its proof is similar to the proof of this lemma.

Proposition 2.3. Let S and X be the same as in Proposition 2.2. Then the natural embedding of X in $\Pi\{X_\alpha\colon \alpha<\tau\}$ is regular.

Below, by $\mathfrak{R}(\tau)$ we denote the class of all spaces with a τ-lattice of d-open mappings. From Theorem 2.1 and Proposition 2.3 we obtain the following results:

Corollary 2.1. The class $\mathfrak{R}(\tau)$ is multiplicative.

Corollary 2.2. Suppose $X\in\mathfrak{R}(\tau)$ and A is regular closed or a regularly embedded subset of X. Then $A\in\mathfrak{R}(\tau)$.

Since every space X is regularly embedded in $\exp X$, it follows from Corollary 2.2 that $X\in\mathfrak{R}(\tau)$ if $\exp X\in\mathfrak{R}(\tau)$. Here $\exp X$ stands for the space of all non-void compact subsets of X endowed with the Vietoris topology.

Corollary 2.3. If $X\in\mathfrak{R}(\tau)$ then $\exp X\in\mathfrak{R}(\tau)$.

Corollary 2.4. Let $X=a-\varprojlim S$ where $S=\{X_\alpha, p_\alpha^\beta\}_{\alpha<\beta<\tau^*}$ is an a. c. inverse system with d-open projections and $X_\alpha\in\mathfrak{R}(\tau)$ for every $\alpha<\tau^*$. Then $X\in\mathfrak{R}(\tau)$.

By $G_\tau(X)$ we denote the space $\{x\in\beta X\colon$every G_τ-subset of βX containing x meets $X\}$. Clearly, $G_{\aleph_0}(X)$ is the Hewitt realcompactification of X. The space X is called τ-realcompact if $X=G_\tau(X)$.

Corollary 2.5. (i) If $X\in\mathfrak{R}(\tau)$ then $G_\tau(X)\in\mathfrak{R}(\tau)$.

(ii) a space X is a limit of a τ-system with d-open projections iff $X\in\mathfrak{R}(\tau)$ and X is τ-realcompact.

Theorem 2.2. Let $X\in\mathfrak{R}(\tau)$ and let Y be a perfect open image of X. Then $Y\in\mathfrak{R}(\tau)$.

Ivanov [1] proved that a compact space X is k-metrizable iff $\lambda(X)$ is a Dugundji space. On the other hand Ščepin [4] characterized Dugundji spaces as spaces with a multiplicative lattice of open mappings. Thus the following theorem is an analogue of Ivanov's result mentioned above.

Theorem 2.3. A compact space $X\in\mathfrak{R}(\tau)$ iff $\lambda(X)$ possesses a multiplicative τ-lattice of open mappings.

Dimov [6] characterized the class of all spaces which have a k-metrizable compactification and proved that this class is multiplicative and hereditary with respect to dense and to regular closed subsets. By Corollary 2.2. $\mathfrak{R}(\aleph_0)$ includes Dimov's class.

Question 1. Does $\mathfrak{R}(\aleph_0)$ coincide with the class of all spaces having a k-metrizable compactification?

Question 2. Is it true that X possesses a multiplicative lattice of d-open mappings iff every $\aleph_0$-embedding of X is d-regular?

3. Spaces with a lattice of *ad*-open mappings

A mapping f from X to Y is called *ad*-open if $\overline{f(U)}$ is regular closed subset of Y for every open set U in X.

Proposition 3.1. For a mapping $f\colon X\to Y$ the following conditions are equivalent:

(i) f is *ad*-open;

(ii) every open subset U of X contains an open and dense subset $V(U)$ such that $f(V(U))$ is dense in some open subset of Y;

(iii) X has a π-base $\mathfrak{B}$ such that $f(U)$ is dense in some open subset of Y for every $U\in\mathfrak{B}$.

If f is closed the above three conditions are equivalent with the following one:

(iv) $\operatorname{Int}(f(U))\neq\emptyset$ for every open U in X.

