EXTENDING HOMEOMORPHISMS ON CANTOR CUBES

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Abstract. We discuss the question of extending homeomorphism between closed subsets of the Cantor discontinuum D^τ. For every set $P \subset D^\tau$ let $\mathfrak{L}_P$ be the set of cardinality λ such that the λ-interior of P is not empty. It is established that any homeomorphism f between two proper closed subsets P and K of D^τ can be extended to an autohomeomorphism of D^τ provided the sets $\mathfrak{L}_P$ and $\mathfrak{L}_K$ do not have so many points of discontinuity and f preserves the λ-interiors of P and K.

1. Introduction

According to [5], in 1951 Ryll-Nardzewski presented his solution of Knaster's problem on extension of homeomorphisms on closed subsets of the Cantor set $D^{\aleph_0}$. Ryll-Nardzewski's proof was based on Boolean algebras and it was not published.

Here is Ryll-Nardzewski's theorem, see [5]: *Let P, K be a proper closed subsets of the Cantor set $D^{\aleph_0}$ and f be a homeomorphism between P and K such that $f(\text{int P}) = \text{int K}$. Then there exists an autohomeomorphism of $D^{\aleph_0}$ extending f.* A topological proof of Ryll-Nardzewski's theorem was established by Knaster-Reichbach [5]. More precisely, Knaster-Reichbach reduced their proof to the case when $\text{int P} = \text{int K} = \varnothing$, see Theorem 2.6 below. The non-metrizable analogue of Knaster-Reichbach's theorem was proved in our paper [8, Theorem 1.2].

The present article is devoted to the proof of a non-metrizable analogue of the Ryll-Nardzewski theorem. In contrast to the metrizable case, the non-metrizable version of Ryll-Nardzewski's theorem cannot be directly obtained from [8, Theorem 1.2]. The key role in our proof plays the concept of λ-interior.

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For a space X, a subset $P \subset X$ and an infinite cardinal λ we denote by $P^{(\lambda)}$ the λ-*interior* of P in X, i.e. the set all $x \in P$ such that there exists a G_λ-subset K of X with $x \in K \subset P$. If λ is finite, then $P^{(\lambda)}$ is the ordinary interior of P and it is denoted by $P^{(0)}$. If there exists $\tau \geq \aleph_0$ such that $P^{(\lambda)}$ is empty for all $\lambda < \tau$, we say that P is τ-*negligible in* X. The sets of all cardinals λ with $P^{(\lambda)} \neq \varnothing$ is denoted by $\mathfrak{L}_P$.

Let $\lambda \in \mathfrak{L}_P$ be a limit cardinal. We say that $\mathfrak{L}_P$ is *discontinuous at* λ if $\overline{\bigcup_{\gamma<\lambda} P^{(\gamma)}} \subsetneqq P^{(\lambda)}$, otherwise $\mathfrak{L}_P$ is said to be *continuous at* λ. Denote by $\mathfrak{L}_P^d$ the cardinals of discontinuity for $\mathfrak{L}_P$. The set $\mathfrak{L}_P^d$ is said to be *discrete in* $\mathfrak{L}_P$ if for every $\lambda \in \mathfrak{L}_P^d$ there is $\gamma < \lambda$ such that the interval (γ, λ) is disjoint from $\mathfrak{L}_P^d$. Obviously, $\mathfrak{L}_P^d$ is discrete in $\mathfrak{L}_P$ if $\mathfrak{L}_P^d$ is finite, in particular that is true provided $\mathfrak{L}_P$ is finite.

Now, we can formulate a non-metrizable version of Ryll-Nardzewski's theorem.

Theorem 1.1. *Let f be a homeomorphism between two proper closed subsets P and K of D^τ such that $\mathfrak{L}_P^d$ is discrete in $\mathfrak{L}_P$ and $f(P^{(\lambda)}) = K^{(\lambda)}$ for every $\lambda \in \mathfrak{L}_P$. Then f can be extended to a homeomorphism of D^τ.*

Observe that the condition $f(P^{(\lambda)}) = K^{(\lambda)}$ for any cardinal number $\lambda \geq 0$ is necessary for the existence of a homeomorphic extension of f. Let's also note that the case $P^{(\lambda)} = \varnothing$ for all $\lambda < \tau$ was settled in [8].

Question 1.2. *Let f be a homeomorphism between two proper closed subsets P and K of D^τ. Is it true that f can be extended to a homeomorphism of D^τ provided $f(P^{(\lambda)}) = K^{(\lambda)}$ for every $\lambda \in \mathfrak{L}_P$?*

2. Some preliminary results

Everywhere below, if not stated otherwise, $X = \prod_{\alpha\in A} X_\alpha$ is an uncountable product of metric compacta, $P \subset X$ is a closed proper subset and $f : P \to P$ is a homeomorphism such that $f(P^{(\mu)}) = P^{(\mu)}$ for every cardinal μ. If $B \subset A$ we denote by $\pi_B : X \to \prod_{\alpha\in B} X_\alpha$ the projection. In case $B = \{\alpha\}$ we write π_α instead of $\pi_{\{\alpha\}}$. We also denote $\pi_B(P)$ by P_B and $\pi_B(P^{(\lambda)})$ by $P_B^{(\lambda)}$. We have to distinguish the sets $P_B^{(\lambda)}$ and $(P_B)^{(\lambda)}$. Since $P^{(\mu)}$ is a union of G_μ-subsets of X, according to [2] the closure of $P^{(\mu)}$ is also a G_μ-set. Hence, $P^{(\mu)}$ is closed in X for every cardinal μ and there is a set $B \subset A$ of cardinality μ with $\pi_B^{-1}(P_B^{(\mu)}) = P^{(\mu)}$. A set $B \subset A$ is called f-*admissible* if there exists a homeomorphism $f_B : P_B \to P_B$ with $\pi_B \circ f = f_B \circ \pi_B$. It is easily seen that an arbitrary union of f-admissible sets is also f-admissible.

According to [8, Proposition 2.1] there is a family $\{B(\alpha) : \alpha \in A\}$ of countable f-admissible sets such that $\alpha \in B(\alpha)$ for every α. We say that a set $C \subset A$ is *saturated* if $C = \bigcup_{\alpha \in C} B(\alpha)$. Every saturated set is f-admissible.

Proposition 2.1. *Every set $C \subset A$ is contained in a saturated set $B \subset A$ of the same cardinality as C.*

Proof. Since the family of f-admissible sets is closed under arbitrary unions, the set $B = \bigcup_{\alpha \in C} B(\alpha)$ is saturated and has the same cardinality as C. □

Lemma 2.2. [8, Lemma 2.2] *Let $K \subset X$ be a closed set. Suppose $\tau > \aleph_0$ and $C \subset A$ is a set of cardinality $< \tau$ such that $(\{z\} \times X_{A \backslash C}) \cap K$ is τ-negligible in $\{z\} \times X_{A \backslash C}$ for every $z \in K_C$. Then $K_{A \backslash C} \neq X_{A \backslash C}$.*

If λ is any cardinal number, then λ^+ denotes the successor of λ.

Proposition 2.3. *Let $C \subset A$ is a saturated set of cardinality $\lambda \geq \aleph_0$ such that $\pi_C^{-1}(P_C^{(\lambda)}) = P^{(\lambda)}$. If $P \neq P^{(\lambda)}$ then there is a saturated set $B \subset A$ of cardinality λ^+ such that*

(i) *$C \subset B$ and $\pi_B^{-1}(P_B^{(\lambda^+)}) = P^{(\lambda^+)}$;*

(ii) *If $D \subset A$ contains B, then every compact set $F \subset P_D \backslash P_D^{(\lambda)}$ is λ^+-negligible in X_D.*

Proof. Since $P_C^{(\lambda)}$ is a closed subset of P_C and the weight of X_C is λ, $P_C \backslash P_C^{(\lambda)}$ is the union of λ many closed sets F_ξ, $\xi < \omega(\lambda)$ (here $\omega(\lambda)$ is the first ordinal of cardinality λ). Let $P_\xi = P \cap \pi_C^{-1}(F_\xi)$. Because P_ξ are disjoint from $P^{(\lambda)}$ and $P \neq P^{(\lambda)}$, each P_ξ is a λ^+-negligible set in X. So are the sets $P_\xi(y) = P_\xi \cap (\{y\} \times X_{A \backslash C})$, $y \in F_\xi$. This observation implies that each $\pi_{A \backslash C}(P_\xi(y))$ is λ^+-negligible in $X_{A \backslash C}$. Indeed, otherwise $\pi_{A \backslash C}(P_\xi(y))$ would contain a closed G_ν-subset of $X_{A \backslash C}$ with $\nu \leq \lambda$, and using that $\{y\} \times X_{A \backslash C}$ is a G_λ-set in X, one can show that $P_\xi(y)$ also contains a closed G_λ-subset of X. This is a contradiction because $P_\xi(y) \subset P \backslash P^{(\lambda)}$.

