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SET-VALUED MAPS AND AE(0)-SPACES

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There are several characterizations of compact AE(0)-spaces in terms of set-valued maps e. g. [3, 4, 6]. On the other hand we know the only result of such type for non-compact spaces [8]. We announce two theorems below. The first one describes the arbitrary AE(0)-spaces, and is new even in the compact case. The second one extends the corresponding result of Dranishnikov and characterizes arbitrary AE(1)-spaces as continuum-valued and in a sense strongly upper semi-continuous retracts of the infinite powers of the real line.

1. Preliminaries. All spaces considered are completely regular and Hausdorff and all single-valued maps are continuous. Below τ always denotes an infinite cardinal. We say that the R-weight of a space X does not exceed a cardinal τ (and write $R\text{-}w(X)\leq\tau$) if X can be embedded in R^τ as a C-embedded subspace. For a map $f:X\to Y$ let $C(f)\to C(Y)\to C(X)$ be the induced by F operator, where $C(X)$ is the set of all real-valued (continuous) maps on X. If $X\subseteq Y$, then $C(Y)/X$ is the set of all elements of $C(X)$ extendable to the whole of Y. Clearly, the equality $C(X)=C(Y)/X$ characterizes the C-embedded subspaces. By dim we denote the dimension defined by finite functionally open covers.

A space X is called [1, 2] an absolute extensor in dimension n ($n=0, 1, \ldots,$) (briefly, AE(n)-space) if for any space Z of dimension $\dim Z\leq n$ and any subspace Z_0 of it each map $f:Z_0\to X$ such that $C(f)(C(X))\subset C(Z)/Z_0$ can be extended to the whole of Z.

It is known that each AE(0)-space is realcompact [2]. Below we need the definition of n-soft maps only between realcompact spaces ([2], Proposition 1.8): A surjection f: $X\to Y$ between realcompact spaces is said to be n-soft ($n=0, 1, \ldots,$) if for any realcompact space Z of dimension $\dim Z\leq n$, any closed subspace Z_0 of it, and any two maps g: $Z_0\to X$ and h: $Z\to Y$ such that $C(g)(C(X))\subset C(Z)/Z_0$, there exists a map k: $Z\to X$ such that $f.k=h$ and $k/Z_0=g$.

All set-valued maps are closed-valued. By usco map we mean an u. s. c. compact-valued map.

2. Maps Having the Selection-Factorization Property and AE(0)-Spaces. Nedev [5] gave a unified method for proving selection and factorization theorems for set-valued maps. The key point in Nedev's paper is the notion of set-valued map having the selection-factorization property. We consider maps with a similar property.

A set-valued map F: $X\to Y$ has weak selection-factorization property (briefly, F is w.s.f.p.) if for every functionally closed subset H of X and every countable family $\mathcal{U}$ consisting of functionally open subsets of Y such that $F^{-1}(\mathcal{U})=\{F^{-1}(U): U\in\mathcal{U}\}$ covers H, there exists a locally finite functionally open (in H) cover of H refining $F^{-1}(\mathcal{U})$.

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The following facts were actually proved by Nedev [5]: A l. s. c. map $F: X \to Y$ s w. s. f. p. if X is paracompact or X is normal and F is compact valued.

Proposition 2.1. The following conditions are equivalent for a space X:

(i) X is a Polish space;

(ii) Any w. s. f. p. map $F: Y \to X$, where $\dim Y = 0$, has a selection.

In the proof of Proposition 2.1 we use the following lemma:

Lemma 2.2. If any w. s. f. p. map $F: Y \to X$, where $\dim Y = 0$, has a selection, then X is an AE(0)-space.

A closed-valued map $F: X \to Y$ is said to have a countable singularity if the collection of all images of F, containing at least two points, is countable and consists of functionally closed subsets of Y.

Theorem 2.3. The following conditions are equivalent for any space X:

(i) X is an AE(0)-space;

(ii) Any w. s. f. p. map $F: Y \to X$ with a countable singularity, where $\dim Y = 0$ has a selection.

3. Upper Semi-Continuous Maps and AE(1)-Spaces.

An u. s. c. map $F: X \to Y$ is said to be strongly u. s. c. if for any functionally open subset U of Y the set $F^{\#}(U) = \{x \in X : F(x) \subset U\}$ is functionally open in X.

Dranishnikov [3] proved that a compactum X is an AE(1)-space iff for every embedding of Y into I^{τ} there exists a continuum-valued upper semi-continuous retraction from I^{τ}_{4} to Y. Theorem 3.1 below is an analog of Dranishnikov's result.

Theorem 3.1. The following conditions are equivalent for any space X:

(i) X is an AE(1)-space;

(ii) For any C-embedding of X into R^{τ} there exists a continuum-valued strongly u. s. c. retraction from R^{τ} onto X;

(iii) There exists a C-embedding of X into R^{τ} (for $\tau = R\text{-}w(X)$) and a continuum-valued strongly u. s. c retraction from R^{τ} onto X.

The proof of theorem 3.1 is based on the following lemmas:

Lemma 3.2. Let $S_X = \{X_\alpha, p^{\beta}_{\alpha}, A\}$ and $S_Y = \{Y_\alpha, q^{\beta}_{\alpha}, A\}$ be two factorizing sigma-spectra with surjective limit projections and $r: \varinjlim S_X \to \varinjlim S_Y$ be a strongly usco map. Then the set $A' = \{\alpha \in A :$ there exists a usco map $r_\alpha: X_\alpha \to Y_\alpha$ such that $q_\alpha . r = r_\alpha . p_\alpha\}$ is cofinal and sigma-complete [7] in A.

Lemma 3.3. Let $f: X \to Y$ be a functionally open surjection between AE(0)-spaces. Suppose that X is C-embedded in the product $Y \times R^{\omega}$ and $f = \pi / X$, where $\pi: Y \times R^{\omega} \to Y$ is the natural projection. If there exists a continuum-valued strongly u. s. c. retraction $r: Y \times R^{\omega} \to X$ such that $f_0 r = \pi$, then f is 1-soft.

Question. Let a space X be a continuum-valued u. s. c. retraction of R^{τ}, $\tau \geq \omega$. Is X an AE(1)-space?

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