An embedding j of X in a space Y is called π-regular if to every open subset U of $j(X)$ we can assign an open subset $e(U)$ of Y such that: (i) $e(U)\cap j(X)$ is dense in U; (ii) $e(U_1)\cap e(U_2)=\emptyset$ iff $U_1\cap U_2=\emptyset$ for each two open subsets U_1 and U_2 of $j(X)$.

Theorem 3.1. For a space X the following conditions are equivalent:

(i) X possesses a (complete) τ-lattice of *ad*-open mappings;

(ii) every τ-embedding of X in any space is π-regular;

(iii) $X=a-\varprojlim S$, where S is a τ-system with *ad*-open projections.

Let $S=\{X_\alpha, p_\alpha^\beta\}_{\alpha<\beta<\tau}$ be an a.c. inverse system with *ad*-open projections and $X=a-\varprojlim S$. For every open subset U of X the set $q(U)=\{\alpha\colon \operatorname{Int}((p_\alpha^{\alpha+1})^{-1}\overline{(p_\alpha(U))}\setminus\overline{p_{\alpha+1}(U)})\neq\emptyset\}$ is called a rank of U.

In the proof of Theorem 3.1 we use the following propositions:

Proposition 3.2. Let $X=a-\varprojlim S$, where $S=\{X_\alpha, p_\alpha^\beta\}_{\alpha<\beta<\tau^*}$ is an a.c. inverse system with *ad*-open projections. Then the family of all open subsets of X having a finite rank is a π-base for X.

Proposition 3.3. Let S and X be the same as in Proposition 3.2. Then the natural embedding of X in $\Pi\{X_\alpha\colon \alpha<\tau^*\}$ is π-regular.

Denote by $\mathfrak{Q}(\tau)$ the class of all spaces with a τ-lattice of *ad*-open mappings. From Theorem 3.1 and Proposition 3.3 we get:

Corollary 3.1. The class $\mathfrak{Q}(\tau)$ is multiplicative.

Corollary 3.2. The class $\mathfrak{Q}(\tau)$ is hereditary with respect to dense, to open and to regular closed subsets.

Corollary 3.3 If $X\in\mathfrak{Q}(\tau)$ then $\exp X\in\mathfrak{Q}(\tau)$ and $\beta X\in\mathfrak{Q}(\tau)$.

Corollary 3.4. Suppose $X=a-\underleftarrow{\lim}\,S$, where $S=\{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau^*}$ is an a. c. inverse systen with *ad*-open projections and $X_\alpha\in\mathfrak{Q}(\tau)$ for every $\alpha<\tau^*$. Then $X\in\mathfrak{Q}(\tau)$.

Obviously, $\mathfrak{R}(\tau)\subset\mathfrak{Q}(\tau)$ for every τ. On the other hand if $\beta X\in\mathfrak{R}(\tau)$ then X i τ-pseudocompact [12] i. e. every continuous image of X with a weight $\leq\tau$ is compact Hence, by Corollary 3.3, for every τ there exists a space $X_\tau\in\mathfrak{Q}(\tau)\setminus\mathfrak{R}(\tau)$.

Theorem 3.2. Let $X\in\mathfrak{Q}(\tau)$ and Y be a perfect open image of X. Then $Y\in\mathfrak{Q}(\tau)$.

Theorem 3.3. A compact space X belongs to $\mathfrak{Q}(\tau)$ iff $\lambda(X)$ possesses a multiplicative τ-lattice of *ad*-open mappings.

Institute of Mathematics
Bulgarian Academy of Science
Sofia, POB 373, Bulgaria

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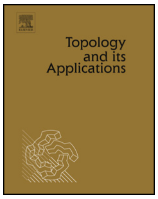

On homogeneity of $\mathbb{N}^\tau$ [☆]

A. Karassev [a], E. Shchepin [b], V. Valov [a,*]

[a] *Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, P.O. Box 5002, North Bay, ON, P1B 8L7, Canada*
[b] *Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina St. Moscow, 119991, Russia*

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ABSTRACT

It is shown that any homeomorphism between two compact subsets of $\mathbb{N}^\tau$ can be extended to an autohomeomorphism of $\mathbb{N}^\tau$.