Therefore, according to Lemma 2.2, $\pi_{A \backslash C}(P_\xi) \neq X_{A \backslash C}$ for every ξ. Hence, we can find a countable set $C_1(\xi) \subset A \backslash C$ such that $\pi_{C_1(\xi)}(P_\xi) \neq X_{C_1(\xi)}$. By Proposition 2.1, we can assume that each $C_1(\xi)$ is saturated. Then $C_1 = \bigcup_\xi C_1(\xi)$ is a saturated set of cardinality λ disjoint from C with $\pi_{C_1}(P_\xi) \neq X_{C_1}$ for all ξ. Moreover, $B_1 = C \cup C_1$ is also a saturated set of cardinality λ, and let $F_\xi^1 = (\pi_C^{B_1})^{-1}(F_\xi)$. Obviously, $P_\xi = P \cap \pi_{B_1}^{-1}(F_\xi^1)$, $\pi_{B_1}^{-1}(P_{B_1}^{(\lambda)}) = P^{(\lambda)}$ and $P_{B_1} \backslash P_{B_1}^{(\lambda)}) = \bigcup_\xi F_\xi^1$. Since all sets $P_\xi(z) = P_\xi \cap (\{z\} \times X_{A \backslash B_1})$, $z \in F_\xi^1$, are λ^+-negligible in X, the arguments from the previous paragraph show that the sets $\pi_{A\backslash B_1}(P_\xi(z))$ are λ^+-negligible in $X_{A\backslash B_1}$. So, by Lemma 2.2 $\pi_{A\backslash B_1}(P_\xi) \neq X_{A\backslash B_1}$ for every ξ. As above, we can find a family of countable saturated sets $C_2(\xi) \subset A\backslash B_1$ with $\pi_{C_2(\xi)}(P_\xi) \neq X_{C_2(\xi)}$. Let $C_2 = \bigcup_\xi C_2(\xi)$. In this way, by transfinite induction we construct for every $\gamma < \omega(\lambda^+)$ a family $\phi_\gamma = \{C_\gamma(\xi) : \xi < \omega(\lambda)\}$ of countable saturated sets such that the sets $C_\gamma = \bigcup_\xi C_\gamma(\xi)$ satisfy the following conditions :

(1) $\{C_\gamma : \gamma < \omega(\lambda^+)\}$ is a disjoint family;
(2) $C_\gamma \subset A\backslash \bigcup_{\beta<\gamma}(C \cup C_\beta)$ if γ is a limit ordinal;
(3) $C_{\gamma+1} \subset A\backslash \bigcup_{\beta\leq\gamma}(C \cup C_\beta)$;
(4) $\pi_{C_\gamma}(P_\xi) \neq X_{C_\gamma}$ for all ξ.

Let $B' = \bigcup_\gamma B_\gamma$, where $B_\gamma = \bigcup_{\beta\leq\gamma} C \cup C_\beta$. Clearly, B' is a saturated set of cardinality λ^+ and $C \subset B'$. Since $P^{(\lambda^+)}$ is a closed G_{λ^+}-subset of X, there is a saturated set $B \subset A$ of cardinality λ^+ containing B' such that $\pi_B^{-1}(P_B^{(\lambda^+)}) = P^{(\lambda^+)}$.

Suppose that $D \subset A$ is a set containing B. For every $\xi < \omega(\lambda)$ let $H_\xi = (\pi_C^D)^{-1}(F_\xi) \cap P_D$. We have $\pi_D^{-1}(P_D^{(\lambda)}) = P^{(\lambda)}$, $P_D\backslash P_D^{(\lambda)} = \bigcup_\xi H_\xi$ and $\pi_{C_\gamma}^D(H_\xi) = \pi_{C_\gamma}(P_\xi)$ for all γ.

Claim 2.4. Every compact set $F \subset P_D\backslash P_D^{(\lambda)}$ is λ^+-negligible in X_D.

Fix a point $y_C^* \in X_C\backslash P_C^{(\lambda)}$ and for every $\gamma < \omega(\lambda^+)$ choose a point $y_\gamma^* \in X_{C_\gamma}\backslash\pi_{C_\gamma}(P_\xi)$. Let $y^* \in X_D$ be a point with $\pi_C^D(y^*) = y_C^*$ and $\pi_{C_\gamma}^D(y^*) = y_\gamma^*$ for all $\gamma < \omega(\lambda^+)$. Consider the set $\sum(y^*)$ of all $y \in X_D$ such that the cardinality of the set $\{\gamma < \omega(\lambda^+) : \pi_{C_\gamma}^D(y) \neq y_\gamma^*)\}$ is $< \lambda^+$. Item (4) implies $H_\xi \subset X_D\backslash \sum(y^*)$ for all ξ, so $\bigcup_\xi H_\xi \subset X_D\backslash \sum(y^*)$. Since $P_D\backslash P_D^{(\lambda)} = \bigcup_\xi H_\xi$, $F \subset X_D\backslash \sum(y^*)$. Suppose F contains a closed G_β-subset of X_D for some $\beta < \lambda^+$. Then there exist a set $\Gamma \subset D$ of cardinality β and a point $z \in X_\Gamma$ with $(\pi_\Gamma^D)^{-1}(z) = \{z\} \times X_{D\backslash\Gamma} \subset F$. Since Γ can contains at most β many sets C_γ, $\{z\} \times X_{D\backslash\Gamma}$ contains points from $\sum(y^*)$, which contradicts $F \subset X_D\backslash \sum(y^*)$. Hence, F is λ^+-negligible in X_D. □

Lemma 2.5. *Let $C \subset B$ be saturated sets satisfying the hypotheses of Proposition* 2.3*. Then for any σ-compact set $F \subset P_C\backslash P_C^{(\lambda)}$ there exists a saturated set $\Lambda \subset B$ containing C such that $\Lambda\backslash C$ is countable and for all $x \in F$ the sets $\pi_{\Lambda\backslash C}^\Lambda((\pi_C^\Lambda)^{-1}(x) \cap P_\Lambda)$, $\pi_{\Lambda\backslash C}^\Lambda((\pi_C^\Lambda)^{-1}(f_C(x)) \cap P_\Lambda)$ are nowhere dense in $X_{\Lambda\backslash C}$. Moreover, we can suppose that Λ contains Γ for a given saturated set $\Gamma \subset B$ containing C with $\Gamma\backslash C$ countable.*

Proof. Let $F' = f_C(F)$. Since C is f-admissible, $f_C(P_C^{(\lambda)}) = P_C^{(\lambda)}$. Hence, $F \cup F'$ is a σ-compact subset of $P_C \backslash P_C^{(\lambda)}$. Represent $F \cup F'$ as a countable union of compact sets $F_n \subset P_C \backslash P_C^{(\lambda)}$. Suppose that $\Gamma \subset B$ is a saturated set containing C with $\Gamma \backslash C$ countable. Then each $K_n = (\pi_C^\Gamma)^{-1}(F_n) \cap P_\Gamma$ is a compact subset of $P_\Gamma \backslash P_\Gamma^{(\lambda)}$. Consequently, all $L_n = (\pi_\Gamma^B)^{-1}(K_n) \cap P_B$ are contained in $P_B \backslash P_B^{(\lambda)}$. We claim that for every n and $z \in K_n$ the sets $\pi_{B\backslash\Gamma}^B(L_n(z))$ are λ^+-negligable subsets of $X_{B\backslash\Gamma}$, where $L_n(z) = (\{z\} \times X_{B\backslash\Gamma}) \cap L_n$. Indeed, otherwise some $\pi_{B\backslash\Gamma}^B(L_n(z))$ would contain a closed G_ν-subset of $X_{B\backslash\Gamma}$ with $\nu < \lambda^+$, and using that $\{z\} \times X_{B\backslash\Gamma}$ is a G_λ-set in X_B (see the proof of Proposition 2.3), one can show that $L_n(z)$ contains a closed G_λ-subset of X_B. Hence, $L_n(z) \subset P_B^{(\lambda)}$, which contradicts the inclusion $L_n(z) \subset L_n \subset P_B \backslash P_B^{(\lambda)}$.

Therefore, according to Lemma 2.2, $\pi_{B\backslash\Gamma}^B(L_n) \neq X_{B\backslash\Gamma}$ for every n, and there exist countable sets $B_1(n) \subset B\backslash\Gamma$ such that $\pi_{B_1(n)}^B(L_n) \neq X_{B_1(n)}$. By Proposition 2.1, we can assume that each $B_1(n)$ is saturated. Then $B_1 = \bigcup_n B_1(n)$ is a countable saturated subset of $B\backslash\Gamma$ with $\pi_{B_1}^B(L_n) \neq X_{B_1}$ for all n. Moreover, $\Lambda(1) = \Gamma \cup B_1$ is also a countable saturated set.

Now, let $K_n^1 = (\pi_\Gamma^{\Lambda(1)})^{-1}(K_n) \cap P_{\Lambda(1)}$ and $L_n^1(z) = (\{z\} \times X_{B\backslash\Lambda(1)}) \cap L_n$ for every $z \in K_n^1$. As above, we can show that $\pi_{B\backslash\Lambda(1)}^B(L_n^1(z))$ are λ^+-negligible subsets of $X_{B\backslash\Lambda(1)}$ for all n and $z \in K_n^1$. Since $\pi_{\Lambda(1)}^B(L_n) = K_n^1$, we can apply Lemma 2.2 to conclude that $\pi_{B\backslash\Lambda(1)}^B(L_n) \neq X_{B\backslash\Lambda(1)}$ for every n. Hence, there exist a sequence $\{B_2(n)\}$ of countable saturated subsets of $B\backslash\Lambda(1)$ with $\pi_{B_2(n)}^B(L_n) \neq X_{B_2(n)}$. This implies $\pi_{B_2}^B(L_n) \neq X_{B_2}$ for all n, where $B_2 = \bigcup_n B_n(n)$. Then $\Lambda(2) = \Gamma \cup B_1 \cup B_2$ is a saturated subset of B such that B_2 and B_1 are disjoint subsets of $B\backslash\Gamma$. In this way we construct a sequence $\{B_k\}$ of disjoint countable saturated subsets of $B\backslash\Gamma$ satisfying the following conditions:

- (a) $B_{k+1} \subset B \backslash \bigcup_{i \leq k}(\Gamma \cup B_i)$;
- (b) $\pi_{B_k}^B(L_n) \neq X_{B_k}$ for all n and k.

