

SOME PROPERTIES of $C_p(X)$
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Abstract: There is found out a necessary and sufficient condition on a space X for $C_p(X)$ to have the following property (τ) : any family of G_τ -subsets of $C_p(X)$ contains a dense subfamily of cardinality $\leq \tau$.

Key words: $C_p(X)$, property (τ) , property $(\tau)^*$.

Classification: 54C35, 54C30

Introduction. B. Efimov [5] proved that every dyadic space X has the following property (referred below as "the property (τ) "): any family \mathcal{F} of G_τ -subsets of X contains a subfamily \mathcal{K} of cardinality $|\mathcal{K}| \leq \tau$ such that the set $\cup\{F: F \in \mathcal{K}\}$ is dense in $\cup\{F: F \in \mathcal{F}\}$ (such a subfamily will be called a *sense subfamily*). Here τ is an infinite cardinal. This result was generalized by E. Ščepin [8] to the class of all continuous images of k -metrizable compacta. In [10] we consider an extension of the class of all k -metrizable compact spaces, namely the class of all completely regular spaces with lattices of d -open [9] mappings. Every element X of the latter class has the following property: any family \mathcal{F} of subsets of X with $\chi(F, X) \leq \tau$ for every $F \in \mathcal{F}$ contains a dense subfamily \mathcal{K} of cardinality $|\mathcal{K}| \leq \tau$. The following question arises in connection with the last fact: Does every space with a lattice of d -open mappings have the property (τ) ?

In the present paper we answer this question in negative by finding a necessary and sufficient condition on a space X for $C_p(X)$ to have the property (τ) (let us note that $C_p(X)$ always has a lattice of d -open mappings [11]). We also give a negative answer to the following problem of A. Arhangel'skii ([1], problem 7.26): Is it true that there exists a continuous injection from X into an Eberlein compactum provided $C_p(X)$ contains a dense

\mathcal{C} -compact space?

Notations. We shall consider only completely regular spaces and continuous mappings.

Given spaces X and Y we denote by $C_p(X, Y)$ the set of all mappings from X to Y , endowed with the topology of pointwise convergence. If $A \subset X$ then p_A will denote the mapping from $C_p(X, Y)$ to $C_p(A, Y)$ defined by $p_A(f) = f|_A$ for every $f \in C_p(X, Y)$. $C_p(X)$ stands for $C_p(X, \mathbb{R})$, where \mathbb{R} is the real line with the usual topology. Let f be a mapping from X to Z . By f^* is denoted the dual mapping from $C_p(Z, Y)$ to $C_p(X, Y)$.

A space X is said to be τ -Lindelöf if every open covering of X contains a subcovering of cardinality $\leq \tau$. We write $hl(X) \leq \tau$ if X is hereditarily τ -Lindelöf. By $d(X)$ is denoted the density of X and by $hd(X)$ the hereditary density of X .

If \mathcal{F} is a family of subsets of X then $\bigcup \mathcal{F}$ denotes the set $\bigcup \{F : F \in \mathcal{F}\}$.

The property (τ) and related properties of $C_p(X)$. We need the following result which is proved by P. Zenor [13].

Lemma 1. For a space X and an infinite cardinal τ the following are equivalent:

- (i) $hl(X^{\omega_0}) \leq \tau$ ($hd(X^{\omega_0}) \leq \tau$, respectively);
- (ii) $hd(C_p(X, Y)) \leq \tau$ ($hl(C_p(X, Y)) \leq \tau$, respectively) for every space Y with $w(Y) \leq \tau$.

Remark 1. Let a set $M \subset C_p(X)$ separate the points from the closed subsets of X i.e. for every closed P in X and every $x \notin P$ there exists $f \in M$ with $f(x) = 0$ and $f(P) = 1$. Suppose $hd(M) \leq \tau$. Using the Veličko arguments ([12], the proof of Theorem 1) one can show that $hl(X^n) \leq \tau$ for each $n \in \mathbb{N}$.

Lemma 2. Let $\mathcal{K} = \bigcup \{\mathcal{K}_n : n \in \mathbb{N}\}$, where \mathcal{K}_n are families of subsets of $C_p(X, \mathbb{R}^\tau)$. Suppose $\{A(n) : n \in \mathbb{N}\}$ is an increasing sequence of closed subsets of X such that $p_{A(n+1)}^{-1}(p_{A(n+1)}(F)) = F$ for every $F \in \mathcal{K}_n$ and every $n \in \mathbb{N}$. Then $f \in \overline{\bigcup \mathcal{K}}$ provided $p_{A(n)}(f) \in p_{A(n)}(\bigcup \mathcal{K}_n)$ for each $n \in \mathbb{N}$.

