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# MILUTIN MAPPINGS AND AE(0)-SPACES

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**1. Introduction.** Dugundji spaces were introduced by Pelczynski [8]. Later Haydon [6] proved that the class of Dugundji spaces coincides with the class of all compact absolute extensors for 0-dimensional spaces (briefly, AE(0)). Recently Chigogidze [2] extended the notion of AE(0) from the class of compact spaces to that of completely regular spaces. In this note a characterization of AE(0)-spaces in the sense of Chigogidze, which is similar to Pelczynski's definition of Dugundji spaces, is given. We also consider mappings having a regular averaging operator and give a linear topological classification of some function spaces.

**2. Notations.** All spaces considered are completely regular and all single-valued mappings are continuous. A mapping $f$ from $Y$ to $X$, where $Y\subset Z$, is called $Z$-normal if for every continuous function $g$ on $X$, the function $g\circ f$ is continuously extendable to $Z$. A space $X$ is called an absolute extensor for 0-dimensional spaces [2] if every $Z$-normal mapping from $Y$ to $X$, where $Y\subset Z$ and $\dim Z=0$, is continuously exendable to $Z$. Here dim stands for the dimensions defined by finite functionally open covers. A mapping $f$ from $X$ to $Y$ will be called 0-soft [2] if for every 0-dimensional space $Z$ and every two $Z$-normal mappings $g: Z_0\to X$, $h: Z_1\to Y$ with $Z_0\subset Z_1\subset Z$ and $f\circ g=h|Z_0$, there exists a $Z$-normal mapping $k: Z_1\to X$ such that $g=k|Z_0$ and $f\circ k=h$.

By $C(X)$ (resp. $C^*(X)$) we denote the real vector space of all continuous (and bounded) real-valued functions on $X$. We will consider two topologies on $C(X)$: the topology of uniform convergence and the compact-open topology. These topologies are distinguished by using subscripts, namely, $C_u(X)$ and $C_k(X)$. It is well known that $C_k(X)$ is a topological vector space, whereas $C_u(X)$ is a topological vector space iff no continuous function on $X$ is unbounded. For a compact space $X$ by $P(X)$ is denoted the space of all regular probability measures on $X$ endowed with the weak-star topology. For a given space $X$, $P_\beta(X)$ (see [3]) denotes the space $\{\mu\in P(\beta X): \operatorname{supp}\mu\subset X\}$, where $\operatorname{supp}\mu$ stands for the support of $\mu$. It follows from the definition of $P_\beta(X)$ that every $\mu\in P_\beta(X)$ is a continuous positive linear functional on $C_u^*(X)$ with $\mu(1_X)=1$. We can consider $\mu$ as a continuous linear functional $\mu'$ on $C_u(X)$, defined by $\mu'(g)=\mu(g|\operatorname{supp}\mu)$. There exists a natural embedding $i: X\hookrightarrow P_\beta(X)$ defined by $i(x)=\delta_x$, where $\delta_x$ is Dirac's measure at the point $x$. One can easily show that $i(X)$ is a closed $C$-embedded subset of $P_\beta(X)$. If $f$ is a mapping from $X$ to $Y$ we denote by $P_\beta(f)$ the natural mapping from $P_\beta(X)$ to $P_\beta(Y)$.

Let $u: C_u(X)\to C_u(Y)$ be a regular operator, i. e. a positive linear mapping with $u(1_X)=1_Y$. We can define a mapping $r: Y\to P(\beta X)$ by letting $r(y)(h)=u(h)(y)$ for every $h\in C^*(X)$. The operator $u$ is said to have compact supports if $r(Y)$ is contained

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in $P_\beta(X)$. On the other hand, if there exists a mapping $r$ from $Y$ to $P_\beta(X)$, the formula $u(h)(y)=r(y)(h)$ for all $h\in C(X)$, defines a regular operator $u: C_u(X)\to C_u(Y)$ with compact supports.

Let $X\subset Y$ and $u: C_u(X)\to C_u(Y)$ be a regular operator such that $u(h)|X=h$ for every $h\in C(X)$. Then $u$ is said to be an extension operator. It is clear that there exists an extension operator $u: C_u(X)\to C_u(Y)$ with compact supports iff there exists a mapping $r$ from $Y$ to $P_\beta(X)$ such that $r(x)=\delta_x$ for every $x\in X$. Such a mapping $r$ will be called a $P_\beta$-valued retraction from $Y$ to $X$.