Let Λ be the union of Γ and all B_k. For every n we have $L_n = (\pi_C^B)^{-1}(F_n) \cap X_B$. So, $\pi_\Lambda^B(L_n) = (\pi_C^\Lambda)^{-1}(F_n) \cap P_\Lambda$ and $\pi_{\Lambda\backslash C}^\Lambda((\pi_C^\Lambda)^{-1}(F_n) \cap P_\Lambda) = \pi_{\Lambda\backslash C}^B(L_n)$. Suppose $\pi_{\Lambda\backslash C}^B(L_n)$ contains an open subset of $X_{\Lambda\backslash C}$ for some n. Since B_k are disjoint subsets of $\Lambda\backslash C$, there exists k with $\pi_{B_k}^{\Lambda\backslash C}(\pi_{\Lambda\backslash C}^B(L_n)) = X_{B(k)}$. On the other hand, $\pi_{B_k}^{\Lambda\backslash C}(\pi_{\Lambda\backslash C}^B(L_n)) = \pi_{B_k}^B(L_n)$, which contradicts condition (b). Therefore, $\pi_{\Lambda\backslash C}^\Lambda((\pi_C^\Lambda)^{-1}(F_n) \cap P_\Lambda)$ are nowhere dense subsets of $X_{\Lambda\backslash C}$. This implies that the sets $\pi_{\Lambda\backslash C}^\Lambda((\pi_C^\Lambda)^{-1}(x) \cap P_\Lambda)$, $x \in F \cup F'$, are also nowhere dense in $X_{\Lambda\backslash C}$. $\square$

Theorem 2.6. [5] *Let X and Y be compact, perfect zero-dimensional metric spaces, and let P and K be closed nowhere dense subsets of X and Y, respectively. If f is a homeomorphism between P and K, then there exists a homeomorphism between X and Y extending f.*

3. Homeomorphisms on product spaces

For any space X let $\mathcal{H}(X)$ denote the set of all homeomorphisms of X equipped with the compact-open topology. In this section we prove that $\mathcal{H}(X)$, where X is a product of compact metric spaces, is an absolute extensor for compact 0-dimensional spaces. This is easily seen (see the proof of Theorem 3.4 below) when X is a countable product. So, the interesting case is when X is an uncountable product of metric compacta. We use the technique developed in [6].

Proposition 3.1. *Let $K \subset \mathcal{H}(X)$ be a Lindelöf subset, where $X = \prod_{\alpha\in A} X_\alpha$ is a product of compact metric spaces with $|A| = \tau > \aleph_0$. Then A can be covered by a family of sets $\{A(\alpha) : \alpha \in \omega(\tau)\}$ such that for every α we have:*

- *$A(\alpha) = \bigcup_{\gamma<\alpha} A(\gamma)$ if α is a limit ordinal;*
- *$A(\alpha) \subset A(\alpha+1)$ and $A(\alpha+1)\backslash A(\alpha)$ is countable for all α;*
- *For every $f \in K$ and $\alpha \in A$ there is $f_\alpha \in \mathcal{H}(X_{A(\alpha)})$ with $\pi_{A(\alpha)} \circ f = f_\alpha \circ \pi_{A(\alpha)}$.*

Proof. Let $B \subset A$ be a countable set. Take a sequence of open covers $\{\mathcal{U}_n\}_{n\geq 1}$ of X_B such that $\mathrm{diam}(\mathcal{U}_n) < 1/n$ for all n. Since X_B is metrizable and X is compact, the compact-open topology on the function space $C(X, X_B)$ coincides with the limitation topology. Recall that $U \subset C(X, X_B)$ is open with respect to the limitation topology if for every $f \in U$ there is $\mathcal{V} \in \mathrm{cov}(X_B)$ such that U contains the set $B(f, \mathcal{V}) = \{g \in C(X, X_B) : g \text{ is } \mathcal{V}-\text{close to } f\}$. Here, $\mathrm{cov}(X_B)$ is the family of all open covers of X_B and g is $\mathcal{V}$-close f provided for any $x \in X$ there is $V \in \mathcal{V}$ containing both points $f(x)$ and $g(x)$. In particular, every $B(f, \mathcal{V})$ contains a neighborhood $B_*(f, \mathcal{V})$ of f, see [1]. There exists a natural map $p_B : \mathcal{H}(X) \to C(X, X_B)$, $p_B(h) = \pi_B \circ h$, which is continuous when both $\mathcal{H}(X)$ and $C(X, X_B)$ carry the compact-open topology.

Claim 3.2. There is a countable set $\Gamma(B) \subset A$ containing B such that for every $f \in K$ there exist $f_{\Gamma(B)}, g_{\Gamma(B)} \in C(X_{\Gamma(B)}, X_B)$ with $\pi_B \circ f = f_{\Gamma(B)} \circ \pi_{\Gamma(B)}$ and $\pi_B \circ f^{-1} = g_{\Gamma(B)} \circ \pi_{\Gamma(B)}$.

Since for each n the family $\{B_*(\pi_B \circ f, \mathcal{U}_n) : f \in K\}$ is an open cover of $p_B(K)$, there is a sequence $\{f_{ni}\}_{i\geq 1} \subset \mathcal{H}(X)$ such that $\{B_*(\pi_B \circ$

$f_{ni}, \mathcal{U}_n) : i \geq 1\}$ covers $p_B(K)$. Because $\mathcal{H}(X)$ is a topological group, the set $K^{-1} = \{f^{-1} : f \in K\}$ is also Lindelöf. Hence, there exists a sequence $\{g_{ni}\}_{i\geq 1} \subset \mathcal{H}(X)$ for each $n \geq 1$ such that $\{B_*(\pi_B \circ g_{ni}, \mathcal{U}_n) : i \geq 1\}$ covers $p_B(K^{-1})$. Because every continuous function on X depends on countably many coordinates (for example, see [4]), there are a countable set $\Gamma(B)$ containing B and corresponding to each n sequences $\{\varphi_{ni}\}_{i\geq 1} \subset C(X_{\Gamma(B)}, X_B)$ and $\{\phi_{ni}\}_{i\geq 1} \subset C(X_{\Gamma(B)}, X_B)$ such that $\pi_B \circ f_{ni} = \varphi_{ni} \circ \pi_{\Gamma(B)}$ and $\pi_B \circ g_{ni} = \phi_{ni} \circ \pi_{\Gamma(B)}$ for all n, i. Then for every $f \in K$ and n there exists i_n such that the map $\pi_B \circ f$ is $\mathcal{U}_n$-close to $\pi_B \circ f_{ni_n}$. Because $\pi_B \circ f_{ni_n} = \varphi_{ni_n} \circ \pi_{\Gamma(B)}$, we obtain that for any $x, y \in X$ with $\pi_{\Gamma(B)}(x) = \pi_{\Gamma(B)}(y)$ we have $\pi_B(f(x)) = \pi_B(f(y))$. This means that there exists a map $f_{\Gamma(B)} \in C(X_{\Gamma(B)}, X_B)$ with $\pi_B \circ f = f_{\Gamma(B)} \circ \pi_{\Gamma(B)}$. Similarly, there exists a map $g_{\Gamma(B)} \in C(X_{\Gamma(B)}, X_B)$ with $\pi_B \circ f^{-1} = g_{\Gamma(B)} \circ \pi_{\Gamma(B)}$.

Claim 3.3. For every countable set $B \subset A$ there is a countable set $\Lambda(B) \subset A$ containing B such that for every $f \in K$ there exist homeomorphisms $f_{\Lambda(B)}, g_{\Lambda(B)} \in \mathcal{H}(X_{\Lambda(B)})$ with $\pi_{\Lambda(B)} \circ f = f_{\Lambda(B)} \circ \pi_{\Lambda(B)}$ and $\pi_{\Lambda(B)} \circ f^{-1} = g_{\Lambda(B)} \circ \pi_{\Lambda(B)}$.

Indeed, using the notations from Claim 3.2, we construct an increasing sequence $B(n)$ of countable subsets of A such that $B(0) = B$ and $B(n) = \Gamma(B(n-1))$ for every $n \geq 1$. Let $\Lambda(B) = \bigcup_{n\geq 1} B(n)$. Then for every $f \in K$ there exist maps f_n and g_n in $C(X_{B(n)}, X_{B(n-1)})$ such that

$$\pi_{B(n-1)} \circ f = f_n \circ \pi_{B(n)} \text{ and } \pi_{B(n-1)} \circ f^{-1} = g_n \circ \pi_{B(n)}.$$

The last condition implies that if $f \in K$, then for every $x, y \in X$ with $\pi_{\Lambda(B)}(x) = \pi_{\Lambda(B)}(y)$ we have $\pi_{\Lambda(B)}(f(x)) = \pi_{\Lambda(B)}(f(y))$ and $\pi_{\Lambda(B)}(f^{-1}(x)) = \pi_{\Lambda(B)}(f^{-1}(y))$. Therefore, there exist homeomorphisms $f_{\Lambda(B)}, g_{\Lambda(B)} \in \mathcal{H}(X_{\Lambda(B)})$ satisfying the required conditions.

