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# YET ANOTHER CHARACTERIZATION OF AE(0)-SPACES

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It is shown that every absolute $P_\beta$-valued retract is an AE(0)-space. This generalizes a previous result of the author.

1. Introduction. The notion of absolute extensor for zerodimensional spaces in the class of all completely regular spaces (br. AE(0)) was introduced by Chigogidze [1]. In the case of a compact space this notion coincides with the notion of Dugundji space introduced by Pelczynski [5]. There are several characterizations of AE(0)-spacevs (see, for example, [1],[3],[9]). Recently the author [10] gave another characterization of AE(0)-spaces, which is similar to Pelczynski's definition of Dugundji spaces. An equivalent form of this characterization is the following one: a completely regular space $X$ is an AE(0)-space if for every C-embedding of $X$ in a space $Y$ there exists a probability-valued retraction $r$ from $Y$ to $X$. In the present note we show that if in the last proposition $r$ is an upper semi-continuous compact-valued (br.usco) mapping $X$ is also an AE(0)-space. This fact is new even in the compact case.

2. Notations. All spaces considered are completely regular and all single-valued mappings - continuous. A mapping $f$ from $Y$ to $X$, where $Y\subset Z$, is called $Z$-normal if for every continuous function $g$ on $X$ the function $g\circ f$ is continuously extendable to $Z$. A space $X$ is called an AE(0)- space [1] if every $Z$-normal mapping from $Y$ to $X$ (where $Y\subset Z$ and $\dim Z=0$) is continuously extendable to $Z$. A mapping f from $X$ to $Y$ will be called $0$-soft [1] if for every $0$-dimensional space $Z$ and every two $Z$-normal mappings $g: Z_0\to X$, $h: Z_1\to Y$ with $Z_0\subset Z_1\subset Z$ and $f\circ g=h/Z_0$, there exists a $Z$-normal mapping $k: Z_1\to X$ such that $g=k/Z_0$ and $f\circ k=h$.

By $C(X)$ (resp. $C^*(X)$) we denote the real vector space of all continuous (and bounded) real-valued functions on $X$ with the topology of uniform convergence. For a compact space $X$ by $P(X)$ is denoted the space of all regular probability measures on $X$ endowed with the weak-star topology. If $X$ is a given space, $P_\beta(X)$ (see [2]) stands for the space $\{\mu\in P(\beta X): \operatorname{supp}\mu\subset X\}$, where $\operatorname{supp}\mu$ is the support of $\mu$. It follows from the definition of $P_\beta(X)$ that every $\mu\in P_\beta(X)$ is a continuous positive linear functional on $C^*(X)$ with $\mu(1_X)=1$. We can consider $\mu$ as a continuous linear functional $\mu'$ on $C(X)$, defined by $\mu'(g)=\mu(g|\operatorname{supp}\mu)$. There exists a natural embedding $i: X\hookrightarrow P_\beta(X)$ defined by letting $i(x)=\delta_x$, where $\delta_x$ is Dirac's measure at the point $x$. One can easily show that $i(X)$ is a closed C-embedded subset of $P_\beta(X)$. If $f$ is a mapping from $X$ to $Y$ we denote by $P_\beta(f)$ the natural mapping from $P_\beta(X)$ to $P_\beta(Y)$. If $X$ is a subspace of $Y$ a (set-valued) mapping $r$ from $Y$ to $P_\beta(X)$ is said to be $P_\beta$-valued retraction from $Y$ to $X$ provided $r(x)=\delta_x$ for every $x\in X$.

We say that a usco mapping $r$ is minimal if every usco selection for $r$ coincides with $r$. It follows from Kuratowski-Zorn lemma that every usco mapping has a minimal usco selection.

If $X = \Pi\{X_\alpha : \alpha \in A\}$ and $B \subset A$, $p_B$ stands for the natural projection from $X$ onto $X_B = \Pi\{X_\alpha : \alpha \in B\}$. For every subset $U$ of $X$, $k(U)$ denotes the family $\{B : p_B^{-1}(p_B(U)) = U\}$.

## 3. AE(0)-spaces.

Lemma 1 [9]. Let $r$ be a usco mapping from $X$ to $Y$ and let $U$ be an open subset of $Y$. Then the following holds:

(i) $r(x) \subset \mathrm{cl}(U)$ for every $x \in \mathrm{Int}(\mathrm{cl}(r^{\#}(U)))$, where $r^{\#}(U) = \{x \in X : r(x) \subset U\}$;

(ii) $\mathrm{cl}(r^{-1}(U)) = \mathrm{cl}(r^{\#}(U))$, vere $r^{-1}(U) = \{x \in X : r(x) \cap U \neq \emptyset\}$.