A mapping $f: X\to Y$ satisfies the zero-dimensional lifting propety for z. d. l. p[10] if for any 0-dimensional space $Z$ and for every mapping $g: Z\to Y$ there exists a mapping $k: Z\to X$ such that $f\circ k=g$. Clearly, every 0-soft mapping satisfies the z. d. l. p.

**3. Milutin Mappings.** A mapping $f: X\to Y$ is called a Milutin mapping if there exists a mapping $g: Y\to P_\beta(X)$ such that $\operatorname{supp} g(y)\subset f^{-1}(y)$ for every $y\in Y$. Such a mapping $g$ will be called associated with $f$. Evidently, every Milutin mapping is a surjection. If $f: X\to Y$ is a Milutin mapping and $g: Y\to P_\beta(X)$ is associated with $f$, we can define a corresponding to $g$ regular operator $u: C_u(X)\to C_u(Y)$ with compact supports by the formula $u(h)(y)=g(y)(h)$. Obviously, $u(l\circ f)=l$ for all $l\in C(Y)$. Such an operator $u$ is said to be an averaging operator for $f$ [8]. Conversely, if $f$ allows a regular averaging operator with compact supports then $f$ is a Milutin mapping. Every Milutin mapping admits an averaging operator in the sense of Čoban [1]. If $X$ and $Y$ are compact spaces, our definition coincides with the definition of a Milutin mapping, given by Pelczynski [8]. Clearly, if $f: X\to Y$ is a Milutin mapping and $Y_0\subset Y$ then the restriction $f|f^{-1}(Y_0)$ is also a Milutin mapping.

**Proposition 3.1.** Let $f: X\to Y$ be a Milutin mapping with an associated mapping $g: Y\to P_\beta(X)$. Then $g$ is an embedding and $g(Y)$ is a closed $C$-embedded subset of $P_\beta(X)$.

**Ppoposition 3.2.** Every Milutin mapping is a quotient mapping.

**Proposition 3.3.** Every product of Milutin mappings is a Milutin mapping.

**Proposition 3.4.** Let $S_1=\{X_\alpha, p_\alpha^\gamma, \alpha, \gamma\in A\}$, $S_2=\{Y_\alpha, q_\alpha^\gamma, \alpha, \gamma\in A\}$ be inverse systems such that $p_\alpha^\gamma(X_\gamma)=X_\alpha$ for all $\gamma>\alpha$. Suppose we are given Milutin mappings $f_\alpha: X_\alpha\to Y_\alpha$ with associated mappings $g_\alpha: Y_\alpha\to P_\beta(X_\alpha)$ sucht hat $q_\alpha^\gamma\circ f_\gamma=f_\alpha\circ p_\alpha^\gamma$ and $g_\alpha\circ q_\alpha^\gamma=P_\beta(p_\alpha^\gamma)\circ g_\gamma$ for every $\gamma>\alpha$. Then the limit mapping $f: \varprojlim S_1\to\varprojlim S_2$ is a Milutin mapping.

**Theorem 3.5.** Let $X$ and $Y$ be metrizable spaces and $f: X\to Y$ be a surjection. Then the following conditions are equivalent:

(i) $f$ is a Milutin mapping;

(ii) the set-valued mapping $f^{-1}: Y\to X$ admits a l. s. c. compact-valued selection;

(iii) $f$ satisfies the z. d. l. p.

In the case when $X$ and $Y$ are compact metric spaces, the equivalences (i)$\leftrightarrow$(ii) and (i)$\leftrightarrow$(iii) were proved by Ditor ([4], Theorem 3.4) and by Hoffmann ([10], Corollary 1), respectively.

**Corollary 3.6.** Let $X$ be a product of metrizable spaces. Then there exists a 0-dimensional space $Y$, which is a product of metrizable spaces, and a perfect Milutin mapping from $Y$ to $X$ satisfying the z. d. l. p.

**Corollary 3.7.** For every $p$ paracompact space $X$ there exists a 0-dimensional $p$-paracompact space $Y$ and a perfect Milutin mapping from $Y$ to $X$ satisfying the z. d. l. p.