Now, for every $\alpha < \omega(\tau)$ take a countable set $\Lambda(\alpha) \subset A$ satisfying the hypotheses of Claim 3.3 with $B = \{\alpha\}$. Let $A(\alpha) = \bigcup_{\gamma<\alpha} \Lambda(\gamma)$ if α is a limit ordinal, and $A(\alpha) = A(\alpha-1) \cup \Lambda(\alpha)$ otherwise. Since for every $\alpha \in A$ and $f \in K$ there exist $f_{\Lambda(\alpha)}, g_{\Lambda(\alpha)} \in \mathcal{H}(X_{\Lambda(\alpha)})$ with $\pi_{\Lambda(\alpha)} \circ f = f_{\Lambda(\alpha)} \circ \pi_{\Lambda(\alpha)}$ and $\pi_{\Lambda(\alpha)} \circ f^{-1} = g_{\Lambda(\alpha)} \circ \pi_{\Lambda(\alpha)}$, we have $\pi_{A(\alpha)}(f(x)) = \pi_{A(\alpha)}(f(y))$ and $\pi_{A(\alpha)}(f^{-1}(x)) = \pi_{A(\alpha)}(f^{-1}(y))$ for any pair $x, y \in X$ with $\pi_{A(\alpha)}(x) = \pi_{A(\alpha)}(y)$. This yields a homeomorphism $f_\alpha \in \mathcal{H}(X_{A(\alpha)})$ such that $\pi_{A(\alpha)} \circ f = f_\alpha \circ \pi_{A(\alpha)}$. □

The next theorem is an analogue of Mednikov's result [6, Corollary 3] stating that $\mathcal{H}([0,1]^A)$ is an absolute extensor for compact spaces.

Theorem 3.4. *Let* $X = \prod_{\alpha\in A} X_\alpha$ *be a product of compact metric spaces. Then* $\mathcal{H}(X)$ *is an absolute extensor for zero-dimensional compact spaces.*

Proof. Suppose Y is a 0-dimensional compact space and $g : P \to \mathcal{H}(X)$ be a map, where P is closed in Y. If A is countable, then $\mathcal{H}(X)$ is a complete separable metric space and we define a set-valued map $\Phi : Y \rightsquigarrow \mathcal{H}(X)$, $\Phi(y) = \{g(y)\}$ if $y \in P$ and $\Phi(y) = \mathcal{H}(X)$ otherwise. Since Φ is lower semi-continuous (the set $\Phi^{-1}(U) = \{y \in Y : \Phi(y) \cap U \neq \varnothing\}$ is open in Y for every open $U \subset \mathcal{H}(X)$), by Michael's 0-dimensional selection theorem [7], Φ admits a continuous selection $\widetilde{g} : Y \to \mathcal{H}(X)$. Obviously, $\widetilde{g}$ extends g.

Assume A is an uncountable set of cardinality τ. Then A can be covered by a family $\xi = \{A(\alpha) : \alpha \in \omega(\tau)\}$ satisfying the hypotheses of Proposition 3.1 with $K = g(P)$. Then for every $\alpha \in \omega(\tau)$ and $f \in K$ we have

$$(*)\ \pi^{A(\alpha+1)}_{A(\alpha)} \circ f_{\alpha+1} = f_\alpha \circ \pi^{A(\alpha+1)}_{A(\alpha)}.$$

Denote by $\mathcal{H}_\xi(X)$ the subspace of $\mathcal{H}(X)$ consisting of all f with the following property: For every α there is $f_\alpha \in \mathcal{H}(X_{A(\alpha)})$ such that f_α and $f_{\alpha+1}$ satisfy $(*)$. Since $K \subset \mathcal{H}_\xi(X)$, it suffice to show that $\mathcal{H}_\xi(X)$ is an absolute extensor for 0-dimensional compacta.

Because $X_{A(\alpha+1)} = X_{A(\alpha)} \times X_{A(\alpha+1)\setminus A(\alpha)}$, it follows from $(*)$ that $f_{\alpha+1}$ is of the form $f_{\alpha+1}(x, y) = (f_\alpha(x), g(x, y))$, where $x \in X_{A(\alpha)}$, $y \in X_{A(\alpha+1)\setminus A(\alpha)}$ and g is a map from $X_{A(\alpha+1)}$ into $X_{A(\alpha+1)\setminus A(\alpha)}$ such that for any $x \in X_{A(\alpha)}$ the map $\varphi_g(x)$, $\varphi_g(x)(y) = g(x, y)$, is a homeomorphism of $X_{A(\alpha)\setminus A(\alpha)}$. Therefore, by [3, Theorem 3.4.9], the correspondence $\varphi_g \to g$ is a homeomorphism between $C(X_{A(\alpha)}, \mathcal{H}(X_{A(\alpha+1)\setminus A(\alpha)}))$ and the subset of $C(X_{A(\alpha+1)}, X_{A(\alpha+1)\setminus A(\alpha)})$ consisting of all g such that for each $x \in X_{A(\alpha)}$ the map $\varphi_g(x)$ belongs to $\mathcal{H}(X_{A(\alpha)\setminus A(\alpha)})$. Hence, the correspondence $(f_\alpha, \varphi_g) \to f_{\alpha+1}$ provides a homeomorphism between the spaces $\mathcal{H}(X_{A(\alpha)}) \times C(X_{A(\alpha)}, \mathcal{H}(X_{A(\alpha)\setminus A(\alpha)}))$ and $\mathcal{H}_\alpha(X_{A(\alpha+1)})$, where $\mathcal{H}_\alpha(X_{A(\alpha+1)})$ consists of all homeomorphisms $f_{\alpha+1}$ on $X_{A(\alpha+1)}$ satisfying equality $(*)$. This means that there is one-to-one correspondence between $\mathcal{H}_\xi(X)$ and the product

$$(**)\ \mathcal{H}(X_{A(0)}) \times \Pi_{\alpha<\omega(\tau)} C(X_{A(\alpha)}, \mathcal{H}(X_{A(\alpha+1)\setminus A(\alpha)})).$$

This correspondence is a homeomorphism when all function spaces carry the compact-open topology.

It remains to show that each multiple in the product $(**)$ is an absolute extensor for 0-dimensional compacta. This is true for $\mathcal{H}(X_{A(0)})$ because $A(0)$ is countable. To show that each $C(X_{A(\alpha)}, \mathcal{H}(X_{A(\alpha+1)\setminus A(\alpha)}))$ is also an absolute extensor for 0-dimensional compacta, take a pair

$L \subset Z$ of 0-dimensional compacta and a map

$$\theta : L \to C(X_{A(\alpha)}, \mathcal{H}(X_{A(\alpha+1)\setminus A(\alpha)})).$$

Since $\mathcal{H}(X_{A(\alpha+1)\setminus A(\alpha)})$ is a separable complete metric space and $X_{A(\alpha)}$ is a compactum, $C(X_{A(\alpha)}, \mathcal{H}(X_{A(\alpha+1)\setminus A(\alpha)}))$ is a complete metric space. Then, as above, we can apply Michael's 0-dimensional selection theorem to find an extension $\widetilde{\theta} : Z \to C(X_{A(\alpha)}, \mathcal{H}(X_{A(\alpha+1)\setminus A(\alpha)}))$ of θ. □

Everywhere below by $\mathfrak{C}$ we denote the Cantor set.

Corollary 3.5. *Let P be a proper closed subset of $\mathfrak{C}^A$ and f be an autohomeomorphism of P. Suppose there exist a proper subset $B \subset A$ and an autohomeomorphism f_B of P_B such that*

- $P = P_B \times \mathfrak{C}^{A\setminus B}$;
- $f_B \circ \pi_B = \pi_B \circ f$;
- *f_B can be extended to a homeomorphism* $\widetilde{f}_B \in \mathcal{H}(\mathfrak{C}^B)$.

Then f can be extended to a homeomorphism $\widetilde{f} \in \mathcal{H}(\mathfrak{C}^A)$ such that $\widetilde{f}_B \circ \pi_B = \pi_B \circ \widetilde{f}$.

Proof. Since $f_B \circ \pi_B = \pi_B \circ f$, f is of the form $f(x,y) = (f_B(x), h(x,y))$ with $(x,y) \in P_B \times \mathfrak{C}^{A\setminus B}$ such that for each $x \in P_B$ the map φ_x, defined by $\varphi_x(y) = h(x,y)$, belongs to $\mathcal{H}(\mathfrak{C}^{A\setminus B})$. So, we have a map $\varphi : P_B \to \mathcal{H}(\mathfrak{C}^{A\setminus B})$, see [3, Theorem 3.4.9]. By Theorem 3.4, we can extend φ to a map $\widetilde{\varphi} : \mathfrak{C}^B \to \mathcal{H}(\mathfrak{C}^{A\setminus B})$ and define $\widetilde{h} : \mathfrak{C}^A \to \mathfrak{C}^{A\setminus B}$, $\widetilde{h}(x,y) = \widetilde{\varphi}(x)(y)$, where $(x,y) \in \mathfrak{C}^B \times \mathfrak{C}^{A\setminus B}$. Finally, $\widetilde{f}(x,y) = (\widetilde{f}_B, \widetilde{h}(x,y))$ provides a homeomorphism in $\mathcal{H}(\mathfrak{C}^A)$ extending f with $\widetilde{f}_B \circ \pi_B = \pi_B \circ \widetilde{f}$. □

4. Proof of Theorem 1.1

The next lemma with a little bit different formulation was proved in [8, Lemma 3.1].

