



On homogeneity of Cantor cubes

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Abstract. We discuss the question of extending homeomorphisms between closed subsets of the Cantor cube D^τ . It is established that any homeomorphism between two closed negligible subsets of D^τ can be extended to an autohomeomorphism of D^τ .

1 Introduction

Knaster and Reichbach [5] established the following theorem, which is considered as a classical result (see also [10] for other types of zero-dimensional separable metric spaces where similar results hold): *Let X and Y be compact, perfect zero-dimensional metric spaces, and let P and K be closed nowhere dense subsets of X and Y , respectively. If f is a homeomorphism between P and K , then there exists a homeomorphism between X and Y extending f .*

If we omit the metrizability condition in Knaster–Reichbach’s theorem, then the conclusion is not anymore true. In order to obtain a correct generalization of the theorem, first of all, it is necessary to find the correct analogue of the condition “nowhere dense.” Moreover, the perfectness condition can be formulated as the nowhere density of the points.

Such an analogue is the following concept of negligibility. A subset of a topological space is called *negligible* if it does not contain a nonempty intersection of a family of open sets such that the cardinality of the family is less than the weight of the space. Note that for metric compacta, the condition of nowhere density is equivalent to the condition of negligibility.

Now we are able to provide a generalization of Knaster–Reichbach’s theorem.

Theorem 1.1 *Let X and Y be compact, zero-dimensional absolute extensors for zero-dimensional spaces of the same weight with negligible points, and let P and K be closed negligible subsets of X and Y , respectively. If f is a homeomorphism between P and K , then there exists a homeomorphism between X and Y extending f .*

This theorem for metric compacta turns into Knaster–Reichbach’s theorem because every metric compact space is an absolute extensor in dimension 0. In general, it is very difficult to avoid the condition of being an absolute extensor in dimension 0 because extending of homeomorphisms is based on the extension of continuous maps.

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Since the negligibility of a point in a compactum is equivalent to having a character at that point equal to the weight of the compactum, Theorem 1 from [11] allows us to assert that the compacta X and Y in Theorem 1.1 are homeomorphic to the Cantor cube D^τ , where τ is the weight of X and Y . Therefore, the above theorem can be obtained from the following special case of its own.

Theorem 1.2 *Let f be a homeomorphism between closed negligible subsets P and K of D^τ . Then f can be extended to a homeomorphism of D^τ .*

Since every subset of D^τ having weight less than τ is negligible, we have the following corollary.

Corollary 1.3 *If P, K are closed subsets of D^τ both of weight $< \tau$, then every homeomorphism between P and K can be extended to homeomorphism of D^τ .*

Similar results for extending homeomorphisms between subsets of the Tychonoff cube \mathbb{I}^τ were established by Chigogidze [2, Corollary 4.10] and Mednikov [7]. Our result is simpler and does not follow from them. Our proof is based on Michael's zero-dimensional selection theorem [9].

2 Some preliminary results

Anywhere below, by a homeomorphism, we always mean a surjective homeomorphism. We need a more precise notion of negligibility. For a space X , a subset $P \subset X$, and an infinite cardinal λ , we denote by $P^{(\lambda)}$ the λ -interior of P in X , i.e., the set all $x \in P$ such that there exists a G_λ -subset K of X with $x \in K \subset P$. If λ is finite, then $P^{(\lambda)}$ is defined to be the ordinary interior of P and it is denoted by $P^{(0)}$. If there exists $\tau \geq \aleph_0$ such that $P^{(\lambda)}$ is empty for all $\lambda < \tau$, we say that P is τ -negligible in X . Let $X = \prod_{\alpha \in A} X_\alpha$ be a product of spaces and $B \subset A$. If $P \subset X$, then P_B denotes $\pi_B(P)$, where $\pi_B : X \rightarrow \prod_{\alpha \in B} X_\alpha$ is the projection.

