Topology and its Applications 329 (2023) 108380

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

On homogeneity of $\mathbb{N}^\tau$ [☆]

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A R T I C L E I N F O

Article history:
Received 5 May 2022
Received in revised form 16 August 2022
Accepted 18 August 2022
Available online 13 December 2022

MSC:
primary 54C20, 54F45
secondary 54B10, 54D30

Keywords:
Compact sets
Extension of homeomorphisms
Powers of the irrationals
0-dimensional spaces

A B S T R A C T

It is shown that any homeomorphism between two compact subsets of $\mathbb{N}^\tau$ can be extended to an autohomeomorphism of $\mathbb{N}^\tau$.

1. Introduction

J. Pollard [4] established the following theorem: *Let X and Y be complete, nowhere locally compact, zero-dimensional separable metric spaces, and let P and K be closed nowhere dense subsets of X and Y, respectively. If f is a homeomorphism between P and K, then there exists a homeomorphism between X and Y extending f.*

Pollard's result is not anymore true for uncountable powers of the irrationals. For example, if P and K are two homeomorphic closed nowhere dense subsets of $\mathbb{N}^{\aleph_1}$ such that only one of them is G_δ, then no homeomorphism between P and K admits an extension to an autohomeomorphism on $\mathbb{N}^{\aleph_1}$. But a non-metrizable analogue of Pollard's theorem remains valid for compact subsets of $\mathbb{N}^\tau$. The technique developed

☆ The first author was partially supported by NSERC Grant 257231-14. The third author was partially supported by NSERC Grant 261914-19.
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https://doi.org/10.1016/j.topol.2022.108380
0166-8641/

in recent papers [5] and [6] allows to prove the following theorem, where $\pi_1 : X \times \mathbb{N}^\tau \to X$ is the projection (by a 0-dimensional space we mean a Tychonoff space having a compactification of covering dimension 0).

Theorem 1.1. *Let X be a 0-dimensional Tychonoff space and P and K be compact subsets of $X \times \mathbb{N}^\tau$ with $\tau \geq \aleph_0$. Then every homeomorphism $f : P \to K$ with $\pi_1 \circ f = \pi_1|P$ can be extended to a homeomorphism $\widetilde{f} : X \times \mathbb{N}^\tau \to X \times \mathbb{N}^\tau$ such that $\pi_1 \circ \widetilde{f} = \pi_1$.*

Theorem 1.1 implies a non-metrizable analogue of mentioned above Pollard's result [4].

Corollary 1.2. *Let P and K be compact subsets of $\mathbb{N}^\tau$ with $\tau \geq \aleph_0$. Then every homeomorphism between P and K can be extended to a homeomorphism of $\mathbb{N}^\tau$.*

2. Proof of Theorem 1.1

For any space X let $\mathcal{H}(X)$ denote the set of all autohomeomorphisms of X. If d be a bounded complete metric on $\mathbb{N}^{\aleph_0}$ we equip $\mathcal{H}(\mathbb{N}^{\aleph_0})$ with the metric $\widetilde{d} = \hat{d}(f,g) + \hat{d}(f^{-1}, g^{-1})$, where $\hat{d}(f,g) = \sup\{d(f(x), g(x)) : x \in \mathbb{N}^{\aleph_0}\}$. It is well known that $\widetilde{d}$ is a complete metric on $\mathcal{H}(\mathbb{N}^{\aleph_0})$.

Next lemma is an analogue of [5, Lemma 3.1].

Lemma 2.1. *Let X be a 0-dimensional space and P, K be compact subsets of $X \times \mathbb{N}^{\aleph_0}$. If $f : P \to K$ and $g \in \mathcal{H}(X)$ are homeomorphisms with $g \circ \pi_X = \pi_X \circ f$, then f can be extended to a homeomorphism $\widetilde{f} \in \mathcal{H}(X \times \mathbb{N}^{\aleph_0})$ such that $g \circ \pi_X = \pi_X \circ \widetilde{f}$.*