Lemma 2. Let $X = \Pi\{X_\alpha : \alpha \in A\}$ be a product of separable metric spaces and $r : X \to Y$ be a minimal usco mapping. Suppose $f$ is a single-valued mapping from $Y$ to $Z$. Then there exists a set $B \subset A$ of cardinality $|B| = w(Z)$ such that $p_B(x) = p_B(y)$ implies $f(r(x)) = f(r(y))$ for every $x, y \in X$.

Proof. Let $\mathcal{B}$ be a finitely additive open base of $Z$ with $|\mathcal{B}| = w(Z)$. For every $U \in \mathcal{B}$ there exists [6] a countable subset $B(U)$ of $A$ such that $B(U)$ $k(\mathrm{cl}(r^{\#}(f^{-1}(U))))$. Put $B = \cup\{B(U) : U \in \mathcal{B}\}$. Obviously $|B| = w(Z)$ and $B \in k(\mathrm{cl}(r^{\#}(f^{-1}(U))))$ for any $U \in \mathcal{B}$. Let $p_B(x) = p_B(y)$ and $f(r(y)) \subset V$ where $x, y \in X$ and $V$ is open in $Z$. Since $f(r(y))$ is compact and $\mathcal{B}$ is finitely additive, we can suppose that $V$ is an element of $\mathcal{B}$. Then $B \in k(\mathrm{cl}(r^{\#}(f^{-1}(V))))$. Consequently, $B \in k(\mathrm{Int}(\mathrm{cl}(r^{\#}(f^{-1}(V)))))$. Thus, $x \in \mathrm{Int}(\mathrm{cl}(r^{\#}(f^{-1}(V))))$, because $y \in r^{\#}(f^{-1}(V))$. Hence, by Lemma 1(i), $r(x) \subset \mathrm{cl}(f^{-1}(V))$ i.e. $f(r(x)) \subset \mathrm{cl}(V)$. The last inclusion shows that $f(r(x)) \subset f(r(y))$. Analogously, $f(r(y)) \subset f(r(x))$. Therefore $f(r(x)) = f(r(y))$.

Let $Y = \Pi\{Y_\alpha : \alpha \in A\}$ be a product of separable metric spaces and $r$ be a usco $P_\beta$-valued retraction from $Y$ to $X$. A subset $B$ of $A$ is called $r$-admissible if $p_B(x) = p_B(y)$ implies $P_B(r(x)) = P_B(r(y))$ for any $x \in X$ and $y \in Y$. Here $P_B$ stands for the mapping $P_\beta(p_B|X) : P_\beta(X) \to P_\beta(X(B))$, where $X(B) = p_B(X)$. Let us note that for every $r$-admissible set $B$ there exists a usco $P_\beta$-valued retraction $r_B$ from $Y_B$ to $X(B)$. Indeed, consider an embedding $j : Y_B \hookrightarrow Y$ such that $p_B \circ j$ is the identity on $Y_B$. Define a usco mapping $r_B : Y_B \longrightarrow P_\beta(X(B))$ by letting $r_B(z) = P_B(r(j(z)))$. It is easy to show that $r_B(z) = \delta_z$ for every $z \in X(B)$ i.e. $r_B$ is a usco $P_\beta$-valued retraction.

Lemma 3. Let $Y = \Pi\{Y_\alpha : \alpha \in A\}$ be a product of separable metric spaces and $r$ be a minimal usco $P_\beta$-valued retraction from $Y$ to $X$. Then for every$ry\alpha \in A$ there exists a countable $r$-admissible subset $B(\alpha)$ of $A$ cotaining $\alpha$.