**Proposition 3.8.** Every 0-soft mapping is a Milutin mapping.

**Theorem 3.9.** Let $X$ be a product of metrizable spaces and $f: X\to Y$ be a Milutin mapping. Then

(i) $Y$ is collectionwise normal with respect to the family of all closed $G_\delta$-subsets of $Y$;

(ii) every closure of a union of $G_\delta$-subsets of $Y$ is a zero-set in $Y$.

**Corollary 3.10.** Every AE(0)-space $X$ is collectionwise normal with respect to the family of all closed $G_\delta$-subsets of $X$.

**Corollary 3.11.** Let a space $Y$ satisfy the assumptions of Theorem 3.9 and each point of $Y$ be a $G_\delta$-set. Then $Y^\omega$ is perfectly normal and collectionwise normal.

**Theorem 3.12.** Let $X$ be a limit space of an inverse system with polish spaces and open projections. The every Milutin image of $X$ is also a limit space of an inverse system with polish spaces and open projections.

**Theorem 3.13** The paracompactness as well as the ($\tau$-)collectionwise normality are preserved by Milutin mappings.

**Corollary 3.14.** Every Milutin image of a (complete) metric space is also a (complete) metric space.

**Theorem 3.15.** Let $f\colon X \to Y$ be a Milutun mapping. Then $f$ preserves the following topological properties: stratifibility, $\delta$-metrizability and perfectly $k$-normality. If in addition $\mathrm{cl}_X(f^{-1}(U)) = f^{-1}(\mathrm{cl}_Y(U))$ for every open subset $U$ of $Y$, then $Y$ is $k$-metrizable (resp., has a $k$-metrizable compactification) provided $X$ is $k$-metrizable (resp., has a $k$-metrizable compactification).

**Corollary 3.16.** Every AE(0)-space has a $k$-metrizable compactification.

**4. AE(0)-spaces.** A space $X$ is said to have the property ($r$) if $X$ is a retract of $P_\beta(X)$.

**Proposition 4.1.** Let $K$ be a convex subset of $R^A$ for some $A$, satisfying the following condition: the closed (in $K$) convex hull of every compact subset of $K$ is also compact. Then $K$ has the property ($r$).

**Theorem 4.2.** For a space $X$ the following conditions are equivalent:

(i) $X$ is an AE(0)-space;

(ii) for every $C$-embedding of $X$ in a space $Y$ there exists a regular extension operator $u\colon C_u(X) \to C_u(Y)$ having compact supports.

If $X$ is compact the condition (ii) coincides with Pelczynski's definition of Dugundji spaces.

**Corollary 4.3.** Let $X$ be an AE(0)-space and $K$ have the property ($r$). Suppose $X$ is $C$-embedded in a space $Y$. Then every mapping from $X$ to $K$ is continuously extendable to $Y$.

**Theorem 4.4.** $P_\beta(X)$ is an absolute retract iff $X$ is a compact AE(0)-space of weight $\leq \omega_1$.

In the proof of Theorem 4.4 we use a result of Ditor and Haydon [6] and the following

**Proposition 4.5** If $P_\beta(X)$ is Čech-complete then $X$ is pseudocompact.

A space $X$ satisfies (*) if we connot write $X = \bigcup_{n=1}^{\infty} X_n$, where the $X_n$ are closed subspaces with $w(X_n) < w(X)$. Haydon [7] proved that a Dugundji space of weight $\tau$ satisfying (*) contains a copy of $D^\tau$.

**Proposition 4.6** Let $X$ be a non-compact AE(0)-space of weight $\tau$ satisfying (*).

(i) If $X$ is locally compact, then $X$ contains a copy of $N \times D^\tau$ as a closed subset.

(ii) If $X$ is nowhere locally compact, then $X$ contains a copy of $N^\omega \times D^\tau$ as a closed subset.

**5. Linear Topological Classification of Spaces of Continuous Functions.**

**Theorem 5.1.** Let an AE(0)-space $X$ of weight $\tau$ contain a $C$-embedded copy of $N^\tau$. Then $C_k(X)$ is linearly homeomorphic to $C_k(N^\tau)$.