Proposition 2.1 *Let $X = \prod_{\alpha \in A} X_\alpha$ be a product of separable metric spaces, let P be a compact subset of X , and let $f : P \rightarrow P$ be a homeomorphism. Then, for any countable set $C \subset A$, there are a countable set $B \subset A$ and a homeomorphism $f_B : P_B \rightarrow P_B$ such that $C \subset B$ and $\pi_B \circ f = f_B \circ \pi_B$.*

Proof Obviously, this is true for a countable set A , so we suppose that A is uncountable. Let f^{-1} be the inverse of f . Using that P is C -embedded in X and any continuous function on X depends on countably many coordinates (see [3, 8]), we construct by induction sequences of countable sets $B(n) \subset A$ and maps $f_{B(2n-1)} : P_{B(2n)} \rightarrow P_{B(2n-1)}$ and $g_{B(2n)} : P_{B(2n+1)} \rightarrow P_{B(2n)}$ such that:

- $B(1) = C, B(n) \subset B(n+1)$.
- $\pi_{B(2n-1)} \circ f = f_{B(2n-1)} \circ \pi_{B(2n)}$.
- $\pi_{B(2n)} \circ f^{-1} = g_{B(2n)} \circ \pi_{B(2n+1)}$.

Then $B = \bigcup_{n=1}^\infty B(n)$ is countable and the equality $\pi_B(x) = \pi_B(y)$ implies $\pi_B(f(x)) = \pi_B(f(y))$ and $\pi_B(f^{-1}(x)) = \pi_B(f^{-1}(y))$ for all $x, y \in P$. Since P_B is compact, there exist maps $f_B : P_B \rightarrow P_B$ and $g_B : P_B \rightarrow P_B$ with $\pi_B \circ f = f_B \circ \pi_B, \pi_B \circ f^{-1} = g_B \circ \pi_B$, and $f_B \circ g_B$ is the identity on P_B . Then f_B is an autohomeomorphism of P_B . ■

In the situation of Proposition 2.1, a subset $B \subset A$ is called *f-admissible* if there exists a homeomorphism $f_B : P_B \rightarrow P_B$ with $\pi_B \circ f = f_B \circ \pi_B$. It is easily seen that arbitrary union of *f*-admissible sets is also *f*-admissible.

In [6], τ -negligible sets with $\tau > \aleph_0$ were considered under the name \tilde{G}_τ -sets. By [6, Lemma 6], if X is a product of metric compacta and $\tau > \aleph_0$, then a closed set $F \subset X$ is τ -negligible in X if and only if the π -character $\pi\chi(F, X)$ of F in X is $\geq \tau$. Recall that $\pi\chi(F, X)$ is the smallest cardinality λ such that there is an open family \mathcal{U} in X of cardinality λ with the following property: Every neighborhood of F in X contains an element of \mathcal{U} .

The next lemma is a modification of [6, Theorem 2].

Lemma 2.2 *Let $X = \prod_{\alpha \in A} X_\alpha$ be a product of compact metric spaces, and let P be a closed set in X . Suppose that $\tau > \aleph_0$ and $C \subset A$ is a set of cardinality $< \tau$ such that $(\{z\} \times X_{A \setminus C}) \cap P$ is τ -negligible in $\{z\} \times X_{A \setminus C}$ for every $z \in P_C$. Then $P_{A \setminus C} \neq X_{A \setminus C}$.*

Proof Since $(\{z\} \times X_{A \setminus C}) \cap P$ is τ -negligible in $\{z\} \times X_{A \setminus C}$ for every $z \in P_C$, the cardinality of $A \setminus C$ is at least τ . Suppose that $P_{A \setminus C} = X_{A \setminus C}$. Passing to a subset of P , we may assume that the projection $\pi_{A \setminus C}$ restricted to P is an irreducible map onto $X_{A \setminus C}$. Denote this map by f and fix $z \in P_C$. Because f is irreducible, we have

$$\pi\chi(f((\{z\} \times X_{A \setminus C}) \cap P), X_{A \setminus C}) \leq \pi\chi((\{z\} \times X_{A \setminus C}) \cap P, P).$$

On the other hand, $\pi\chi((\{z\} \times X_{A \setminus C}) \cap P, P) \leq \pi\chi(z, P_C) < \tau$. So, $\pi\chi(f((\{z\} \times X_{A \setminus C}) \cap P), X_{A \setminus C}) < \tau$. This, according to [6, Lemma 6], means that $f((\{z\} \times X_{A \setminus C}) \cap P)$ is not τ -negligible in $X_{A \setminus C}$. Since $f((\{z\} \times X_{A \setminus C}) \cap P)$ is homeomorphic to $(\{z\} \times X_{A \setminus C}) \cap P$ and $X_{A \setminus C}$ is homeomorphic to $\{z\} \times X_{A \setminus C}$, $(\{z\} \times X_{A \setminus C}) \cap P$ is not τ -negligible in $\{z\} \times X_{A \setminus C}$, a contradiction. ■