1. Introduction

J. Pollard [4] established the following theorem: *Let X and Y be complete, nowhere locally compact, zero-dimensional separable metric spaces, and let P and K be closed nowhere dense subsets of X and Y, respectively. If f is a homeomorphism between P and K, then there exists a homeomorphism between X and Y extending f.*

Pollard's result is not anymore true for uncountable powers of the irrationals. For example, if P and K are two homeomorphic closed nowhere dense subsets of $\mathbb{N}^{\aleph_1}$ such that only one of them is G_δ, then no homeomorphism between P and K admits an extension to an autohomeomorphism on $\mathbb{N}^{\aleph_1}$. But a non-metrizable analogue of Pollard's theorem remains valid for compact subsets of $\mathbb{N}^\tau$. The technique developed

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[*] Corresponding author.
E-mail addresses: alexandk@nipissingu.ca (A. Karassev), scepin@yandex.ru (E. Shchepin), veskov@nipissingu.ca (V. Valov).

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in recent papers [5] and [6] allows to prove the following theorem, where $\pi_1 : X \times \mathbb{N}^\tau \to X$ is the projection (by a 0-dimensional space we mean a Tychonoff space having a compactification of covering dimension 0).

Theorem 1.1. *Let X be a 0-dimensional Tychonoff space and P and K be compact subsets of $X \times \mathbb{N}^\tau$ with $\tau \geq \aleph_0$. Then every homeomorphism $f : P \to K$ with $\pi_1 \circ f = \pi_1|P$ can be extended to a homeomorphism $\widetilde{f} : X \times \mathbb{N}^\tau \to X \times \mathbb{N}^\tau$ such that $\pi_1 \circ \widetilde{f} = \pi_1$.*

Theorem 1.1 implies a non-metrizable analogue of mentioned above Pollard's result [4].

Corollary 1.2. *Let P and K be compact subsets of $\mathbb{N}^\tau$ with $\tau \geq \aleph_0$. Then every homeomorphism between P and K can be extended to a homeomorphism of $\mathbb{N}^\tau$.*

2. Proof of Theorem 1.1

For any space X let $\mathcal{H}(X)$ denote the set of all autohomeomorphisms of X. If d be a bounded complete metric on $\mathbb{N}^{\aleph_0}$ we equip $\mathcal{H}(\mathbb{N}^{\aleph_0})$ with the metric $\widetilde{d} = \hat{d}(f,g) + \hat{d}(f^{-1}, g^{-1})$, where $\hat{d}(f,g) = \sup\{d(f(x), g(x)) : x \in \mathbb{N}^{\aleph_0}\}$. It is well known that $\widetilde{d}$ is a complete metric on $\mathcal{H}(\mathbb{N}^{\aleph_0})$.

Next lemma is an analogue of [5, Lemma 3.1].

Lemma 2.1. *Let X be a 0-dimensional space and P, K be compact subsets of $X \times \mathbb{N}^{\aleph_0}$. If $f : P \to K$ and $g \in \mathcal{H}(X)$ are homeomorphisms with $g \circ \pi_X = \pi_X \circ f$, then f can be extended to a homeomorphism $\widetilde{f} \in \mathcal{H}(X \times \mathbb{N}^{\aleph_0})$ such that $g \circ \pi_X = \pi_X \circ \widetilde{f}$.*