Proof. Obviously, $g(P_X) = K_X$, where $P_X = \pi_X(P)$ and $K_X = \pi_X(K)$. Denote by $\pi : X \times \mathbb{N}^{\aleph_0} \to \mathbb{N}^{\aleph_0}$ the projection. For any $x \in P_X$ let $\Phi(x)$ be the set of all $h \in \mathcal{H}(\mathbb{N}^{\aleph_0})$ such that $f(x,c) = (g(x), h(c))$ for every $c \in \pi_X^{-1}(x) \cap P$. Since $f|(\pi_X^{-1}(x) \cap P)$ is a homeomorphism between the compact subsets $\pi((\{x\} \times \mathbb{N}^{\aleph_0}) \cap P)$ and $\pi((\{g(x)\} \times \mathbb{N}^{\aleph_0}) \cap K)$ of $\mathbb{N}^{\aleph_0}$, the Pollard's theorem [4] cited above yields a homeomorphism $h_x \in \mathcal{H}(\mathbb{N}^{\aleph_0})$ extending $f|(\pi_X^{-1}(x) \cap P)$. Hence, $\Phi(x) \neq \varnothing$ for all $x \in P_X$. Moreover, the sets $\Phi(x)$ are closed in $\mathcal{H}(\mathbb{N}^{\aleph_0})$ equipped with the metric $\widetilde{d}$. So, we have a set-valued map $\Phi : P_X \rightsquigarrow \mathcal{H}(\mathbb{N}^{\aleph_0})$. One can show that if Φ admits a continuous selection $\phi : P_X \to \mathcal{H}(\mathbb{N}^{\aleph_0})$, then the map $f_1 : P_X \times \mathbb{N}^{\aleph_0} \to K_X \times \mathbb{N}^{\aleph_0}$, defined by $f_1(x,c) = (g(x), \phi(x)(c))$, is a homeomorphism between $P_X \times \mathbb{N}^{\aleph_0}$ and $K_X \times \mathbb{N}^{\aleph_0}$ extending f (see [2, Proposition 2.6.11]) with $\pi_X \circ f_1 = g \circ \pi_X$. On the other hand, since X is 0-dimensional and P_X is a compact subset of X, the map ϕ has a continuous extension $\widetilde{\phi} : X \to \mathcal{H}(\mathbb{N}^{\aleph_0})$. This is true because $\mathcal{H}(\mathbb{N}^{\aleph_0})$ is a metric space. Indeed, then $\phi(P_X)$ is a compact subset of $\mathcal{H}(\mathbb{N}^{\aleph_0})$, hence $\phi(P_X)$ is itself a compact metric space. So, it is an absolute extensor for 0-dimensional spaces in the sense of Chigogidze [1, Definition 6.1.3], and we can extend ϕ to a map $\widetilde{\phi} : X \to \mathcal{H}(\mathbb{N}^{\aleph_0})$. Next, let $\widetilde{f} : X \times \mathbb{N}^{\aleph_0} \to X \times \mathbb{N}^{\aleph_0}$ be the map defined by $\widetilde{f}(x,c) = (g(x), \widetilde{\phi}(x)(c))$. Then $\widetilde{f}$ is a homeomorphism extending f. Therefore, according to Michael's [3] zero-dimensional selection theorem, it suffices to show that Φ is lower semi-continuous.

To prove that, let $x^* \in P_X$ be a fixed point and $h^* \in \Phi(x^*) \cap W$, where W is open in $\mathcal{H}(\mathbb{N}^{\aleph_0})$. We can assume that W is of the form $\{h \in \mathcal{H}(\mathbb{N}^{\aleph_0}) : \{x^*\} \times h(U_i) = \{g(x^*)\} \times V_i \,, i = 1, 2, ..\}$, where $\{U_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ are clopen disjoint countable covers of $\mathbb{N}^{\aleph_0}$. Because $P(x^*) = (\{x^*\} \times \mathbb{N}^{\aleph_0}) \cap P$ and $K(g(x^*)) = (\{g(x^*)\} \times \mathbb{N}^{\aleph_0}) \cap K$ are compact, there is k such that $(\{x^*\} \times U_i) \cap P(x^*) \neq \varnothing$ and $(\{g(x^*)\} \times V_i) \cap K(g(x^*)) \neq \varnothing$ if and only if $i \leq k$.