Let us note that the condition $\tau > \aleph_0$ in Lemma 2.2 is essential. The following example was provided by van Mill [12]: Let $X = \prod_{n=0}^{\infty} X_n$ with $X_n = [0, 1]$ for every n , $C = \{0\}$, and let $f : X_0 \rightarrow X_{A \setminus C} = \prod_{n=1}^{\infty} X_n$ be a continuous surjection. Then the graph $G(f)$ of f meets every vertical slice in a single point and hence is negligible, but $\pi_{A \setminus C}(G(f)) = X_{A \setminus C}$.

Proposition 2.3 *Let $X = \prod_{\alpha \in A} X_\alpha$ be a product of compact metric spaces, and let P be a closed τ -negligible set in X with $\tau > \aleph_0$. If $C \subset A$ is a set of cardinality $< \tau$, then there is a set $B \subset A$ containing C such that $B \setminus C$ is countable and $P_{B \setminus C}$ is nowhere dense in $X_{B \setminus C}$. If, in addition, $f : P \rightarrow P$ is a homeomorphism and C is *f*-admissible, then we can assume that B is also *f*-admissible.*

Proof Let $\Gamma \subset A$ be a set of cardinality $< \tau$. Since P is a τ -negligible set in X , so are the sets $P(z) = (\{z\} \times X_{A \setminus \Gamma}) \cap P$ for all $z \in P_\Gamma$. This implies that each $P(z)$ is τ -negligible in $\{z\} \times X_{A \setminus \Gamma}$. Otherwise, $P(z^*)$ would contain a closed G_λ -set in $\{z^*\} \times X_{A \setminus \Gamma}$ for some $z^* \in P_\Gamma$ and $\lambda < \tau$. Because $\{z^*\} \times X_{A \setminus \Gamma}$ is G_μ -set in X , where μ is the cardinality of Γ , $P(z^*)$ contains a $G_{\lambda'}$ -subset of X with $\lambda' = \max\{\lambda, \mu\} < \tau$, a contradiction.

Using the above observation, we can apply Lemma 2.2 countably many times to construct by induction a disjoint sequence $\{C_n\}$ of finite subsets of $A \setminus C$ such that:

- $C_1 \subset A \setminus C$.
- $C_{n+1} \subset A \setminus \bigcup_{k \leq n} C \cup C_k$.
- $P_{C_n} \neq X_{C_n}$ for all n .

Indeed, suppose that we already constructed the sets C_k , $k = 1, 2, \dots, n$. Since the cardinality of $C'_n = \bigcup_{k \leq n} C \cup C_k$ is $< \tau$, $(\{z\} \times X_{A \setminus C'_n}) \cap P$ is τ -negligible in $\{z\} \times X_{A \setminus C'_n}$ for every $z \in P_{C'_n}$. Then, by Lemma 2.2, $P_{A \setminus C'_n} \neq X_{A \setminus C'_n}$. Hence, we can choose a finite set $C_{n+1} \subset A \setminus C'_n$ and an open set $V \subset X_{C_{n+1}}$ such that $V \times X_{A \setminus (C'_n \cup C_{n+1})}$ is disjoint from $P_{A \setminus C'_n}$. This implies $P_{C_{n+1}} \neq X_{C_{n+1}}$.

One can show that $B = \bigcup_{n \geq 1} C \cup C_n$ is the required set.

If $f : P \rightarrow P$ is a homeomorphism and C is f -admissible, then for every $\alpha \in A$, fix a countable f -admissible set $B(\alpha)$ containing α (see Proposition 2.1). Next, using Lemma 2.2, we construct a disjoint sequence $\{C_n\}$ of finite sets with:

- $C_1 \subset A \setminus C$;
- $C_{n+1} \subset A \setminus \bigcup_{k \leq n} C \cup C'_k$, where $C'_k = \bigcup_{\alpha \in C_k} B(\alpha)$;
- $P_{C_n} \neq X_{C_n}$ for all n .

Then $B = \bigcup_{n \geq 1} C \cup C'_n$ is f -admissible and satisfies the required conditions. ■

Everywhere below by $\mathcal{H}(X)$ we denote the space of all autohomeomorphisms of X with the compact-open topology.