Lemma 4.1. *Let X, Y be 0-dimensional paracompact spaces, $\mathfrak{C}$ be the Cantor set and $P' \subset X \times \mathfrak{C}$, $K' \subset Y \times \mathfrak{C}$ be closed sets such that $\pi_X(P') = X$ and $\pi_Y(K') = Y$. Suppose $f : P' \to K'$ and $g : X \to Y$ are homeomorphisms with $g \circ \pi_X = \pi_Y \circ f$, and there are proper closed sets $F_X \subset X$ and $F_Y \subset Y$ such that:*

(i) *$g(F_X) = F_Y$;*
(ii) *$F_X \times \mathfrak{C} \subset P'$ and $F_Y \times \mathfrak{C} \subset K'$;*
(iii) *$\pi_{\mathfrak{C}}((\{x\} \times \mathfrak{C}) \cap P')$ and $\pi_{\mathfrak{C}}((\{y\} \times \mathfrak{C}) \cap K')$ are nowhere dense in $\mathfrak{C}$ for all $x \in X\setminus F_X$ and $y \in Y\setminus F_Y$.*

Then f can be extended to a homeomorphism $\widetilde{f} : X \times \mathfrak{C} \to Y \times \mathfrak{C}$ such that $g \circ \pi_X = \pi_Y \circ \widetilde{f}$.

Suppose $g : X \to Y$ and $f : X\times Z \to Y\times Z$ are two homeomorphisms and let $\pi_X : X \times Z \to X$, $\pi_Y : Y \times Z \to Y$ be the corresponding projections. We say that f is a *fiberwise homeomorphism with respect to g* if $\pi_Y \circ f = g \circ \pi_X$.

Everywhere below we suppose that X is an uncountable power of the Cantor set, $P \subset X$ is closed and $f \in \mathcal{H}(P)$ with $f(P^{(\lambda)}) = P^{(\lambda)}$ for every cardinal λ. We follow the notations and the definitions from Section 2.

Lemma 4.2. *Let $X = \mathfrak{C}^A$ and $\Gamma \subset \Lambda$ be f-admissible subsets of A with $\Lambda\backslash\Gamma$ countable. Suppose there exists a closed set $F \subset P_\Gamma$ such that $F \times X_{\Lambda\backslash\Gamma} \subset P_\Lambda$ and $f_\Lambda(F \times X_{\Lambda\backslash\Gamma}) = f_\Gamma(F) \times X_{\Lambda\backslash\Gamma}$. Then there exists a closed G_δ-set L in P_Γ containing F and a fiberwise homeomorphism $f'_\Lambda : L \times X_{\Lambda\backslash\Gamma} \to f_\Gamma(L) \times X_{\Lambda\backslash\Gamma}$ with respect to $f_\Gamma|L$ extending the homeomorphism $f_\Lambda|((L \times X_{\Lambda\backslash\Gamma}) \cap P_\Lambda)$.*

Proof. The case $\Gamma = \Lambda$ is trivial, so let $\Lambda\backslash\Gamma \neq \varnothing$. Since $\pi^\Lambda_\Gamma \circ f_\Lambda = f_\Gamma \circ \pi^\Lambda_\Gamma$, there is a map $\phi : F \times X_{\Lambda\backslash\Gamma} \to X_{\Lambda\backslash\Gamma}$ such that $f_\Lambda(x, y) = (f_\Gamma(x), \phi(x, y))$ for all $(x, y) \in F \times X_{\Lambda\backslash\Gamma}$ and the equality $\varphi_x(y) = \phi(x, y)$ defines a homeomorphism of $X_{\Lambda\backslash\Gamma}$ for every $x \in F$. Therefore, we have a continuous map $\varphi : F \to \mathcal{H}(X_{\Lambda\backslash\Gamma})$, $\varphi(x) = \varphi_x$. Then by Theorem 3.4, φ can be extended to a map $\widetilde{\varphi} : P_\Gamma \to \mathcal{H}(X_{\Lambda\backslash\Gamma})$. Fix a countable, finitely additive base $\mathcal{B}$ of $X_{\Lambda\backslash\Gamma}$ consisting of clopen sets and let Ω be the family of all finite, disjoint clopen covers of $X_{\Lambda\backslash\Gamma}$ whose elements belong to $\mathcal{B}$. Because $\mathcal{B}$ is countable and finitely additive, Ω is countable and contains every disjoint clopen cover of $X_{\Lambda\backslash\Gamma}$. Moreover, for every $\omega = \{U_1, U_2, .., U_k\} \in \Omega$ and $x \in F$ the family $\{\{x\}\times U_1, \{x\}\times U_2, .., \{x\} \times U_k\}$ is a clopen disjoint cover of $\{x\} \times X_{\Lambda\backslash\Gamma}$ and the set

$$O_\omega(\varphi_x) = \{h \in \mathcal{H}(X_{\Lambda\backslash\Gamma}) : h(U_i) \subset V_i, i = 1, .., k\},$$

is a neighborhood of φ_x in $\mathcal{H}(X_{\Lambda\backslash\Gamma})$, where $V_i = \varphi_x(U_i)$. We have $f_\Lambda(\{x\} \times U_i) = f_\Gamma(x) \times V_i$, $i = 1, 2, .., k$. Therefore, for every $\omega \in \Omega$ and $x \in F$ there exists a neighborhood $O_\omega(x)$ of x in P_Γ such that $f_\Lambda((\{z\} \times U) \cap P_\Lambda) \subset \{f_\Gamma(z)\} \times \varphi_x(U)$ for all $U \in \omega$ and $z \in O_\omega(x)$.

Claim 4.3. If $h \in O_\omega(\varphi_x)$, then $h(U) = \varphi_x(U)$ for every $U \in \omega$.

Obviously, $h(U) \subset \varphi_x(U)$ for every $U \in \omega$. So, the claim follows from the fact that ω is a disjoint cover of $X_{\Lambda\backslash\Gamma}$ and h, φ_x are homeomorphisms of $X_{\Lambda\backslash\Gamma}$.

Then $G_\omega(x) = \widetilde{\varphi}^{-1}(O_\omega(\varphi_x)) \cap O_\omega(x)$ is a neighborhood of x in P_Γ. Let W_ω be a clopen neighborhood of F in P_Γ with $W_\omega \subset \bigcup_{x\in F} G_\omega(x)$. Finally, let $L = \bigcap_{\omega\in\Omega} W_\omega$. Then for every $z \in L$ the map $\widetilde{\varphi}_z = \widetilde{\varphi}(z)$ is a homeomorphism of $X_{\Lambda\backslash\Gamma}$ such that $\widetilde{\varphi}_z(U) \subset \varphi_x(U)$ for each $U \in \omega$

provided $z \in \widetilde{\varphi}^{-1}(O_\omega(\varphi_x))$. Now we define the homeomorphism $f'_\Lambda : L \times X_{\Lambda\setminus\Gamma} \to f_\Gamma(L) \times X_{\Lambda\setminus\Gamma}$, $f'_\Lambda(z,y) = (f_\Gamma(z), \widetilde{\varphi}_z(y))$. Let show that f'_Λ extends $f_\Lambda|((L \times X_{\Lambda\setminus\Gamma}) \cap P_\Lambda)$. If $(z,y) \in (L \times X_{\Lambda\setminus\Gamma}) \cap P_\Lambda$, then for every $\omega \in \Omega$ there is $x_\omega \in F$ and $U_\omega \in \omega$ such that $z \in G_\omega(x_\omega)$ and $y \in U_\omega$. Consequently, $\widetilde{\varphi}_z \in O_\omega(\varphi_{x_\omega})$ which, by Claim 4.3, implies $\widetilde{\varphi}_z(U_\omega) = \varphi_{x_\omega}(U_\omega)$. Hence, $f'_\Lambda(z,y) \in \{f_\Gamma(z)\} \times \widetilde{\varphi}_z(U_\omega) = \{f_\Gamma(z)\} \times \varphi_{x_\omega}(U_\omega)$. On the other hand, $z \in O_\omega(x_\omega)$ yields $f_\Lambda(z,y) \in \{f_\Gamma(z)\} \times \varphi_{x_\omega}(U_\omega)$. Therefore,

$$f'_\Lambda(z,y), f_\Lambda(z,y) \in \{f_\Gamma(z)\} \times \widetilde{\varphi}_z(U_\omega)$$

for every $\omega \in \Omega$, where U_ω is the only element from ω containing y. This means that $f'_\Lambda(z,y) = f_\Lambda(z,y)$. Indeed, otherwise there would be $\omega' \in \Omega$ and two different elements $U'_i, U'_j \in \omega'$ with $\widetilde{\varphi}_z(y) \in U'_i$ and $\pi_{\Lambda\setminus\Gamma}(f_\Lambda(z,y)) \in U'_j$. Then $\omega = \widetilde{\varphi}_z^{-1}(\omega') \in \Omega$ and $U_\omega = \widetilde{\varphi}_z^{-1}(U'_i)$, but $f_\Lambda(z,y) \notin \{f_\Gamma(z)\} \times \widetilde{\varphi}_z(U_\omega)$ while $f'_\Lambda(z,y) \in \{f_\Gamma(z)\} \times \widetilde{\varphi}_z(U_\omega)$. □