Proof. By a result of Chigogidze [2] we have $w(P_\beta(M)) = w(M)$ for every space $M$. Hence, by Lemma 2, there exists a countable subset $B(1)$ of $A$ containing $\alpha$ such that $p_{B(1)}(x) = p_{B(1)}(y)$ implies $P_\alpha(r(x)) = P_\alpha(r(y))$ for any $x, y \in Y$. In this way we construct an increasing sequence $\{B(n) : n \in N\}$ of countable subsets of $A$ such that $p_{B(n+1)}(x) = p_{B(n+1)}(y)$ implies $P_{B(n)}(r(x)) = P_{B(n)}(r(y))$ for every $x, y \in Y$. Put $B(\alpha) = \cup\{B(n) : n \in N\}$. Obviously $B(\alpha)$ is countable and $\alpha \in B(\alpha)$. Suppose $p_{B(\alpha)}(x) = p_{B(\alpha)}(y)$, where $x \in X$ and $y \in Y$. Let $\mu \in r(y)$. Since $r(x) = \delta_x$ and $P_{B(n)}(r(x)) = P_{B(n)}(r(y))$, we have $P_{B(n)}(r(x)) = P_{B(n)}(\mu)$ for every $n \in N$, i.e. supp $P_{B(n)}(\mu)$ is the one-point set $p_{B(n)}(x)$ for each $n \in N$. Hence, supp $P_{B(\alpha)}(\mu)$ is the one-point set $p_{B(\alpha)}(x)$. Consequently, $P_{B(\alpha)}(\mu) = P_{B(\alpha)}(r(x))$. Thus $P_{B(\alpha)}(r(x)) = P_{B(\alpha)}(r(y))$. Therefore $B(\alpha))$ is $r$-admissible.

Lemma 4. Let $Y$, $r$ and $X$ be as in Lemma 3. Suppose $B$ is a $r$-admissible subset of $A$. Then the following conditions are fulfilled:

(i) every union of $r$-admissible subsets of $A$ is also $r$-admissible;

(ii) the mapping $p_B/X : X \to X(B)$ is functionally open;

(iii) $X(B)$ is closed in $Y_B$. If in addition $B$ is a union of countable $r$-admissible sets, then $X(B)$ is C-embedded in $Y_B$.

Proof. (i) Let $B'=\cup\{B(s):s\in S\}$, where each $B(s)$ is $r$-admissible. Assume $x\in X$, $y\in Y$ and $p_{B'}(x)=p_{B'}(y)$ Then $P_{B(s)}(r(x))=P_{B(s)}(r(y))$ for every $s\in S$. Hence, supp $P_{B(s)}(\mu)=p_{B(s)}(x)$ for each $s\in S$, where $\mu\in r(y)$. Therefore, supp $P_{B'}(\mu)$ is the one-point set $p_{B'}(x)$ for any $\mu\in r(y)$, i.e. $P_{B'}(\mu)=P_{B'}(r(x))$. Thus $P_{B'}(r(x))=P_{B'}(r(y))$ Consequently, $B'$ is $r$-admissible.

(ii) Let $U$ be a functionally open subset of $X$. So there exists a function $f:X\to[0,1]$ such that $U=f^{-1}(0,1]$. Consider the continuous extension $f_1:P_\beta(X)\to[0,1]$ of $f$, defined by $f_1(\mu)=\mu(f)$. Put $U_1=f_1^{-1}(0,1]$ and $V=r^{\#}(U_1)$. Since the projection $\mu_B$ is functionally open [9], the set $p_B(V)\cap X(B)$ is functionally open in $X(B)$. Therefore, the proof of (ii) will be completed if we show that $p_B(U)=p_B(V)\cap X(B)$. Let $z\in p_B(V)\cap X(B)$. Then there are points $x\in X$ and $y\in Y$ such that $z=p_B(x)=p_B(y)$. Hence, $P_B(r(x))=P_B(r(y))=\delta_z$. Consequently, for each $\mu\in r(y)$ we have supp $P_B(\mu)=z$. Thus, $p_B$ (supp$\mu$)$=z$ for every $\mu\in r(y)$. The last implies $z\in p_B(U)$ provided there exists $\mu\in r(y)$ with supp $\mu\cap U\neq\emptyset$. We prove that this is true for all $\mu\in r(y)$. Indeed, in the opposite case there is $\mu^*\in r(y)$ such that supp$\mu^*\cap U=\emptyset$. Then $f_1(\mu^*)=\mu^*(f)=0$ because $f/(X\setminus U)\equiv 0$. But this is in contradiction with $\mu^*\in r(y)\subset U_1$. Therefore $p_B(V)\cap X(B)\subset p_B(U)$. The inverse inclusion is obvious.