Proof. Put $A = \cup \{A(n) : n \in \mathbb{N}\}$. Let $U = \{f \in C_p(X, R^\tau) : f(x_i) \in V_i, i=1, \dots, k\}$ be a neighbourhood of f in $C_p(X, R^\tau)$. We can suppose that A meets $\{x_i : i=1, \dots, k\}$. Then $U = U_1 \cap U_2$, where $U_1 = \{f \in C_p(X, R^\tau) : f(x_i) \in V_i \text{ for each } x_i \in A\}$ and $U_2 = \{f \in C_p(X, R^\tau) : f(x_i) \in V_i \text{ for each } x_i \notin A\}$. Choose m such that $A \cap \{x_i : i=1, \dots, k\} \subset A(m)$. Since $p_{A(m)}(U)$ is open in $p_{A(m)}(C_p(X, R^\tau))$ we have $p_{A(m)}(U) \cap p_{A(m)}(\cup \mathcal{K}_m) \neq \emptyset$. Hence $p_{A(m)}(F_0) \cap p_{A(m)}(U) \neq \emptyset$ for some $F_0 \in \mathcal{K}_m$. Then $U_1 \cap F_0 \neq \emptyset$. Let $g_1 \in U_1 \cap F_0$. There exists $g_2 \in U_2$ such that $g_2|_{A(m+1)} = g_1|_{A(m+1)}$. Thus, $g_2 \in F_0 \cap U$. Therefore $f \in \overline{\cup \mathcal{K}}$.

Let τ be an infinite cardinal. A space X has the property $(\tau)^*$ if $hl(A^n) \leq \tau$ for every closed A in X with $d(A) \leq \tau$ and every $n \in \mathbb{N}$.

Theorem 1. For a space X and an infinite cardinal τ the following are equivalent:

- (i) X has the property $(\tau)^*$;
- (ii) $C_p(X, R^\lambda)$ has the property (τ) for every $\lambda \leq \tau$;
- (iii) $C_p(X, R)$ has the property (τ) .

Proof. (i) \rightarrow (ii). Let \mathcal{F} be a family of G_τ -subsets of $C_p(X, R^\lambda)$, where $\lambda \leq \tau$. We can suppose that for every $F \in \mathcal{F}$ there exists a set $A(F) \subset X$ of cardinality $\leq \tau$ with $p_{A(F)}^{-1}(p_{A(F)}(F)) = F$.

For every $n \in \mathbb{N}$ we construct a set $B(n) \subset X$ and a subfamily $\mathcal{K}_n \subset \mathcal{F}$ such that:

- 1) $B(n) \subset B(n+1)$, $|B(n)| \leq \tau$ and $|\mathcal{K}_n| \leq \tau$;
- 2) $p_{A(n+1)}^{-1}(p_{A(n+1)}(F)) = F$ for every $F \in \mathcal{K}_n$, where $A(k) = \overline{B(k)}$ for $k \in \mathbb{N}$;
- 3) $p_{A(n)}(\cup \mathcal{K}_n)$ is dense in $p_{A(n)}(\cup \mathcal{F})$.

Assume we have already defined $B(n)$ and \mathcal{K}_n satisfying 1) - 3). Put $B(n+1) = B(n) \cup (\cup \{A(F) : F \in \mathcal{K}_n\})$. Obviously $|B(n+1)| \leq \tau$ and $F = p_{A(n+1)}^{-1}(p_{A(n+1)}(F))$ for every $F \in \mathcal{K}_n$. By Lemma 1 we have $hd(C_p(A(n+1), R^\lambda)) \leq \tau$. So, there exists a subfamily \mathcal{K}_{n+1} of \mathcal{F} such that $|\mathcal{K}_{n+1}| \leq \tau$ and $p_{A(n+1)}(\cup \mathcal{K}_{n+1})$ is dense in $p_{A(n+1)}(\cup \mathcal{F})$. Denote $\mathcal{K} = \cup \{\mathcal{K}_n : n \in \mathbb{N}\}$, $B = \cup \{B(n) : n \in \mathbb{N}\}$ and $A = \cup \{A(n) : n \in \mathbb{N}\}$. Let $f \in p_A^{-1}(p_A(\overline{\cup \mathcal{F}}))$. Since