We extend each of the sets $\{x^*\} \times U_i$ and $\{g(x^*)\} \times V_i$, $i \leq k$, to clopen sets $\widetilde{U}_i \subset P_X \times \mathbb{N}^{\aleph_0}$ and $\widetilde{V}_i \subset K_X \times \mathbb{N}^{\aleph_0}$ such that

(1) $\widetilde{U}_i = O(x^*) \times U_i$ and $\widetilde{V}_i = g(O(x^*)) \times V_i$, where $O(x^*)$ is a clopen neighborhood of x^* in P_X such that $P(x) \subset \bigcup_{i=1}^{k} \widetilde{U}_i$ for all $x \in O(x^*)$;

(2) $O(x^*)$ is so small that $f(\widetilde{U}_i \cap P) \subset \widetilde{V}_i \cap K$.

We are going to show that for every $x \in O(x^*)$ there exists $h_x \in \Phi(x) \cap W$. We fix such x and observe that all sets $\widetilde{U}_i(x) = \widetilde{U}_i \cap (\{x\} \times \mathbb{N}^{\aleph_0})$ and $\widetilde{V}_i(x) = \widetilde{V}_i \cap (\{g(x)\} \times \mathbb{N}^{\aleph_0})$ are nowhere locally compact and complete. Moreover, $\widetilde{U}_i(x) \cap P$ and $\widetilde{V}_i(x) \cap K$ are compact sets in $\widetilde{U}_i(x)$ and $\widetilde{V}_i(x)$, respectively, and $f_i^x = f|(\widetilde{U}_i(x) \cap P)$ is a homeomorphism between them for every $i \leq k$. Hence, by Pollard's theorem [4], there exist homeomorphisms $\widetilde{f}_i^x : \widetilde{U}_i(x) \to \widetilde{V}_i(x)$ extending f_i^x, $i \leq k$. For every $(x, c) \notin \bigcup_{i=1}^k \widetilde{U}_i(x)$ there is exactly one $i > k$ with $c \in U_i$, and we define $\widetilde{f}_i^x(x, c) = h^*(x^*, c)$. The homeomorphisms $\widetilde{f}_i^x$, $i = 1, 2, ..$, provide a homeomorphism h'_x between $\pi_X^{-1}(x)$ and $\pi_X^{-1}(g(x))$ extending $f|\pi_X^{-1}(x) \cap P$. Then the equality $h_x(c) = h'_x(x, c)$, $c \in \mathbb{N}^{\aleph_0}$, defines a homeomorphism from $\mathcal{H}(\mathbb{N}^{\aleph_0})$ with $h_x \in \Phi(x) \cap W$. Therefore, Φ is lower semi-continuous. □

Everywhere below we suppose that P, K are compact subsets of $X \times \mathbb{N}^A$ and $f : P \to K$ is a homeomorphism such that $\pi_1|P = \pi_1 \circ f$, where X is a 0-dimensional space. A set $B \subset A$ is called *f-admissible* if there exists a homeomorphism $f_B : P_B \to K_B$ such that $(f_B \circ p_B)|P = p_B \circ f$ and $q_B|P_B = q_B \circ f_B$, where $\pi_B : \mathbb{N}^A \to \mathbb{N}^B$, $p_B : Y \times \mathbb{N}^A \to X \times \mathbb{N}^B$, and $q_B : X \times \mathbb{N}^B \to X$ denote the projections, $P_B = p_B(P)$ and $K_B = p_B(K)$.

Lemma 2.2. *If A is an uncountable set, then for every $\alpha \in A$ there is an f-admissible countable set $B(\alpha) \subset A$ containing α*

Proof. Since $\pi_1|P = \pi_1 \circ f$, $f(y, x) = (y, h_1(y, x))$ for all $(y, x) \in P$, where $h_1 : P \to \mathbb{N}^A$. Similarly, $f^{-1}(y, x) = (y, h_2(y, x))$ for all $(y, x) \in K$ with $h_2 : K \to \mathbb{N}^A$.