Lemma 4.4. *Let the saturated sets $C \subset B$ satisfy the hypotheses of Proposition* 2.3 *with $X = \mathfrak{C}^A$. Then for every saturated set $\Gamma \subset B$ of cardinality $\leq \lambda$ containing C there exists a saturated set $\Lambda(\Gamma) \subset B$ containing Γ such that $\Lambda(\Gamma)\setminus\Gamma \neq \varnothing$ is countable and the homeomorphism $f_{\Lambda(\Gamma)}$ can be extended to a fiberwise homeomorphism $\widetilde{f}_{\Lambda(\Gamma)} : P_\Gamma \times X_{\Lambda(\Gamma)\setminus\Gamma} \to P_\Gamma \times X_{\Lambda(\Gamma)\setminus\Gamma}$ with respect to f_Γ.*

Proof. Recall that for every f-admissible set $T \subset A$ we denote by $P_T^{(\lambda)}$ the set $\pi_T(P^{(\lambda)})$. Since B and Γ are saturated and B is of cardinality λ^+, there is a saturated set $\Lambda(1) \subset B$ containing Γ such that $\Lambda(1)\setminus\Gamma \neq \varnothing$ is countable. Observe that $\pi_\Gamma^{-1}(P_\Gamma^{(\lambda)}) = P^{(\lambda)}$ and $\pi_{\Lambda(1)}^{-1}(P_{\Lambda(1)}^{(\lambda)}) = P^{(\lambda)}$ because $\pi_C^{-1}(P_C^{(\lambda)}) = P^{(\lambda)}$. The last two equalities together with $f(P^{(\lambda)}) = P^{(\lambda)}$ imply that $P_{\Lambda(1)}^{(\lambda)} = P_\Gamma^{(\lambda)} \times X_{\Lambda(1)\setminus\Gamma}$, $f_\Gamma(P_\Gamma^{(\lambda)}) = P_\Gamma^{(\lambda)}$ and $f_{\Lambda(1)}(P_{\Lambda(1)}^{(\lambda)}) = P_{\Lambda(1)}^{(\lambda)}$.

Then, we can apply Lemma 4.2 for the pair $\Gamma \subset \Lambda(1)$ to obtain a closed G_δ-set L_1 in P_Γ containing $P_\Gamma^{(\lambda)}$ and a fiberwise homeomorphism $f_1 : L_1 \times X_{\Lambda(1)\setminus\Gamma} \to f_\Gamma(L_0) \times X_{\Lambda(1)\setminus\Gamma}$ with respect to $f_\Gamma|L_1$ extending $f_{\Lambda(1)}|((L_1 \times X_{\Lambda(1)\setminus\Gamma}) \cap P_{\Lambda(1)})$. Since the sets $F_1 = P_\Gamma\setminus L_1$ and $F'_1 = P_\Gamma\setminus f_\Gamma(L_1)$ are σ-compact disjoint from $P_\Gamma^{(\lambda)}$, according to Lemma 2.5 there is a saturated set $\Lambda(2) \subset B$ containing $\Lambda(1)$ such that $\Lambda(2)\setminus\Lambda(1) \neq \varnothing$ is countable and the sets $\pi_{\Lambda(2)\setminus\Gamma}^{\Lambda(2)}((\pi_\Gamma^{\Lambda(2)})^{-1}(x) \cap P_{\Lambda(2)})$ and $\pi_{\Lambda(2)\setminus\Gamma}^{\Lambda(2)}((\pi_\Gamma^{\Lambda(2)})^{-1}(y) \cap P_{\Lambda(2)})$ are nowhere dense in $X_{\Lambda(2)\setminus\Gamma}$ for all $x \in F_1$ and $y \in F'_1$.

Since $P^{(\lambda)}_{\Lambda(2)}) = P^{(\lambda)}_{\Lambda(1)} \times X_{\Lambda(2)\backslash\Lambda(1)}$ and $f_{\Lambda(2)}(P^{(\lambda)}_{\Lambda(2)}) = P^{(\lambda)}_{\Lambda(2)}$, we can apply Lemma 4.2 (with $F = P^{(\lambda)}_{\Lambda(1)})$ for the saturated sets $\Lambda(1) \subset \Lambda(2)$. Therefore, there exist a closed G_δ-set L'_2 in $P_{\Lambda(1)}$ containing $P^{(\lambda)}_{\Lambda(1)})$ and a fiberwise homeomorphism $f'_2 : L'_2 \times X_{\Lambda(2)\backslash\Lambda(1)} \to f_{\Lambda(1)}(L'_2) \times X_{\Lambda(2)\backslash\Lambda(1)}$ with respect to $f_{\Lambda(1)}|L'_2$ extending $f_{\Lambda(2)}|((L'_2 \times X_{\Lambda(2)\backslash\Lambda(1)}) \cap P_{\Lambda(2)})$. Since $L_1 \times X_{\Lambda(1)\backslash\Gamma}$ is a G_δ-set in $P_\Gamma \times X_{\Lambda(1)\backslash\Gamma}$ and $(\pi^{\Lambda(1)}_\Gamma)^{-1}(P^{(\lambda)}_\Gamma) = P^{(\lambda)}_{\Lambda(1)}$, there exists a closed G_δ-set L_2 in P_Γ containing $P^{(\lambda)}_\Gamma$ such that $L_2 \subset L_1$ and $L_2 \times X_{\Lambda(1)\backslash\Gamma} \subset L'_2$. So, $L_2 \times X_{\Lambda(2)\backslash\Gamma} \subset L'_2 \times X_{\Lambda(2)\backslash\Lambda(1)}$. Let f_2 be the restriction of f'_2 on $L_2 \times X_{\Lambda(2)\backslash\Gamma}$. Then f_2 is a fiberwise homeomorphism between $L_2 \times X_{\Lambda(2)\backslash\Gamma}$ and $f_\Gamma(L_2) \times X_{\Lambda(2)\backslash\Gamma}$ with respect to $f_{\Lambda(1)}|(L_2 \times X_{\Lambda(1)\backslash\Gamma})$ extending $f_{\Lambda(2)}|((L_2 \times X_{\Lambda(2)\backslash\Gamma}) \cap P_{\Lambda(2)})$.

In this way we can construct an increasing sequence $\{\Lambda(n)\}$ of saturated subsets of B each containing Γ, a decreasing sequence $\{L_n\}$ of closed G_δ-sets in P_Γ each containing $P^{(\lambda)}_\Gamma$, and homeomorphisms f_n between $L_n \times X_{\Lambda(n)\backslash\Gamma}$ and $f_\Gamma(L_n) \times X_{\Lambda(n)\backslash\Gamma}$ extending $f_{\Lambda(n)}|((L_n \times X_{\Lambda(n)\backslash\Gamma}) \cap P_{\Lambda(n)})$ such that for each n we have:

(a) $\Lambda(n+1)\backslash\Lambda(n) \neq \varnothing$ is countable;
(b) The set $\pi^{\Lambda(n+1)}_{\Lambda(n+1)\backslash\Gamma}((\pi^{\Lambda(n+1)}_\Gamma)^{-1}(x) \cap P_{\Lambda(n+1)})$ is nowhere dense in $X_{\Lambda(n+1)\backslash\Gamma}$ for all $x \in F_n = P_\Gamma\backslash L_n$;
(c) The set $\pi^{\Lambda(n+1)}_{\Lambda(n+1)\backslash\Gamma}((\pi^{\Lambda(n+1)}_\Gamma)^{-1}(y) \cap P_{\Lambda(n+1)})$ is nowhere dense in $X_{\Lambda(n+1)\backslash\Gamma}$ for all $y \in F'_n = P_\Gamma\backslash f_\Gamma(L_n)$;
(d) Each f_{n+1} is a fiberwise homeomorphism with respect to the restriction $f_n|(L_{n+1} \times X_{\Lambda(n)\backslash\Gamma})$.

Let $\Lambda(\Gamma) = \bigcup_{n\geq 0} \Lambda(n)$ and $L = \bigcap_{n\geq 0} L(n)$. For every n consider the map $p_n : L \times X_{\Lambda(n+1)\backslash\Gamma} \to L \times X_{\Lambda(n)\backslash\Gamma}$ defined by $p_n(x,y) = (x, \pi^{\Lambda(n+1)}_{\Lambda(n)}(y))$. Then $L \times X_{\Lambda(\Gamma)\backslash\Gamma}$ is the limit of the inverse sequence $\{L \times X_{\Lambda(n)\backslash\Gamma}, p_n\}$ and $f_\Gamma(L) \times X_{\Lambda(\Gamma)\backslash\Gamma}$ is the limit of the inverse sequence $\{f_\Gamma(L) \times X_{\Lambda(n)\backslash\Gamma}, p_n\}$. Since $p_n \circ f_{n+1} = f_n \circ p_n$ for every n, the homeomorphisms f_n provide a homeomorphism f_∞ between $L \times X_{\Lambda(\Gamma)\backslash\Gamma}$ and $f_\Gamma(L) \times X_{\Lambda(\Gamma)\backslash\Gamma}$ extending $f_{\Lambda(\Gamma)}|((L \times X_{\Lambda(\Gamma)\backslash\Gamma}) \cap P_{\Lambda(\Gamma)})$. Moreover, items (b) and (c) imply that the sets $\pi^{\Lambda(\Gamma)}_{\Lambda(\Gamma)\backslash\Gamma}((\pi^{\Lambda(\Gamma)}_\Gamma)^{-1}(x) \cap P_{\Lambda(\Gamma)})$ and $\pi^{\Lambda(\Gamma)}_{\Lambda(\Gamma)\backslash\Gamma}((\pi^{\Lambda(\Gamma)}_\Gamma)^{-1}(y) \cap P_{\Lambda(\Gamma)})$ are nowhere dense in $X_{\Lambda(\Gamma)\backslash\Gamma}$ for all $x \in P_\Gamma\backslash L$ and $y \in P_\Gamma\backslash f_\Gamma(L)$. Finally, by Lemma 4.1, f_∞ can be extended to a homeomorphism $\widetilde{f}_{\Lambda(\Gamma)}$ between $P_\Gamma \times X_{\Lambda(\Gamma)\backslash\Gamma}$ and $P_\Gamma \times X_{\Lambda(\Gamma)\backslash\Gamma}$. Obviously, $\widetilde{f}_{\Lambda(\Gamma)}$ is a fiberwise extension of $f_{\Lambda(\Gamma)}$ with respect to f_Γ. □

Corollary 4.5. *Let the saturated sets $C \subset B$ satisfy the hypotheses of Proposition* 2.3 *with $X = \mathfrak{C}^A$. Suppose also that f_C can be extended* *to a homeomorphism* $\widetilde{f}_C$ *of* X_C*. Then the homeomorphism* f_B *can be extended to a homeomorphism* $\widetilde{f}_B$ *of* X_B *such that* $\pi^B_C \circ \widetilde{f}_B = \widetilde{f}_C \circ \pi^B_C$.