Proof. Obviously, $g(P_X) = K_X$, where $P_X = \pi_X(P)$ and $K_X = \pi_X(K)$. Denote by $\pi : X \times \mathbb{N}^{\aleph_0} \to \mathbb{N}^{\aleph_0}$ the projection. For any $x \in P_X$ let $\Phi(x)$ be the set of all $h \in \mathcal{H}(\mathbb{N}^{\aleph_0})$ such that $f(x,c) = (g(x), h(c))$ for every $c \in \pi_X^{-1}(x) \cap P$. Since $f|(\pi_X^{-1}(x) \cap P)$ is a homeomorphism between the compact subsets $\pi((\{x\} \times \mathbb{N}^{\aleph_0}) \cap P)$ and $\pi((\{g(x)\} \times \mathbb{N}^{\aleph_0}) \cap K)$ of $\mathbb{N}^{\aleph_0}$, the Pollard's theorem [4] cited above yields a homeomorphism $h_x \in \mathcal{H}(\mathbb{N}^{\aleph_0})$ extending $f|(\pi_X^{-1}(x) \cap P)$. Hence, $\Phi(x) \neq \varnothing$ for all $x \in P_X$. Moreover, the sets $\Phi(x)$ are closed in $\mathcal{H}(\mathbb{N}^{\aleph_0})$ equipped with the metric $\widetilde{d}$. So, we have a set-valued map $\Phi : P_X \rightsquigarrow \mathcal{H}(\mathbb{N}^{\aleph_0})$. One can show that if Φ admits a continuous selection $\phi : P_X \to \mathcal{H}(\mathbb{N}^{\aleph_0})$, then the map $f_1 : P_X \times \mathbb{N}^{\aleph_0} \to K_X \times \mathbb{N}^{\aleph_0}$, defined by $f_1(x,c) = (g(x), \phi(x)(c))$, is a homeomorphism between $P_X \times \mathbb{N}^{\aleph_0}$ and $K_X \times \mathbb{N}^{\aleph_0}$ extending f (see [2, Proposition 2.6.11]) with $\pi_X \circ f_1 = g \circ \pi_X$. On the other hand, since X is 0-dimensional and P_X is a compact subset of X, the map ϕ has a continuous extension $\widetilde{\phi} : X \to \mathcal{H}(\mathbb{N}^{\aleph_0})$. This is true because $\mathcal{H}(\mathbb{N}^{\aleph_0})$ is a metric space. Indeed, then $\phi(P_X)$ is a compact subset of $\mathcal{H}(\mathbb{N}^{\aleph_0})$, hence $\phi(P_X)$ is itself a compact metric space. So, it is an absolute extensor for 0-dimensional spaces in the sense of Chigogidze [1, Definition 6.1.3], and we can extend ϕ to a map $\widetilde{\phi} : X \to \mathcal{H}(\mathbb{N}^{\aleph_0})$. Next, let $\widetilde{f} : X \times \mathbb{N}^{\aleph_0} \to X \times \mathbb{N}^{\aleph_0}$ be the map defined by $\widetilde{f}(x,c) = (g(x), \widetilde{\phi}(x)(c))$. Then $\widetilde{f}$ is a homeomorphism extending f. Therefore, according to Michael's [3] zero-dimensional selection theorem, it suffices to show that Φ is lower semi-continuous.

To prove that, let $x^* \in P_X$ be a fixed point and $h^* \in \Phi(x^*) \cap W$, where W is open in $\mathcal{H}(\mathbb{N}^{\aleph_0})$. We can assume that W is of the form $\{h \in \mathcal{H}(\mathbb{N}^{\aleph_0}) : \{x^*\} \times h(U_i) = \{g(x^*)\} \times V_i \ , i = 1, 2, ..\}$, where $\{U_i\}_{i=1}^\infty$ and $\{V_i\}_{i=1}^\infty$ are clopen disjoint countable covers of $\mathbb{N}^{\aleph_0}$. Because $P(x^*) = (\{x^*\} \times \mathbb{N}^{\aleph_0}) \cap P$ and $K(g(x^*)) = (\{g(x^*)\} \times \mathbb{N}^{\aleph_0}) \cap K$ are compact, there is k such that $(\{x^*\} \times U_i) \cap P(x^*) \neq \varnothing$ and $(\{g(x^*)\} \times V_i) \cap K(g(x^*)) \neq \varnothing$ if and only if $i \leq k$.