Claim 2.3. For every $i = 1, 2$ and a countable set $C \subset A$, there is countable $D(i) \subset A$ containing C and a continuous maps $g_1 : P_{D(1)} \to \mathbb{N}^C$, $g_2 : K_{D(2)} \to \mathbb{N}^C$ with $(g_1 \circ p_{D(1)})|P = \pi_C \circ h_1$ and $(g_2 \circ p_{D(2)})|K = \pi_C \circ h_2$.

Let $\mathcal{B}$ be a countable base for $\mathbb{N}^C$. Then $G_U = h_1^{-1}(\pi_C^{-1}(U))$ is a σ-compact open subset of P for every $U \in \mathcal{B}$. So, there is a sequence $\{W_U(n)\}_{n \geq 1}$ of standard open sets in $X \times \mathbb{N}^A$ such that G_U is the union of all $W_U(n) \cap P$, $n \geq 1$. Therefore, for every U there exists a countable set $C_U \subset A$ with $p_{C_U}^{-1}(p_{C_U}(W_U(n))) = W_U(n)$, $n \geq 1$. We can assume that each C_U contains C. Then $D(1) = \bigcup_{U \in \mathcal{B}} C_U$ is a countable set containing C and $p_{D(1)}(y, x) = p_{D(1)}(y', x')$ implies $\pi_C(h_1(y, x)) = \pi_C(h_1(y', x'))$, where $(y, x), (y', x') \in P$. Because $P_{D(1)}$ is compact, this yields the existence of a map $g_1 : P_{D(1)} \to \mathbb{N}^C$ with $(g_1 \circ p_{D(1)})|P = \pi_C \circ h_1$. Similarly, we can find a countable set $D(2) \subset A$ which contains C and a map $g_2 : K_{D(2)} \to \mathbb{N}^C$ satisfying the claim.

Using Claim 2.3, we construct by induction an increasing sequence $\{B(n)\}_{n \geq 0}$ of countable sets $B(n) \subset A$ and maps $\varphi_n : P_{B(n+1)} \to \mathbb{N}^{B(n)}$ for $n = 2k$ and $\psi_n : K_{B(n+1)} \to \mathbb{N}^{B(n)}$ for $n = 2k + 1$ such that

- $B(0) = \{\alpha\}$;
- $\pi_{B(n)} \circ h_1 = (\varphi_n \circ p_{B(n+1)})|P$ if $n = 2k$;
- $\pi_{B(n)} \circ h_2 = (\psi_n \circ p_{B(n+1)})|K$ if $n = 2k + 1$.

Let $B(\alpha) = \bigcup_{n=0}^{\infty} B(n)$. Then we have maps $\varphi_{B(\alpha)} : P_{B(\alpha)} \to \mathbb{N}^{B(\alpha)}$ and $\psi_{B(\alpha)} : K_{B(\alpha)} \to \mathbb{N}^{B(\alpha)}$ such that $\pi_{B(\alpha)} \circ h_1 = (\varphi_{B(\alpha)} \circ p_{B(\alpha)})|P$ and $\pi_{B(\alpha)} \circ h_2 = (\psi_{B(\alpha)} \circ p_{B(\alpha)})|K$. Observe that $P_{B(\alpha)}$ and $K_{B(\alpha)}$ are subsets of $X \times \mathbb{N}^{B(\alpha)}$. Then $f_{B(\alpha)} : P_{B(\alpha)} \to X \times \mathbb{N}^{B(\alpha)}$, defined by $f_{B(\alpha)}(y, x) = (y, \varphi_{B(\alpha)}(y, x))$ for every $(y, x) \in P_{B(\alpha)}$, is a homeomorphism between $P_{B(\alpha)}$ and $K_{B(\alpha)}$ whose inverse is the map $g_{B(\alpha)} : K_{B(\alpha)} \to X \times \mathbb{N}^{B(\alpha)}$, defined by $g_{B(\alpha)}(y, x) = (y, \psi_{B(\alpha)}(y, x))$ for $(y, x) \in K_{B(\alpha)}$. Therefore, $B(\alpha)$ is f-admissible. □