(iii) Suppose there exists a point $y^*\in \mathrm{cl}(X(B))\setminus X(B)$. Let $r_B:Y_B\to P_\beta(X(B))$ be a usco $P_\beta$-valued retraction. One can easily show that $r_B(y^*)\cap i(X(B))\neq 0$, where $i:X(B)\hookrightarrow P_\beta(X(B))$ is the natural embedding. Consider the compact set $H=r_B(y^*)\cap i(X(B))$. Choose disjoint open in $Y_B$ sets $U$ and $V$ such that $y^*\in U$ and $i^{-1}(H)\subset V$. There exists an open set $V$ in $P_\beta(X(B))$ such that $V_1\cap i(X(B)=i(V\cap X(B))$ and $r_B(y^*)\subset V_1$. Therefore, we can assume that $U\subset r_B^{\#}(V_1)$. Take a point $x\in X(B)\cap U$. Then $r_B(x)=\delta_x\in V_1$. So $x\in V\cap X(B)$, which is impossible. Thus $X(B)$ is closed in $Y_B$.

Suppose now that $B$ is a union of a family $\mathcal{K}$ consisting of countable $r$-admissible sets. Then $X(B)$ is a limit of the inverse system $\{X(C),p_C^{C'}:C\subset C'$ and $C,C'\in\mathcal{K}\}$, where $p_C^{C'}$ is the natural projection from $X(C')$ onto $X(C)$. By (ii), $p_C^B$ is functionally open for every $C\in\mathcal{K}$. Let $f$ be a continuous function on $X(B)$. Then, by a result of Scepin [7], there exists $C^*\in\mathcal{K}$ and a continuous function $g$ on $X(C^*)$ such that $f=g\circ p_{C^*}^B$. Since $X(C^*)$ is closed in $Y_{C^*}$, $g$ is continuously extendable to $Y_{C^*}$. Therefore, $f$ is continuously extendable to $Y_B$.

Theorem. For a space $X$ the following conditions are equivalent:

(i) $X$ is an AE(0)-space;

(ii) for every C-embedding of $X$ in a space $Y$ there exists a usco $P_\beta$-valued retraction from $Y$ to $X$.

Proof. The implication (i)$\to$(ii) follows from [9].

(ii)$\to$(i) Consider $X$ as a C-embedded subset of $R^\tau$ for some $\tau$. Then there exists a usco $P_\beta$-valued retraction $r$ from $R^\tau$ to $X$. Therefore, we can assume that $r$ is a minimal usco mapping and $\tau$ is identified with the set $\{\alpha:\alpha<\omega(\tau)\}$, where $\omega(\tau)$ is the initial ordinal of cardinality $\tau$. It follows from Lemma 3 and Lemma 4(i) that $X$ can be supposed to be of weight $\tau$. By Lemma 3, for every $\alpha<\omega(\tau)$ there exists a countable $r$-admissible set $B(\alpha)$ containing $\alpha$. Put $A(\alpha)=\cup\{B(\beta):\beta<\alpha\}$ if $\alpha$ is a limit ordinal and $A(\alpha)=\cup\{B(\beta):\beta\le\alpha\}$ otherwise, $q_\alpha=p_{A(\alpha)}/X$ and $X_\alpha=q_\alpha(X)$. Let $q_\beta^\alpha=q_\beta\circ q_\alpha^{-1}$, where $\beta<\alpha$. By Lemma 4, each $X_\alpha$ is a closed C-embedded subset of $R^{A(\alpha)}$ and all projections $q_\beta^\alpha$ are functionally open. Hence, we construct a continuous [9] inverse system $S=\{X_\alpha, q_\beta^\alpha, \beta<\alpha<\omega(\tau)\}$ such that $X=\varprojlim S$ and every $q_\alpha^{\alpha+1}$ has a Polish kernel [1]. Since $A(\alpha)$ is a union of countable $r$-admissible sets, every $X_\alpha$ is a limite space of a $\sigma$-system in the sense of Scepin, consisting of Polish spaces and open projections. According to Corollary 1.20 from [1] and Lemma 4[11], we have that all mappings $q_\alpha^{\alpha+1}$ are 0-soft. The space $X_1$ being a closed set in $R^{A(1)}$, is an AE(0). By a transfinite induction, using the fact that every mapping $q_\alpha^{\alpha+1}$ is 0-soft, we can show that each $X_\alpha$ is an AE(0)-space. The last implies $X\in$ AE(0).

Corollary. Let $P_\beta(X)$ is an AE(0)-space. Then $X$ is also AE(0)-space.

Proof. It follows from [9] that $P_\beta(X)$ is absolute usco retract. Then $X$ is an absolute usco $P_\beta$-valued retract and, by our Theorem, $X\in$ AE(0).

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## ОЩЕ ЕДНА ХАРАКТЕРИСТИКА НА AE(0)-ПРОСТРАНСТВАТА

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Показано е, че всеки абсолютен $P_\beta$-значен ретракт е AE(0)-пространство. Това обобщава един предишен резултат на автора.