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SELECTION THEOREMS FOR AE(n)-SPACES*

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A well known theorem of Kuratowski-Dugundji asserts that a metrizable space X is an absolute extensor for n-dimensional spaces (briefly, $AE(n)$) iff X is LC^{n-1} and C^{n-1}. The formula $AE(n)=C^{n-1}\cap LC^{n-1}$ is not true for non-metrizable spaces, only the inclusion $AE(n)\subset C^{n-1}\cap LC^{n-1}$ remains valid. In the present note we show that the class of all $AE(n)$-spaces coincides with the class of all $C^{n-1}\cap LC^{n-1}$-spaces satisfying a selection condition of Michael's type. A similar result for $AE(0)$-spaces is proved in [2].

1. **Preliminaries.** All spaces considered are completely regular and all single-valued maps continuous. Below, dim stands for the dimension defined by finite functionally open covers. For a map $f: X\to Y$ let $C(f): C(Y)\to C(X)$ be the operator induced by f, where $C(X)$ is the set of all real-valued (continuous) functions on X. If $X\subset Y$ then $C(Y)/X$ is the set of all elements of $C(X)$ extendable to Y.

A space X is called (see [1]) an absolute extensor for n-dimensional spaces $(n=0, 1, \dots .)$ if for any space Z of dimension $\dim Z\leq n$ and any subset Z_0 of it each map $f: Z_0\to X$ such that $C(f)(C(X))\subset C(Z)/Z_0$ can be extended to the whole of Z.

A collection $\mathscr{L}$ of subsets of a space X is called equi-LC^n in X [4] if, for every $x\in X$, every neighbourhood V of x contains a neighbourhood W of x such that for any $L\in\mathscr{L}$, every continuous image of an i-sphere $(i\leq n)$ in $W\cap L$ is contractible in $V\cap L$.

A closed-valued set-valued map $F: X\to Y$ has the weak selection-factorization property [2] (briefly, F is w. s. f. p.) if for every functionally closed subset H of X and every countable family $\mathscr{U}$ consisting of functionally open subsets of Y such that $F^{-1}(\mathscr{U})=\{F^{-1}(U): U\in\mathscr{U}\}$ covers H, there exists a locally finite functionally open (in H) cover of H refining $F^{-1}(\mathscr{U})$. Here $F^{-1}(U)$ is the set $\{x\in X: F(x)\cap U\neq\varnothing\}$.

A set-valued map $F: X\to Y$ is said to have a countable singularity [2] if the collection of all images of F, containing at least two points, is countable and consists of functionally closed subsets of Y.

Let ω and γ be two families of subsets of a given space. A mapping $k: \gamma\to\omega$ is called refining if $U\subset k(U)$ for every $U\in\gamma$.

By a selection for a set-valued map $F: X\to Y$ we mean a single-valued map g: $X\to Y$ such that $g(x)\in F(x)$ for every $x\in X$.

2. **Selection Theorems for AE(n)-Spaces.** A set-valued map $F: X\to Y$ is said to have the property (n), where $n=0,1, \dots,$ if the following conditions are fulfilled:

(A_1) the family $\{F(x): x\in X\}$ is equi-LC^{n-1} in Y and every image $F(x)$ is C^{n-1};

(A_2) for every countable functionally open cover ω of X, every countable functionally open cover φ of Y and any collection $\{y(x)\in K(F(x)): x\in X\}$ there exist a countable functionally open cover γ of X and a refining mapping $k: \gamma\to\omega$ such that for any $U\in\gamma$ there is a point $x_U\in k(U)$ with $y(x_U)\subset\cap\{\mathrm{St}_\varphi(F(x)): x\in U\}$.

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Here, for every $x \in X$, $\mathrm{St}_\varphi(F(x))$ is the set $\cup\{V \in \varphi : V \cap F(x) \neq \emptyset\}$ and $K(F(x))$ stands for the family of all finite subsets of $F(x)$.

Let us note that the condition (A_2) is an analog of the notion of an s. s. f. s. c. map [3] in the completely regular case. It is easy to show that if $F: X \to Y$ is lower semi-continuous and X is paracompact then F satisfies the condition (A_2).

Theorem 2.1. For an $LC^{n-1} \cap C^{n-1}$-space X the following conditions are equivalent:

(*i*) X is an $AE(n)$-space;

(*ii*) Any set-valued map $F: Y \to X$ having a countable singularity and satisfying the property (n), where $\dim Y \leq n$, has a selection.

A similar theorem for $n=0$ is proved in [2] Theorem 2.3.

A set-valued map $F: X \to Y$ has the quasi selection-factorization property (briefly F is q. s. f. p.) if for every countable functionally open cover ω of X, for every countable functionally open cover φ of Y and for any collection $\{y(x) \in F(x) : x \in X\}$ there exist a countable functionally open cover γ of X and a refining map $k: \gamma \to \omega$ such that for any $U \in \gamma$ there is a point $x_U \in k(U)$ with $y(x_U) \in \cap\{\mathrm{St}_\varphi(F(x)) : x \in U\}$.

Proposition 2.2. Every w. s. f. p. map is q. s. f. p. and there is a q. s. f. p. map which is not w. s. f. p. (see [3]).

Theorem 2.3. For a space X the following conditions are equivalent:

(*i*) X is an $AE(0)$-space;

(*ii*) Every q. s. f. p. map $F: Y \to X$ with a countable singularity, where $\dim Y = 0$, has a selection.

Obviously, every set-valued map satisfying the property (0) is a q. s. f. p. map. Hence, the implication (*i*)→(*ii*) of Theorem 2.3 is stronger than the corresponding implication of Theorem 2.1 in the case $n=0$. It follows also from Proposition 2.2 that the implication (*i*)→(*ii*) of Theorem 2.3 is an improvement of the implication (*i*)→(*ii*) of Theorem 2.3 from [2].

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