$p_{A(n)}(f) \in \overline{p_{A(n)}(\mathcal{U}\mathcal{F})} = \overline{p_{A(n)}(\bigcup \mathcal{K}_n)}$, it follows from Lemma 2 that $f \in \overline{\mathcal{K}}$, i.e. $p_A^{-1}(p_A(\overline{\mathcal{U}\mathcal{F}})) = \overline{\mathcal{K}} = \overline{\mathcal{U}\mathcal{F}}$. The implication is proved. Let us note that $p_B^{-1}(\overline{\mathcal{U}\mathcal{F}}) = \overline{\mathcal{U}\mathcal{F}}$ because B is dense in A ; hence $\overline{\mathcal{U}\mathcal{F}}$ is a union of G_τ -subsets of $C_p(X, R^\lambda)$.

(ii) \rightarrow (iii). This implication is obvious.

(iii) \rightarrow (i). Suppose A is a closed subset of X with $d(A) \leq \tau$. Then there is an injection from $p_A(C_p(X))$ into a space of weight $\leq \tau$, so that every point of $p_A(C_p(X))$ is a G_τ -set in $p_A(C_p(X))$. Hence $hd(p_A(C_p(X))) \leq \tau$. It follows from Remark 1 that $hl(A^n) \leq \tau$ for each $n \in N$.

Corollary 1. Let X be a monolithic space in the sense of Arhangel'skii [2]. Then $C_p(X)$ has the property (τ) for every infinite cardinal τ .

Corollary 2. For a space X and an infinite cardinal τ the following conditions are equivalent:

(i) $C_p(X)$ has the property (τ) ;

(ii) $C_p(X^\lambda)$ has the property (τ) for every $\lambda \leq \tau$.

Proof. In view of Theorem 1 it is enough to prove that X^λ has the property (τ) provided X does. Let A be a closed subset of X^λ with $d(A) \leq \tau$. Put $B = \bigcup \{q_s(A) : s < \lambda\}$, where $q_s : X^\lambda \rightarrow X$ is the projection onto the s -th factor. Obviously, $d(B) \leq \tau$, so $hl(B^n) \leq \tau$ for every $n \in N$. Hence, $hl(B^\lambda) \leq \tau$ (see [13], Theorem 3). Since $A \subset B^\lambda$ we have $hl(A^n) \leq \tau$ for each $n \in N$.

Lemma 3. Suppose $X = \prod \{X_s : s \in S\}$, where $|S| \leq \tau$. Then X has the property (τ) provided $\prod \{X_s : s \in B\}$ does for every finite set $B \subset S$.

The proof of Lemma 3 is similar to the proof of Theorem 3* from [13].

Theorem 2. Let X^n have the property (τ) for every $n \in N$. Then $C_p(X, Z)$ has the property $(\tau)^*$ for every space Z with $w(Z) \leq \tau$.

Proof. Suppose A is closed in $C_p(X, Z)$ and $d(A) \leq \tau$. Put $p = \Delta \{f : f \in A\}$ and $Y = p(X)$. Since $d(A) \leq \tau$, there exists an injection from Y to a space of weight not greater than τ . So,

every point of Y^{ω_0} is a G_τ -set in Y^{ω_0} . By Lemma 3, X^{ω_0} has the property (τ) . Thus, $hd(Y^{\omega_0}) \leq \tau$ because Y^{ω_0} is a continuous image of X^{ω_0} . It follows from Lemma 1 that $hl(C_p(X, Z^{\omega_0})) \leq \tau$. Next, for every $f \in A$ there exists $h_f \in C_p(Y, Z)$ such that $f = h_f \circ p$. Denote $B = \{h_f : f \in A\}$. Clearly, B^{ω_0} can be considered as a subspace of $C_p(Y, Z^{\omega_0})$; hence $hl(B^{\omega_0}) \leq \tau$. But $p^*(B) = A$, so $hl(A^{\omega_0}) \leq \tau$.

Corollary 3. A space X has the property $(\tau)^*$ if and only if $C_p(C_p(X))$ has the property $(\tau)^*$.

Proof. Since $C_p(C_p(X))$ contains X , all we have to prove is the "only if" part. Suppose X has the property $(\tau)^*$. By Theorem 1, $C_p(X, R^{\omega_0})$ has the property (τ) , so the space $C_p(X)^{\omega_0}$ does, too. The Theorem 2 completes the proof.