Proof of Theorem 1.1. We identify $\mathbb{N}^\tau$ with $\mathbb{N}^A$, where A is a set of cardinality τ. The case $\tau = \aleph_0$ follows from Lemma 2.1. So, let $A = \{\alpha : \alpha < \omega(\tau)\}$ be uncountable. By Lemma 2.2, we can cover A by a family $\{B(\alpha) : \alpha < \omega(\tau)\}$ of countable f-admissible sets. Since any union of f-admissible sets is also f-admissible, from the family $\{B(\alpha) : \alpha < \omega(\tau)\}$ we obtain an increasing family of f-admissible sets $A(\alpha)$ and homeomorphisms $f_{A(\alpha)} : P_{A(\alpha)} \to K_{A(\alpha)}$ such that:

(3) $A(1)$ is countable, and the cardinality of each $A(\alpha)$ is less than τ;
(4) $A(\alpha) = \bigcup_{\beta<\alpha} A(\beta)$ if α is a limit ordinal;
(5) $A(\alpha+1)\backslash A(\alpha)$ is countable but infinite for all α;
(6) $p_{A(\alpha)} \circ f = (f_{A(\alpha)} \circ p_{A(\alpha)})|P$.

We need to prove that each $f_{A(\alpha)}$ can be extended to a homeomorphism $\widetilde{f}_{A(\alpha)} : X \times \mathbb{N}^{A(\alpha)} \to X \times \mathbb{N}^{A(\alpha)}$ such that

(7) $p^{A(\alpha+1)}_{A(\alpha)} \circ \widetilde{f}_{A(\alpha+1)} = (\widetilde{f}_{A(\alpha)} \circ p^{A(\alpha+1)}_{A(\alpha)})$;
(8) $q_{A(\alpha)} = q_{A(\alpha)} \circ \widetilde{f}_{A(\alpha)}$, where $q_{A(\alpha)} : X \times \mathbb{N}^{A(\alpha)} \to X$ denotes the projection.

The proof is by transfinite induction. The first extension $\widetilde{f}_{A(1)}$ exists by Lemma 2.1 because $P_{A(1)}$ and $K_{A(1)}$ are compact subsets of $X \times \mathbb{N}^{A(1)}$ and $q_{A(1)}|P_{A(1)} = q_{A(1)} \circ f_{A(1)}$. If β is a limit ordinal and $\widetilde{f}_{A(\alpha)}$ is already defined for all $\alpha < \beta$, then item (4) implies the existence of $\widetilde{f}_{A(\beta)}$. Therefore, we need only to define $\widetilde{f}_{A(\alpha+1)}$ provided $\widetilde{f}_{A(\alpha)}$ exists.

To this end, we apply again Lemma 2.1 for the space $X \times \mathbb{N}^{A(\alpha+1)} = X \times \mathbb{N}^{A(\alpha)} \times \mathbb{N}^{A(\alpha+1)\backslash A(\alpha)}$, the sets $P_{A(\alpha+1)}, K_{A(\alpha+1)}$, the projection $\pi : X \times \mathbb{N}^{A(\alpha)} \times \mathbb{N}^{A(\alpha+1)\backslash A(\alpha)} \to X \times \mathbb{N}^{A(\alpha)}$, and the homeomorphisms $f_{A(\alpha+1)}$ and $\widetilde{f}_{A(\alpha)}$. Moreover, $X \times \mathbb{N}^{A(\alpha)}$ has a 0-dimensional compactification because both X and $N^{A(\alpha)}$ have such compactifications. Hence, there is a homeomorphism $\widetilde{f}_{A(\alpha+1)} \in \mathcal{H}(X \times \mathbb{N}^{A(\alpha+1)})$ extending $f_{A(\alpha+1)}$ and satisfying condition (7). Since $q_{A(\alpha)} = q_{A(\alpha)} \circ \widetilde{f}_{A(\alpha)}$, we also have $q_{A(\alpha+1)} = q_{A(\alpha+1)} \circ \widetilde{f}_{A(\alpha+1)}$. □

Proof of Corollary 1.2. The corollary is obtained from Theorem 1.1 by letting X to be the one-point space. □

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