Lemma 2.4 *Let $X = \prod_{\alpha \in A} X_\alpha$ be a product of zero-dimensional compact metric spaces, and let P be a closed set in X . Suppose that f is an autohomeomorphism of P and that there exist a proper subset $B \subset A$ and an autohomeomorphism f_B of P_B such that:*

- $A \setminus B$ is countable and $P = P_B \times X_{A \setminus B}$.
- $f_B \circ \pi_B = \pi_B \circ f$.
- f_B can be extended to a homeomorphism $\tilde{f}_B \in \mathcal{H}(X_B)$.

Then f can be extended to a homeomorphism $\tilde{f} \in \mathcal{H}(X)$ such that $\tilde{f}_B \circ \pi_B = \pi_B \circ \tilde{f}$.

Proof Since $f_B \circ \pi_B = \pi_B \circ f$, f is of the form $f(x, y) = (f_B(x), h(x, y))$ with $(x, y) \in P_B \times X_{A \setminus B}$ such that for each $x \in P_B$, the map φ_x , defined by $\varphi_x(y) = h(x, y)$, belongs to $\mathcal{H}(X_{A \setminus B})$. So, we have a map $\varphi : P_B \rightarrow \mathcal{H}(X_{A \setminus B})$ (see [4, Theorem 3.4.9]). Because $\mathcal{H}(X_{A \setminus B})$ is a complete separable metric space, it is an absolute extensor for zero-dimensional compacta (for example, this follows from Michael's zero-dimensional selection theorem [9]). Hence, φ can be extended to a map $\tilde{\varphi} : X_B \rightarrow \mathcal{H}(X_{A \setminus B})$. Define $\tilde{h} : X \rightarrow X_{A \setminus B}$, $\tilde{h}(x, y) = \tilde{\varphi}(x)(y)$, where $(x, y) \in X_B \times X_{A \setminus B}$. Finally, $\tilde{f}(x, y) = (\tilde{f}_B, \tilde{h}(x, y))$ provides a homeomorphism in $\mathcal{H}(X)$ extending f such that $\tilde{f}_B \circ \pi_B = \pi_B \circ \tilde{f}$. ■

3 Extending homeomorphisms

In this section, we provide a proof of Theorem 1.2. Everywhere below, we denote by \mathfrak{C} the Cantor set. Recall that \mathfrak{C} is the unique zero-dimensional perfect compact metrizable space [1].

Lemma 3.1 *Let X be a zero-dimensional paracompact space. Suppose that $P' \subset X \times \mathfrak{C}$ is a closed set such that $\pi_X(P') = X$ and that $f \in \mathcal{H}(P')$ and $g \in \mathcal{H}(X)$ are homeomorphisms with $g \circ \pi_X = \pi_X \circ f$. If the set $\pi_{\mathfrak{C}}((\{x\} \times \mathfrak{C}) \cap P')$ is nowhere dense in \mathfrak{C} for all $x \in X$, then f can be extended to a homeomorphism $\tilde{f} \in \mathcal{H}(X \times \mathfrak{C})$ such that $g \circ \pi_X = \pi_X \circ \tilde{f}$.*

Proof For any $x \in X$, let $\Phi(x)$ be the set of all $h \in \mathcal{H}(\mathfrak{C})$ such that $f(x, c) = (g(x), h(c))$ for all $c \in \pi_X^{-1}(x) \cap P'$. Since $f|(\pi_X^{-1}(x) \cap P')$ is a homeomorphism between the nowhere dense subsets $\pi_{\mathfrak{C}}((\{x\} \times \mathfrak{C}) \cap P')$ and $\pi_{\mathfrak{C}}((\{g(x)\} \times \mathfrak{C}) \cap P')$ of \mathfrak{C} , Knaster–Reichbach’s theorem [5] cited above yields a homeomorphism $h_x \in \mathcal{H}(\mathfrak{C})$ extending $f|(\pi_X^{-1}(x) \cap P')$. Hence, $\Phi(x) \neq \emptyset$ for all $x \in X$. Moreover, the sets $\Phi(x)$ are closed in $\mathcal{H}(\mathfrak{C})$ equipped with the compact-open topology. So, we have a set-valued map $\Phi : X \rightarrow \mathcal{H}(\mathfrak{C})$. One can show that if Φ admits a continuous selection $\phi : X \rightarrow \mathcal{H}(\mathfrak{C})$, then the map $\tilde{f} : X \times \mathfrak{C} \rightarrow X \times \mathfrak{C}$, defined by $\tilde{f}(x, c) = (g(x), \phi(x)(c))$, is the required homeomorphism extending f . Therefore, according to Michael’s [9] zero-dimensional selection theorem, it suffices to show that Φ is lower semi-continuous, i.e., the set $\{x \in X : \Phi(x) \cap W \neq \emptyset\}$ is open in X for any open $W \subset \mathcal{H}(\mathfrak{C})$.