We extend each of the sets $\{x^*\} \times U_i$ and $\{g(x^*)\} \times V_i$, $i \leq k$, to clopen sets $\widetilde{U}_i \subset P_X \times \mathbb{N}^{\aleph_0}$ and $\widetilde{V}_i \subset K_X \times \mathbb{N}^{\aleph_0}$ such that

(1) $\widetilde{U}_i = O(x^*) \times U_i$ and $\widetilde{V}_i = g(O(x^*)) \times V_i$, where $O(x^*)$ is a clopen neighborhood of x^* in P_X such that $P(x) \subset \bigcup_{i=1}^k \widetilde{U}_i$ for all $x \in O(x^*)$;

(2) $O(x^*)$ is so small that $f(\widetilde{U}_i \cap P) \subset \widetilde{V}_i \cap K$.

We are going to show that for every $x \in O(x^*)$ there exists $h_x \in \Phi(x) \cap W$. We fix such x and observe that all sets $\widetilde{U}_i(x) = \widetilde{U}_i \cap (\{x\} \times \mathbb{N}^{\aleph_0})$ and $\widetilde{V}_i(x) = \widetilde{V}_i \cap (\{g(x)\} \times \mathbb{N}^{\aleph_0})$ are nowhere locally compact and complete. Moreover, $\widetilde{U}_i(x) \cap P$ and $\widetilde{V}_i(x) \cap K$ are compact sets in $\widetilde{U}_i(x)$ and $\widetilde{V}_i(x)$, respectively, and $f_i^x = f|(\widetilde{U}_i(x) \cap P)$ is a homeomorphism between them for every $i \leq k$. Hence, by Pollard's theorem [4], there exist homeomorphisms $\widetilde{f}_i^x : \widetilde{U}_i(x) \to \widetilde{V}_i(x)$ extending f_i^x, $i \leq k$. For every $(x, c) \notin \bigcup_{i=1}^k \widetilde{U}_i(x)$ there is exactly one $i > k$ with $c \in U_i$, and we define $\widetilde{f}_i^x(x, c) = h^*(x^*, c)$. The homeomorphisms $\widetilde{f}_i^x$, $i = 1, 2, ..$, provide a homeomorphism h'_x between $\pi_X^{-1}(x)$ and $\pi_X^{-1}(g(x))$ extending $f|\pi_X^{-1}(x) \cap P$. Then the equality $h_x(c) = h'_x(x, c)$, $c \in \mathbb{N}^{\aleph_0}$, defines a homeomorphism from $\mathcal{H}(\mathbb{N}^{\aleph_0})$ with $h_x \in \Phi(x) \cap W$. Therefore, Φ is lower semi-continuous. □

Everywhere below we suppose that P, K are compact subsets of $X \times \mathbb{N}^A$ and $f : P \to K$ is a homeomorphism such that $\pi_1|P = \pi_1 \circ f$, where X is a 0-dimensional space. A set $B \subset A$ is called *f-admissible* if there exists a homeomorphism $f_B : P_B \to K_B$ such that $(f_B \circ p_B)|P = p_B \circ f$ and $q_B|P_B = q_B \circ f_B$, where $\pi_B : \mathbb{N}^A \to \mathbb{N}^B$, $p_B : Y \times \mathbb{N}^A \to X \times \mathbb{N}^B$, and $q_B : X \times \mathbb{N}^B \to X$ denote the projections, $P_B = p_B(P)$ and $K_B = p_B(K)$.

Lemma 2.2. *If A is an uncountable set, then for every $\alpha \in A$ there is an f-admissible countable set $B(\alpha) \subset A$ containing α*

Proof. Since $\pi_1|P = \pi_1 \circ f$, $f(y, x) = (y, h_1(y, x))$ for all $(y, x) \in P$, where $h_1 : P \to \mathbb{N}^A$. Similarly, $f^{-1}(y, x) = (y, h_2(y, x))$ for all $(y, x) \in K$ with $h_2 : K \to \mathbb{N}^A$.