**Theorem 5.2.** Let a $p$-paracompact AE(0)-space $X$ of weight $\tau$ contain a closed copy of $N^\omega \times D^\tau$. Then $C_k(X)$ is linearly homeomorphic to $C_k(N^\omega \times D^\tau)$.

**Corollary 5.3.** Let $X$ be a $p$-paracompact, nowhere locally compact AE(0)-space of weight $\tau$ satisfying (*). Then $C_k(X)$ is linearly homeomorphic to $C_k(N^\omega \times D^\tau)$.

**Corollary 5.4.** Let $X$ be a non-compact, locally compact AE(0)-space of weight $\tau$ satisfying (*). Then $C_k(X)$ is linearly homeomorphic to $C_k(N \times D^\tau)$.

**6. AE(0, $\tau$)-spaces.** A mapping $f\colon Y \to X$ is called ($Z$, $\tau$)-normal, where $Y \subset Z$, if for every mapping $g\colon X \to B$ into a Banach space of weight $\leq \tau$, the mapping $g.f$ is continuously extendable to $Z$. A space $X$ is called $\tau$-absolute extensor for 0-dimensional spaces (br., AE(0, $\tau$)) if every $(Z, \tau)$-normal mapping from $Y$ to $X$, where $Y \subset Z$ and $\dim Z = 0$, is continuously extendable to $Z$. Obviously, the class of all AE(0)-spaces coincides with the class of all AE (0, $\omega$)-spaces and if $\lambda < \tau$ then every AE(0, $\lambda$)-space is an AE(0, $\tau$)-space.

**Proposition 6.1 (i).** Every pinnate AE(0, $\tau$)-space is Čech-complete.

(ii) A metric space $X$ is an AE(0, $\tau$)-space iff $X$ is Čech-complete and $w(X) \leq \tau$.

A subset $A$ of $X$ is $P^{\tau}$-embedded in $X$ [9] if every mapping from $A$ into a Banach space of weight $\leq \tau$ can be continuously extended onto X. For a space $X$ denote $w_{\tau}(X) = \min\{\operatorname{card} A: X$ is $P^{\tau}$-embedded in a product $\Pi\{M_{\alpha}: \alpha \in A\}$, where each $M_{\alpha}$ is a complete metric space of weight $\tau$. Below $B(\tau)$ stands for the Baire space of weight $\tau$.

**Proposition 6.2.** Let an AE(0, $\tau$)-space $X$ with $w_{\tau}(X) = \lambda$ contain a $P^{\tau}$-embedded copy of $B(\tau)^{\lambda}$. Then $C_k(X)$ is linearly homeomorphic to $C_k(B(\tau)^{\lambda})$.

**Corollary 6.3.** Let $X = \Pi\{X_{\alpha};\ \alpha \in A\}$, where each $X_{\alpha}$ is a complete metric space with $w(X_{\alpha}) = \tau$ and $\operatorname{card} A = \lambda \geq \omega$. Then $C_k(X)$ is linearly homeomorphic to $C_k(B(\tau)^{\lambda})$.

**Proposition 6.4.** Let $X$ be a $p$-paracompact AE(0, $\tau$)-space with $w_{\tau}(X) = \lambda$. Then

(i) $C_k(X)$ is linearly homeomorphic to $C_k(D^{\lambda} \times B(\tau))$ provided $X$ contains a closed copy of $B(\tau) \times D^{\lambda}$;

(ii) $C_k(X)$ is linearly homeomorphic to $C_k(T \times D^{\lambda})$ provided $X$ is locally compact and $X$ contains a closed copy of $T \times D^{\lambda}$, where $T$ is a discrete set of cardinality $\tau$.

**Corollary 6.5.** Suppose $X = M \times \Pi\{X_{\alpha}: \alpha \in A\}$, where $\operatorname{card} A = \lambda$, $M$ is a complete metric space of weight $\tau$ and each $X_{\alpha}$ is a compact metric space. Then

(i) $C_k(X)$ is linearly homeomorphic to $C_k(B(\tau) \times D^{\lambda})$ provided $M$ is nowhere locally compact and $w(U) = \tau$ for every open subset $U$ of $M$;

(ii) $C_k(X)$ is linearly homeomorphic to $C_k(T \times D^{\lambda})$ provided $M$ is locally compact

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