Proof. Using Lemma 4.4 (and the notations from that lemma), we can represent B as an increasing family $\{B(\alpha) : \alpha < \omega(\lambda^+)\}$ of saturated sets $B(\alpha)$ each of cardinality $\leq \lambda$ such that:

(3) $B(0) = C$;
(4) $B(\alpha) = \bigcup_{\beta<\alpha} B(\beta)$ if α is a limit ordinal;
(5) $B(\alpha+1) = \Lambda(B(\alpha))$, so $B(\alpha+1)\backslash B(\alpha)$ is countable for all α;
(6) $f_{B(\alpha+1)}$ can be extended to a fiberwise homeomorphism $f'_{B(\alpha+1)} : P_{B(\alpha)} \times X_{B(\alpha+1)\backslash B(\alpha)} \to P_{B(\alpha)} \times X_{B(\alpha+1)\backslash B(\alpha)}$ with respect to $f_{B(\alpha)}$.

We are going to prove that each $f_{B(\alpha)}$ can be extended to a homeomorphism $\widetilde{f}_{B(\alpha)}$ of $X_{B(\alpha)}$ satisfying the following equality

(7) $\pi^{B(\alpha+1)}_{B(\alpha)} \circ \widetilde{f}_{B(\alpha+1)} = \widetilde{f}_{B(\alpha)} \circ \pi^{B(\alpha+1)}_{B(\alpha)}$.

The proof is by transfinite induction. The extension $\widetilde{f}_{B(0)}$ exists according to our assumption since $B(0) = C$. If $\widetilde{f}_{B(\beta)}$ is defined for each $\beta < \alpha$, where α is a limit ordinal, then item (4) implies the existence of $\widetilde{f}_{B(\alpha)}$. Therefore, we need only to define $\widetilde{f}_{B(\alpha+1)}$ provided $\widetilde{f}_{B(\alpha)}$ exists. To that end, since $B(\alpha+1) = \Lambda(B(\alpha))$, according to Lemma 4.4, $f_{B(\alpha+1)}$ can be extended to a fiberwise homeomorphism $f'_{B(\alpha+1)}$ of $P_{B(\alpha)} \times X_{B(\alpha+1)\backslash B(\alpha)}$ with respect to $f_{B(\alpha)}$. Then, by Corollary 3.5, $f'_{B(\alpha+1)}$ is extended to a homeomorphism $\widetilde{f}_{B(\alpha+1)}$ satisfying condition (7). □

Proof of Theorem 1.1. Let show first that the proof is reduced to the case of one subset $P \subset D^A$ and an autohomeomorphism $f \in \mathcal{H}(P)$. Indeed, take two disjoint copies X and Y of D^A with $P \subset X$ and $K \subset Y$, and let $Q = P \biguplus K$ be the disjoint union of P and K. Obviously, $X \biguplus Y$ is homeomorphic to D^A and $g = f \biguplus f^{-1}$ is an autohomeomorphism of Q with $g(Q^{(\lambda)}) = Q^{(\lambda)}$ for all cardinals λ. Suppose $f \biguplus f^{-1}$ can be extended to a homeomorphism $F : X \biguplus Y \to X \biguplus Y$. Choose two clopen neighborhoods X' and Y' of P and K in X and Y, respectively, with $X\backslash X' \neq \varnothing \neq Y\backslash Y'$ such that $F(X') = Y'$. Then there is a homeomorphism $G : X\backslash X' \to Y\backslash Y'$, and $F|X'$ and G provide a homeomorphism $\widetilde{f} : X \to Y$ extending f. Therefore, we can suppose that we have one subset P of D^A and a homeomorphism $f \in \mathcal{H}(P)$. Because f preserves the interior of P, we can also assume P is nowhere dense in D^A. Moreover, we identify D^A with $X = \mathfrak{C}^A$, where A is an uncountable set of cardinality τ (we also identify any infinite cardinal number λ with the first ordinal of cardinality λ). Recall that $\mathfrak{L} = \mathfrak{L}_P$ denotes the set of all infinite cardinal numbers $\mu \leq \tau$ with $P^{(\mu)} \neq \varnothing$, and let $\mathfrak{L}' = \mathfrak{L} \cup \{\aleph_0\}$. Obviously, if $\lambda \in \mathfrak{L}$, then $\mu \in \mathfrak{L}$ for all $\mu \geq \lambda$, in particular, $\tau \in \mathfrak{L}$. Take a functionally open and dense subset U of $X \backslash P$ and a countable saturated set $C \subset A$ with $\pi_C^{-1}(\pi_C(U)) = U$ (that is possible because every continuous function on X depends on countably many coordinates). Then P_C is nowhere dense in X_C. If $\aleph_0 \notin \mathfrak{L}$, we put $\Gamma(\aleph_0) = C$. In case $\aleph_0 \in \mathfrak{L}$, by Proposition 2.1 and [2], there exits a countable saturated set $C_1 \subset A$ with $\pi_{C_1}^{-1}(\pi_{C_1}(P^{(\aleph_0)})) = P^{(\aleph_0)}$, and we denote $\Gamma(0) = C \cup C_1$. Therefore, in both cases $P_{\Gamma(\aleph_0)}$ is nowhere dense in $X_{\Gamma(0)}$, $\Gamma(0)$ is a countable saturated set and $\pi_{\Gamma(0)}^{-1}(P_{\Gamma(0)}^{(\aleph_0)}) = P^{(\aleph_0)}$.

The case when $P^{(\lambda)} = \varnothing$ for all $\lambda < \tau$ was settled in [8]. Hence, there are another two passible cases: either $P \neq P^{(\lambda)}$ for all $\lambda < \tau$ or $P = P^{(\lambda)}$ for some $\lambda < \tau$.

Claim 4.6. The case $P = P^{(\lambda)}$ for some $\lambda < \tau$ can be reduced to one of the following two cases: (i) $P^{(\lambda)} = \varnothing$ for all $\lambda < \tau$; (ii) $P \neq P^{(\lambda)}$ for all $\lambda < \tau$.

Indeed, suppose $P = P^{(\lambda_0)}$ for some $\lambda_0 < \tau$ and $\mathfrak{L}^d$ is discrete in $\mathfrak{L}$. We can assume that λ_0 is the minimal cardinal with $P = P^{(\lambda_0)}$. Then $P = P^{(\lambda)}$ for all $\lambda \geq \lambda_0$ and $P = P_{A(\lambda_0)} \times X_{A \backslash A(\lambda_0)}$. So, $\mathfrak{L}_{P_{A(\lambda_0)}}$ is a subset of $\mathfrak{L}_P$ and consists either of λ_0 or, except λ_0, contains some $\lambda < \lambda_0$. If $\mathfrak{L}_{P_{A(\lambda_0)}} = \{\lambda_0\}$, then $(P_{A(\lambda_0)})^{(\lambda)} = \varnothing$ for all $\lambda < \lambda_0$. Hence, we can apply [8] to extend the homeomorphism $f_{\lambda_0} = f_{A(\lambda_0)}$ to a homeomorphism $\widetilde{f}_{\lambda_0} \in H(X_{A(\lambda_0)})$. Finally, Corollary 3.5 provides a homeomorphism $\widetilde{f} \in H(X)$ extending f.