Claim 2.3. For every $i = 1, 2$ and a countable set $C \subset A$, there is countable $D(i) \subset A$ containing C and a continuous maps $g_1 : P_{D(1)} \to \mathbb{N}^C$, $g_2 : K_{D(2)} \to \mathbb{N}^C$ with $(g_1 \circ p_{D(1)})|P = \pi_C \circ h_1$ and $(g_2 \circ p_{D(2)})|K = \pi_C \circ h_2$.

Let $\mathcal{B}$ be a countable base for $\mathbb{N}^C$. Then $G_U = h_1^{-1}(\pi_C^{-1}(U))$ is a σ-compact open subset of P for every $U \in \mathcal{B}$. So, there is a sequence $\{W_U(n)\}_{n \geq 1}$ of standard open sets in $X \times \mathbb{N}^A$ such that G_U is the union of all $W_U(n) \cap P$, $n \geq 1$. Therefore, for every U there exists a countable set $C_U \subset A$ with $p_{C_U}^{-1}(p_{C_U}(W_U(n))) = W_U(n)$, $n \geq 1$. We can assume that each C_U contains C. Then $D(1) = \bigcup_{U \in \mathcal{B}} C_U$ is a countable set containing C and $p_{D(1)}(y, x) = p_{D(1)}(y', x')$ implies $\pi_C(h_1(y, x)) = \pi_C(h_1(y', x'))$, where $(y, x), (y', x') \in P$. Because $P_{D(1)}$ is compact, this yields the existence of a map $g_1 : P_{D(1)} \to \mathbb{N}^C$ with $(g_1 \circ p_{D(1)})|P = \pi_C \circ h_1$. Similarly, we can find a countable set $D(2) \subset A$ which contains C and a map $g_2 : K_{D(2)} \to \mathbb{N}^C$ satisfying the claim.

Using Claim 2.3, we construct by induction an increasing sequence $\{B(n)\}_{n \geq 0}$ of countable sets $B(n) \subset A$ and maps $\varphi_n : P_{B(n+1)} \to \mathbb{N}^{B(n)}$ for $n = 2k$ and $\psi_n : K_{B(n+1)} \to \mathbb{N}^{B(n)}$ for $n = 2k + 1$ such that

- $B(0) = \{\alpha\}$;
- $\pi_{B(n)} \circ h_1 = (\varphi_n \circ p_{B(n+1)})|P$ if $n = 2k$;
- $\pi_{B(n)} \circ h_2 = (\psi_n \circ p_{B(n+1)})|K$ if $n = 2k + 1$.

Let $B(\alpha) = \bigcup_{n=0}^{\infty} B(n)$. Then we have maps $\varphi_{B(\alpha)} : P_{B(\alpha)} \to \mathbb{N}^{B(\alpha)}$ and $\psi_{B(\alpha)} : K_{B(\alpha)} \to \mathbb{N}^{B(\alpha)}$ such that $\pi_{B(\alpha)} \circ h_1 = (\varphi_{B(\alpha)} \circ p_{B(\alpha)})|P$ and $\pi_{B(\alpha)} \circ h_2 = (\psi_{B(\alpha)} \circ p_{B(\alpha)})|K$. Observe that $P_{B(\alpha)}$ and $K_{B(\alpha)}$ are subsets of $X \times \mathbb{N}^{B(\alpha)}$. Then $f_{B(\alpha)} : P_{B(\alpha)} \to X \times \mathbb{N}^{B(\alpha)}$, defined by $f_{B(\alpha)}(y, x) = (y, \varphi_{B(\alpha)}(y, x))$ for every $(y, x) \in P_{B(\alpha)}$, is a homeomorphism between $P_{B(\alpha)}$ and $K_{B(\alpha)}$ whose inverse is the map $g_{B(\alpha)} : K_{B(\alpha)} \to X \times \mathbb{N}^{B(\alpha)}$, defined by $g_{B(\alpha)}(y, x) = (y, \psi_{B(\alpha)}(y, x))$ for $(y, x) \in K_{B(\alpha)}$. Therefore, $B(\alpha)$ is f-admissible. □