Combining Corollary 3 and Theorem 1 we get: $C_p(X)$ has the property (τ) if and only if $C_p(C_p(X))$ has the property $(\tau)^*$. Therefore, Theorem 2 is inversable in the case $X = C_p(Y)$ for some Y . The following example shows that this is not the case in general.

Example 1. Let X be a nonmetrizable Eberlein compactum. Then $C_p(X)$ has the property $(\omega_0)^*$ being a monolithic space (see [1]). Suppose X has the property (ω_0) . Then X is separable, because X contains a dense metrizable space (ss [6]). But every separable Eberlein compactum is metrizable [3] - a contradiction.

Let τ be an infinite cardinal. A space X is said to have the property $[\tau]$ if for every family \mathcal{F} of G_τ -subsets of X and every $x \in \overline{\bigcup \mathcal{F}}$ there is a subfamily \mathcal{K} of \mathcal{F} such that $x \in \overline{\bigcup \mathcal{K}}$ and $|\mathcal{K}| \leq \tau$. X has the property $[\tau]^*$ if $l(A^n) \leq \tau$ for every $n \in \mathbb{N}$, where A is a closed subset of X with $d(A) \leq \tau$.

Theorem 3. For a space X and an infinite cardinal τ the following conditions are equivalent:

- (i) X has the property $[\tau]^*$;
- (ii) $C_p(X, R^\lambda)$ has the property $[\tau]$ for every $\lambda \leq \tau$;
- (iii) $C_p(X)$ has the property $[\tau]$.

Proof. (i) \rightarrow (ii). Let \mathcal{F} be a family of G_τ -subsets of

$C_p(X, R^\lambda)$ and $f \in \overline{\cup \mathcal{F}}$. Using the same notations as in the proof of Theorem 1, for every $n \in \mathbb{N}$ we construct a set $B(n) \subset X$ and a family $\mathcal{K}_n \subset \mathcal{F}$ such that:

- 1) $|B(n)| \leq \tau$, $|\mathcal{K}_n| \leq \tau$ and $B(n+1) = B(n) \cup (\cup \{A(F) : F \in \mathcal{K}_n\})$;
- 2) $p_{A(n+1)}^{-1} p_{A(n+1)}(F) = F$ for every $F \in \mathcal{K}_n$;
- 3) $p_{A(n)}(f) \in \overline{p_{A(n)}(\cup \mathcal{K}_n)}$.

The following result (actually established by Arhangel'skii [2]) will be used below: If $l(X^n) \leq \tau$ for every $n \in \mathbb{N}$, then $t(C_p(X, R^\lambda)) \leq \tau$ where $\lambda \leq \tau$.

Suppose we have defined $B(n)$ and \mathcal{K}_n . Put $B(n+1) = B(n) \cup (\cup \{A(F) : F \in \mathcal{K}_n\})$ and $A(n+1) = \overline{B(n+1)}$. Then $l(A(n+1))^k \leq \tau$ for every $k \in \mathbb{N}$ because $d(A(n+1)) \leq \tau$. By the result mentioned above (Arhangel'skii [2]), $t(C_p(A(n+1), R^\lambda)) \leq \tau$. Since $p_{A(n+1)}(f) \in \overline{p_{A(n+1)}(\cup \mathcal{F})}$, there exists a subfamily \mathcal{K}_{n+1} of \mathcal{F} such that $p_{A(n+1)}(f) \in \overline{p_{A(n+1)}(\cup \mathcal{K}_{n+1})}$ and $|\mathcal{K}_{n+1}| \leq \tau$. Thus, $B(n)$ and \mathcal{K}_n are constructed for every $n \in \mathbb{N}$. Let $\mathcal{K} = \cup \{\mathcal{K}_n : n \in \mathbb{N}\}$. Obviously $|\mathcal{K}| \leq \tau$. It follows from Lemma 2 that $f \in \overline{\cup \mathcal{K}}$.

(ii) \rightarrow (iii). This implication is obvious.