To prove that, let $x^* \in X$ be a fixed point and $h^* \in \Phi(x^*) \cap W$, where W is open in $\mathcal{H}(\mathfrak{C})$. We can assume that W is of the form $\{h \in \mathcal{H}(\mathfrak{C}) : h(U_i) \subset V_i, i = 1, 2, \dots, k\}$, where $\{U_i\}_{i=1}^k$ is a clopen disjoint cover of $\{x^*\} \times \mathfrak{C}$ and $\{V_i\}_{i=1}^k$ is a disjoint clopen cover of $\{g(x^*)\} \times \mathfrak{C}$. We extend the sets U_i and V_i to clopen sets $\tilde{U}_i, \tilde{V}_i \subset X \times \mathfrak{C}$ such that:

- (1) $\tilde{U}_i = O(x^*) \times U_i$ and $\tilde{V}_i = g(O(x^*)) \times V_i$, where $O(x^*)$ is a clopen neighborhood of x^* in X .
- (2) $O(x^*)$ can be chosen so small that $f(\tilde{U}_i \cap P') \subset \tilde{V}_i \cap P'$.

We are going to show that for every $x \in O(x^*)$, there exists $h_x \in \Phi(x) \cap W$. We fix such x and observe that all sets $\tilde{U}_i(x) = \tilde{U}_i \cap (\{x\} \times \mathfrak{C})$ and $\tilde{V}_i(x) = \tilde{V}_i \cap (\{g(x)\} \times \mathfrak{C})$ are compact and perfect. Moreover, $\tilde{U}_i(x) \cap P'$ and $\tilde{V}_i(x) \cap P'$ are nowhere dense sets in $\tilde{U}_i(x)$ and $\tilde{V}_i(x)$, respectively, and $f_i^x = f|(\tilde{U}_i(x) \cap P')$ is a homeomorphism between $\tilde{U}_i(x) \cap P'$ and $\tilde{V}_i(x) \cap P'$. Hence, by Knaster–Reichbach’s theorem [5], for every i , there exists a homeomorphism $\tilde{f}_i^x : \tilde{U}_i(x) \rightarrow \tilde{V}_i(x)$ extending f_i^x . Because $\{\tilde{U}_i(x)\}_{i=1}^k$ and $\{\tilde{V}_i(x)\}_{i=1}^k$ are disjoint clopen covers of $\pi_X^{-1}(x)$ and $\pi_X^{-1}(g(x))$, respectively, the homeomorphisms $\tilde{f}_i^x, i = 1, 2, \dots, k$, provide a homeomorphism h'_x between $\pi_X^{-1}(x)$ and $\pi_X^{-1}(g(x))$ extending $f|(\pi_X^{-1}(x) \cap P')$. Then the equality $h_x(c) = h'_x(x, c), c \in \mathfrak{C}$, defines a homeomorphism $h_x \in \mathcal{H}(\mathfrak{C})$ with $h_x \in \Phi(x) \cap W$. Therefore, Φ is lower semi-continuous. ■

Proof of Theorem 1.2 We identify D^τ with D^A , where A is a set of cardinality τ . We already observed that the theorem is true when A is countable. So, let $A = \{\alpha : \alpha < \omega(\tau)\}$ be uncountable. Let show that the proof is reduced to the case of one negligible subset $P \subset D^A$ and an autohomeomorphism $f \in \mathcal{H}(P)$. Indeed, take two disjoint copies X and Y of D^A with $P \subset X$ and $K \subset Y$, and let $Q = P \uplus K$ be the disjoint union of P and K . Obviously, $X \uplus Y$ is homeomorphic to D^A , Q is negligible in $X \uplus Y$, and $f \uplus f^{-1}$ is an autohomeomorphism of Q . Suppose that $f \uplus f^{-1}$ can be extended