Proof of Theorem 1.1. We identify $\mathbb{N}^\tau$ with $\mathbb{N}^A$, where A is a set of cardinality τ. The case $\tau = \aleph_0$ follows from Lemma 2.1. So, let $A = \{\alpha : \alpha < \omega(\tau)\}$ be uncountable. By Lemma 2.2, we can cover A by a family $\{B(\alpha) : \alpha < \omega(\tau)\}$ of countable f-admissible sets. Since any union of f-admissible sets is also f-admissible, from the family $\{B(\alpha) : \alpha < \omega(\tau)\}$ we obtain an increasing family of f-admissible sets $A(\alpha)$ and homeomorphisms $f_{A(\alpha)} : P_{A(\alpha)} \to K_{A(\alpha)}$ such that:

(3) $A(1)$ is countable, and the cardinality of each $A(\alpha)$ is less than τ;
(4) $A(\alpha) = \bigcup_{\beta<\alpha} A(\beta)$ if α is a limit ordinal;
(5) $A(\alpha+1)\backslash A(\alpha)$ is countable but infinite for all α;
(6) $p_{A(\alpha)} \circ f = (f_{A(\alpha)} \circ p_{A(\alpha)})|P$.

We need to prove that each $f_{A(\alpha)}$ can be extended to a homeomorphism $\widetilde{f}_{A(\alpha)} : X \times \mathbb{N}^{A(\alpha)} \to X \times \mathbb{N}^{A(\alpha)}$ such that

(7) $p^{A(\alpha+1)}_{A(\alpha)} \circ \widetilde{f}_{A(\alpha+1)} = (\widetilde{f}_{A(\alpha)} \circ p^{A(\alpha+1)}_{A(\alpha)})$;
(8) $q_{A(\alpha)} = q_{A(\alpha)} \circ \widetilde{f}_{A(\alpha)}$, where $q_{A(\alpha)} : X \times \mathbb{N}^{A(\alpha)} \to X$ denotes the projection.

The proof is by transfinite induction. The first extension $\widetilde{f}_{A(1)}$ exists by Lemma 2.1 because $P_{A(1)}$ and $K_{A(1)}$ are compact subsets of $X \times \mathbb{N}^{A(1)}$ and $q_{A(1)}|P_{A(1)} = q_{A(1)} \circ f_{A(1)}$. If β is a limit ordinal and $\widetilde{f}_{A(\alpha)}$ is already defined for all $\alpha < \beta$, then item (4) implies the existence of $\widetilde{f}_{A(\beta)}$. Therefore, we need only to define $\widetilde{f}_{A(\alpha+1)}$ provided $\widetilde{f}_{A(\alpha)}$ exists.

To this end, we apply again Lemma 2.1 for the space $X \times \mathbb{N}^{A(\alpha+1)} = X \times \mathbb{N}^{A(\alpha)} \times \mathbb{N}^{A(\alpha+1)\backslash A(\alpha)}$, the sets $P_{A(\alpha+1)}, K_{A(\alpha+1)}$, the projection $\pi : X \times \mathbb{N}^{A(\alpha)} \times \mathbb{N}^{A(\alpha+1)\backslash A(\alpha)} \to X \times \mathbb{N}^{A(\alpha)}$, and the homeomorphisms $f_{A(\alpha+1)}$ and $\widetilde{f}_{A(\alpha)}$. Moreover, $X \times \mathbb{N}^{A(\alpha)}$ has a 0-dimensional compactification because both X and $N^{A(\alpha)}$ have such compactifications. Hence, there is a homeomorphism $\widetilde{f}_{A(\alpha+1)} \in \mathcal{H}(X \times \mathbb{N}^{A(\alpha+1)})$ extending $f_{A(\alpha+1)}$ and satisfying condition (7). Since $q_{A(\alpha)} = q_{A(\alpha)} \circ \widetilde{f}_{A(\alpha)}$, we also have $q_{A(\alpha+1)} = q_{A(\alpha+1)} \circ \widetilde{f}_{A(\alpha+1)}$. □

Proof of Corollary 1.2. The corollary is obtained from Theorem 1.1 by letting X to be the one-point space. □

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