Suppose $\mathfrak{L}_{P_{A(\lambda_0)}}$, except λ_0, contains some $\lambda < \lambda_0$. Then, we construct by transfinite induction saturated sets $A(\lambda) \subset A$, $\lambda \leq \lambda_0$, such that each $A(\lambda)$ has cardinality λ, $A(\lambda) \subset A(\lambda_0)$ and $\pi_{A(\lambda)}^{-1}(P_{A(\lambda)}^{(\lambda)}) = P^{(\lambda)}$. We claim that the homeomorphism f_{λ_0} preserves the λ-interiors of $P_{A(\lambda_0)}$ for all $\lambda \leq \lambda_0$. This is obvious for $\lambda = \lambda_0$ because $(P_{A(\lambda_0)})^{(\lambda_0)} = P_{A(\lambda_0)}$. Suppose $\lambda < \lambda_0$. Since $P^{(\lambda)} = \pi_{A(\lambda)}^{-1}(P_{A(\lambda)}^{(\lambda)})$ and $P_{A(\lambda_0)}^{(\lambda)} = (\pi_{A(\lambda)}^{A(\lambda_0)})^{-1}(P_{A(\lambda)}^{(\lambda)})$ we have $P_{A(\lambda_0)}^{(\lambda)} \subset (P_{A(\lambda_0)})^{(\lambda)}$. On the other hand, the inclusion $(\pi_{A(\lambda_0)})^{-1}((P_{A(\lambda_0)})^{(\lambda)}) \subset P^{(\lambda)}$ implies $(P_{A(\lambda_0)})^{(\lambda)} \subset P_{A(\lambda_0)}^{(\lambda)}$. Hence, $(P_{A(\lambda_0)})^{(\lambda)} = P_{A(\lambda_0)}^{(\lambda)}$ and, because $f(P^{(\lambda)}) = P^{(\lambda)}$ and $\pi_{A(\lambda_0)} \circ f = f_{\lambda_0} \circ \pi_{A(\lambda_0)}$, f_{λ_0} preserves the λ-interior of $P_{A(\lambda_0)}$. Moreover, the equalities $(P_{A(\lambda_0)})^{(\lambda)} = P_{A(\lambda_0)}^{(\lambda)}$ and $\pi_{A(\lambda_0)}^{-1}(\pi_{A(\lambda_0)}(P^{(\lambda)})) = P^{(\lambda)}$ for $\lambda \leq \lambda_0$ imply that $\mathfrak{L}_{P_{A(\lambda_0)}}^d$ is discrete in $\mathfrak{L}_{P_{A(\lambda_0)}}$. Hence, assuming that the case (ii)

is established, we can extend f_{λ_0} to a homeomorphism $\widetilde{f}_{\lambda_0} \in H(X_{A(\lambda_0)})$. As before, Corollary 3.5 provides a homeomorphism $\widetilde{f} \in H(X)$ extending f. That completes the proof of Claim 4.6.

Therefore, we can assume that $P \neq P^{(\lambda)}$ for all $\lambda < \tau$. Denote by $\mathfrak{L}^c$ the limit cardinals at which $\mathfrak{L}$ is continuous. We are going first to construct an increasing family $\{\Gamma(\lambda) : \aleph_0 \leq \lambda < \tau\}$ of saturated subsets of A covering A such that:

(8) $\Gamma(\lambda)$ is of cardinality λ and $\Gamma(\aleph_0)$ is the set constructed above;

(9 Every compact set $F \subset P_{\Gamma(\lambda^+)} \backslash P^{(\lambda)}_{\Gamma(\lambda^+)}$ is λ^+-negligible in $X_{\Gamma(\lambda^+)}$;

(10) $\pi^{-1}_{\Gamma(\lambda)}(P^{(\lambda)}_{\Gamma(\lambda)}) = P^{(\lambda)}$;

(11) $\Gamma(\lambda) = \bigcup_{\gamma<\lambda} \Gamma(\gamma)$ if $\lambda \in \mathfrak{L}^c$.

The construction is by transfinite induction. We cover A by an increasing family $\{A(\lambda) : \aleph_0 \leq \lambda \leq \tau\}$ with each $A(\lambda)$ having cardinality λ. Suppose the sets $\Gamma(\lambda)$ are already constructed for all $\lambda < \gamma$, where $\gamma < \tau$, such that $\Gamma(\aleph_0)$ is the set constructed above. If $\gamma = \lambda^+$ for some λ, by Proposition 2.3, there exists a saturated set $\Gamma(\gamma)$ of cardinality γ containing $\Gamma(\lambda) \cup A(\gamma)$ such that $\pi^{-1}_{\Gamma(\gamma)}(P^{(\gamma)}_{\Gamma(\gamma)}) = P^{(\gamma)}$ and every compact set in $P_{\Gamma(\gamma)} \backslash P^{(\lambda)}_{\Gamma(\gamma)}$ is γ-negligible in $X_{\Gamma(\gamma)}$. If γ is a limit cardinal and $\gamma \in \mathfrak{L}^c$, let $\Gamma(\gamma)$ be the union of all $\Gamma(\lambda)$, $\lambda < \gamma$. In this case, since each $\Gamma(\lambda)$, $\lambda < \gamma$, satisfies condition (10) and $\gamma \in \mathfrak{L}^c$, $\Gamma(\gamma)$ also satisfies (10). If γ is a limit cardinal and $\gamma \in \mathfrak{L}^d$, we take $\Gamma(\gamma)$ to be a set containing $\bigcup_{\lambda<\gamma} \Gamma(\lambda)$ with $\pi^{-1}_{\Gamma(\gamma)}(P^{(\gamma)}_{\Gamma(\gamma)}) = P^{(\gamma)}$.

Since $\mathfrak{L}^d$ is discrete in $\mathfrak{L}$ for every $\lambda \in \mathfrak{L}^d$ there exists $\delta(\lambda) < \lambda$ such that the interval $[\delta(\lambda), \lambda)$ is disjoint from $\mathfrak{L}^d$. This implies that if $\lambda_1, \lambda_2 \in \mathfrak{L}^d$ and $\lambda_1 < \lambda_2$, then $\delta(\lambda_1) < \delta(\lambda_2)$. Now, we are going to refine the construction of the sets $\Gamma(\lambda)$ and obtain a new increasing family $\{\Lambda(\lambda) : \lambda < \tau\}$ of saturated sets covering A such that:

(12) $\Lambda(\lambda)$ is of cardinality λ and $\Lambda(\aleph_0) = \Gamma(\aleph_0)$;

(13) Every compact set $F \subset P_{\Lambda(\lambda^+)} \backslash P^{(\lambda)}_{\Lambda(\lambda^+)}$ is λ^+-negligible in $X_{\Lambda(\lambda^+)}$;

(14) $\pi^{-1}_{\Lambda(\lambda)}(P^{(\lambda)}_{\Lambda(\lambda)}) = P^{(\lambda)}$;

(15) $\Lambda(\lambda) = \bigcup_{\gamma<\lambda} \Lambda(\gamma)$ for every limit λ;

(16) If $\lambda \in \mathfrak{L}^d$, then $\Gamma(\gamma) \subset \Lambda(\gamma)$ for all $\gamma \in (\delta(\lambda), \lambda)$ such that $\bigcup_{\gamma \in (\delta(\lambda),\lambda)} \Lambda(\gamma) = \Gamma(\lambda)$.

We fix $\lambda \in \mathfrak{L}^d$ and let $\Gamma(\lambda) \backslash \Gamma(\delta(\lambda)) = \bigcup_{\delta(\lambda)<\gamma<\lambda} C(\gamma)$, where $\{C(\gamma) : \delta(\lambda) < \gamma < \lambda\}$ is an increasing family. Since $\Gamma(\lambda)$ is saturated, $\Gamma(\lambda) = \bigcup_{\alpha \in \Gamma(\lambda)} B(\alpha)$ with all $B(\alpha)$ being countable saturated sets. Define $\Lambda(\delta(\lambda))$ to be $\Gamma(\delta(\lambda))$ and assume that the sets $\Lambda(\eta)$ have been defined for all $\eta < \gamma$, where $\gamma \in (\delta(\lambda), \lambda)$. If $\gamma = \eta^+$, we take $\Lambda(\gamma) \subset \Gamma(\lambda)$ to be a saturated set of cardinality γ containing $\Gamma(\gamma)\cup C(\gamma)$ such that $\Lambda(\gamma)$ satisfies conditions $(13)-(14)$, see Proposition 2.3(ii). If $\gamma<\lambda$ is a limit cardinal, then let $\Gamma(\gamma)=\bigcup_{\delta(\lambda)<\eta<\gamma}\Lambda(\eta)$. In this case, since $\gamma\in\mathfrak{L}^c$ and $\pi^{-1}_{\Lambda(\eta)}(P^{(\eta)}_{\Lambda(\eta)})=P^{(\eta)}$, we also have $\pi^{-1}_{\Lambda(\gamma)}(P^{(\gamma)}_{\Lambda(\gamma)})=P^{(\gamma)}$. Observe that all limit cardinals outside the intervals $(\delta(\lambda),\lambda]$, $\lambda\in\mathfrak{L}^d$, belong to $\mathfrak{L}^c$. So, conditions (11) and (16) imply condition (15). Hence, repeating this construction for every $\lambda\in\mathfrak{L}^d$, we complete the constructions of the sets $\Lambda(\lambda)$.

Now, we can finalize the proof of Theorem 1.1. Denote by f_λ the homeomorphism $f_{\Lambda(\lambda)}$. Since A is the union of all $\Lambda(\lambda)$, it suffices to extend each f_λ to a homeomorphism $\widetilde{f}_\lambda\in H(X_{\Lambda(\lambda)})$ such that

(17) $\pi^{\Lambda(\lambda)}_{\Lambda(\mu)}\circ\widetilde{f}_\lambda=\widetilde{f}_\mu\circ\pi^{\Lambda(\lambda)}_{\lambda(\mu)}$ for all $\mu<\lambda\leq\tau$.

Suppose the homeomorphisms $\widetilde{f}_\lambda$ are defined for all $\lambda<\gamma$. Assume γ is a limit cardinal. Then by (15), $\Lambda(\gamma)=\bigcup_{\lambda<\gamma}\Lambda(\lambda)$, and we define $\widetilde{f}_\gamma$ to be the limit of all $\widetilde{f}_\lambda$, $\lambda<\gamma$. If $\gamma=\lambda^+$ for some λ, then by conditions $(13)-(14)$ we can apply Corollary 4.5 to find an extension $\widetilde{f}_\gamma$ of f_γ satisfying condition (17). □

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