to a homeomorphism $F : X \uplus Y \rightarrow X \uplus Y$. Choose two clopen neighborhoods X' and Y' of P and K in X and Y , respectively, with $X \setminus X' \neq \emptyset \neq Y \setminus Y'$ such that $F(X') = Y'$. Then there is a homeomorphism $G : X \setminus X' \rightarrow Y \setminus Y'$. Hence, $F|_{X'}$ and G provide a homeomorphism $\tilde{f} : X \rightarrow Y$ extending f . Therefore, we can suppose that we have one negligible subset P of D^A and an autohomeomorphism $f \in \mathcal{H}(P)$.

We identify D^A with $X = \mathfrak{C}^A$ and take a functionally open set $V(P)$ in X which is dense in $X \setminus P$. Because every continuous function on X depends on countably many coordinates, we can choose a countable set $C \subset A$ such that $\pi_C^{-1}(\pi_C(V(P))) = V(P)$. Hence, P_B is a nowhere dense subset of X_B for any set $B \subset A$ containing C . Next, using Proposition 2.3, we can cover A by an increasing transfinite family $\{A(\alpha) : \alpha < \omega(\tau)\}$ and find homeomorphisms $f_\alpha \in \mathcal{H}(P_{A(\alpha)})$ satisfying the following conditions:

- (3) $A(1)$ is countable and the cardinality of each $A(\alpha)$ is less than τ .
- (4) $A_\alpha = \bigcup_{\beta < \alpha} A(\beta)$ if α is a limit ordinal.
- (5) $A(\alpha + 1) \setminus A(\alpha)$ is countable and $C \subset A(\alpha)$ for all α .
- (6) $\pi_{A(\alpha)} \circ f = f_\alpha \circ \pi_{A(\alpha)}$.
- (7) Each $P_{A(\alpha+1) \setminus A(\alpha)}$ is a nowhere dense set in $X_{A(\alpha+1) \setminus A(\alpha)}$.

It remains to prove that each f_α can be extended to a homeomorphism $\tilde{f}_\alpha \in \mathcal{H}(X_{A(\alpha)})$ such that

$$(8) \quad \pi_{A(\alpha)}^{A(\alpha+1)} \circ \tilde{f}_{\alpha+1} = \tilde{f}_\alpha \circ \pi_{A(\alpha)}^{A(\alpha+1)}.$$

The proof is by transfinite induction. The first extension \tilde{f}_1 exists by Knaster-Reichbach's theorem [5] because $P_{A(1)}$ is nowhere dense in $X_{A(1)}$. If \tilde{f}_α is already defined for all $\alpha < \beta$, where β is a limit ordinal, then item (4) implies the existence of \tilde{f}_β . Therefore, we need only to define $\tilde{f}_{\alpha+1}$ provided \tilde{f}_α exists.

To this end, consider the space $P_{A(\alpha)} \times X_{A(\alpha+1) \setminus A(\alpha)}$, the set $P' = P_{A(\alpha+1)} \subset P_{A(\alpha)} \times X_{A(\alpha+1) \setminus A(\alpha)}$, and the homeomorphisms $f_{\alpha+1}, f_\alpha$. For any $x \in P_{A(\alpha)}$, consider the set

$$P'(x) = P' \cap (\{x\} \times X_{A(\alpha+1) \setminus A(\alpha)}).$$

Item (7) yields that $\pi_{A(\alpha+1) \setminus A(\alpha)}(P'(x))$ is nowhere dense in $X_{A(\alpha+1) \setminus A(\alpha)}$ for every $x \in P_{A(\alpha)}$. Therefore, by Lemma 3.1, the homeomorphism $f_{\alpha+1}$ can be extended to a homeomorphism

$$f'_{\alpha+1} : P_{A(\alpha)} \times X_{A(\alpha+1) \setminus A(\alpha)} \rightarrow P_{A(\alpha)} \times X_{A(\alpha+1) \setminus A(\alpha)}$$

such that $\pi_{A(\alpha)} \circ f'_{\alpha+1} = f_\alpha \circ \pi_{A(\alpha)}$. Finally, by Lemma 2.4, there is a homeomorphism $\tilde{f}_{\alpha+1} \in \mathcal{H}(\mathfrak{C}^{A(\alpha+1)})$ satisfying condition (8). ■

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