(iii) \rightarrow (i). Suppose A is closed in X with $d(A) \leq \tau$. Since every point of $p_A(C_p(X))$ is a G_τ -set in $p_A(C_p(X))$ and the mapping p_A is open (see [1]), we have $t(p_A(C_p(X))) \leq \tau$. Observe that $p_A(C_p(X))$ separates the points from the closed sets in A . Now, the following result (actually established by E. Pytkeev [7]) completes the proof: Let $t(M) \leq \tau$ and M separate the points from the closed subsets in Y , where $M \subset C_p(Y)$. Then $l(Y^n) \leq \tau$ for every $n \in \mathbb{N}$.

Corollary 4. For a space X and an infinite cardinal τ the following are equivalent:

- (i) $C_p(X)$ has the property $[\tau]$;
- (ii) $C_p(X^n)$ has the property $[\tau]$ for every $n \in \mathbb{N}$.

Proof. In view of Theorem 3, it suffices to show that X^n has the property $[\tau]^*$ for every $n \in \mathbb{N}$ provided X does. The last proposition can be proved by the same arguments as Corollary 2 was.

Theorem 4. Let τ be an infinite cardinal and $C_p(C_p(X))$ have the property $[\tau]^*$. Then $C_p(X)$ has the property $[\tau]$.

Proof. Let $f \in \overline{\cup \mathcal{F}}$, where \mathcal{F} is a family of G_τ -subsets of $C_p(X)$. Using the same notations as in the proof of Theorem 1, we define sets $B(n) \subset X$ and subfamilies \mathcal{K}_n of \mathcal{F} with the following properties:

- 1) $B(n) \subset B(n+1)$, $|B(n)| \leq \tau$ and $|\mathcal{K}_n| \leq \tau$;
- 2) $p_{A(n+1)}^{-1}(p_{A(n+1)}(F)) = F$ for every $F \in \mathcal{K}_n$;
- 3) $p_{A(n)}(f) \in \overline{p_{A(n)}(\cup \mathcal{K}_n)}$.

Suppose we have already constructed $B(i)$ and \mathcal{K}_i for $i \leq n$.

Put $B(n+1) = B(n) \cup (\cup \{A(F) : F \in \mathcal{K}_n\})$ and $A(n+1) = \overline{B(n+1)}$. Since $d(A(n+1)) \leq \tau$, there exists a one-to-one mapping from $P(n+1) = p_{A(n+1)}(C_p(X))$ onto a space of weight $\leq \tau$, so $d(C_p(P(n+1))) \leq \tau$. Hence, $l(C_p(P(n+1))) \leq \tau$ because $C_p(P(n+1))$ is closed in $C_p(C_p(X))$. By a result of Asanov [4], we have $t(P(n+1)) \leq \tau$. Therefore, there is a subfamily \mathcal{K}_{n+1} of \mathcal{F} such that $p_{A(n+1)}(f) \in \overline{p_{A(n+1)}(\cup \mathcal{K}_{n+1})}$ and $|\mathcal{K}_{n+1}| \leq \tau$. Hence $B(n)$ and \mathcal{K}_n are defined for every $n \in \mathbb{N}$. Denote $\mathcal{K} = \cup \{\mathcal{K}_n : n \in \mathbb{N}\}$. Obviously $|\mathcal{K}| \leq \tau$. It follows from Lemma 2 that $f \in \overline{\cup \mathcal{K}}$.

Corollary 5. A space X has the property $[\tau]^*$ provided $C_p(C_p(X))$ does.

Example 2. There exists a space X with the property $[\omega_0]$ but $C_p(X)$ does not have the property $[\omega_0]^*$.

Let X be a discrete space of cardinality c . Then $C_p(X) = R^c$. The space R^c does not have the property $[\omega_0]^*$ because it is separable and non-normal.

Finally, we give a negative answer to the question of Arhangel'skii ([1], problem 7.26), mentioned above. Let X be a non-metrizable Eberlein compactum. Using the arguments of Arhangel'skii ([1], the proof of Proposition 6.9) one can show that $C_p(C_p(X))$ contains a \mathfrak{c} -compact subspace (this follows also from a more general result of V. Tkačuk). Suppose that there exists an injection j from $C_p(X)$ into an Eberlein compactum Y . Then the closure of $j(C_p(X))$ in Y is also an Eberlein compactum satisfying the countable chain condition. Hence $w(j(C_p(X))) \leq \omega_0$ (see [3]). Next, it is well known that $C_p(X)$ contains a \mathfrak{c} -compact dense subset M . Therefore $d(M) \leq \omega_0$ because M is a countable

union of compact metrizable spaces. Consequently X is metrizable - a contradiction.

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