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SURJECTIVE CHARACTERIZATIONS OF METRIZABLE LC^∞-SPACES

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Веско Вылов. СЮРЕКТИВНЫЕ ХАРАКТЕРИСТИКИ МЕТРИЗУЕМЫХ LC^∞-ПРОСТРАНСТВ

В работе доказывается следующая теорема (теорема 1. 1):

Метризуемое пространство Y является LC^∞ (соответственно $LC^\infty \& C^\infty$) тогда и только тогда, когда для каждого паракомпактного p-пространства X и каждого его замкнутого локально–конечномерно вложенного подмножества A, любое непрерывное отображение $f\colon A \to Y$ имеет непрерывное продолжение на окрестность множества A (соответственно на X).

При помощи этой теореме получается положительный ответ следующего вопроса А. Чигогидзе: Верно ли что метризуемые $LC^\infty \& C^\infty$-пространства характеризуются как непрерывные образы абсолютных ретрактов при индуктивно ∞-мягких отображениях?

Vesko Valov. SURJECTIVE CHARACTERIZATIONS OF METRIZABLE LC^∞-SPACES

In this note the following theorem is proved (theorem 1. 1):

A metrizable space Y is LC^∞ (resp. $LC^\infty \& C^\infty$) if and only if for any paracompact p-space X and any closed locally finite–dimensionally embedded subset A of X, any map $f\colon A \to Y$ can be continuously extended to a neighborhood of A in X (resp. to X).

Using this theorem we give a positive answer of the following question of A. Chigogidze: Is it true that a metrizable space Y is $LC^\infty \& C^\infty$ if and only if Y is an image of an absolute extensor for metrizable spaces under a ∞-soft map?

INTRODUCTION

In this note we prove the following theorem:

Theorem 1.1. *A metrizable space Y is LC^∞ (resp. $LC^\infty \& C^\infty$) if and only* *if for any paracompact p-space X, any closed locally finite-dimensionally embedded subset A of X, any map $f\colon A \to Y$ can be continuously extended to a neighborhood of A in X (resp. to X).*

Let us note that all maps are assumed to be continuous and all spaces—normal. A space Y is LC^∞ (resp. $LC^\infty\&C^\infty$) if Y is LC^n (resp. $LC^n\&C^n$) for every natural n. A subset H of a space X is said to be locally finite-dimensionally embedded in X [6] if every point $x \in X$ has a neighborhood $O(x)$ in X such that $rd_X(H \cap O(x)) < \infty$, where

$$rd_X(H \cap O(x)) = \sup\{\dim(P) : P \text{ is closed in X and } P \subset H \cap O(x)\}.$$

The first motivation for the above result was the obvious parallelism between the following theorems:

Theorem 2. ([2]) *A metrizable space Y is LC^n (resp. $LC^n\&C^n$) if and only if for any metrizable space X and any closed subspace A of X with* $\dim(X-A) \leqq n+1$, *any map $f\colon A \to Y$ can be continuously extended to a neighborhood of A in X (resp. to the whole of X).*

Theorem 3. ([1]) *A metrizable space Y is LC^n (resp. $LC^n\&C^n$) if and only if for every metrizable space X and any closed subspace A of X with* $\dim(A) \leqq n$, *any map $f\colon A \to Y$ can be continuously extended to a neighborhood of A in X (resp. to the whole of X).*

In the class of compact metrizable spaces the above theorem was proved by Dranishnikov [3].

The second motivation was the following result:

Theorem 4. ([7]) *A metrizable space Y is LC^∞ (resp. $LC^\infty\&C^\infty$) if and only if for every paracompact p-space X and any closed subset A of X, having a neighborhood U in X such that $U - A$ is locally finite-dimensionally embedded in U, any map $f\colon A \to Y$ can be continuously extended to a neighborhood of A in X (resp. to the whole of X).*

As a consequence of Theorem 1.1 we get surjective characterizations of metrizable $LC^\infty\&C^\infty$-spaces. It is proved in [1] that the metrizable space X is $LC^n\&C^n$ (resp. LC^n) if and only if X is an inductively n-soft image of an AE (resp. of an ANE). In this connection A. Chigogidze asked whether a similar characterization is true for metrizable $LC^\infty\&C^\infty$-spaces. The following theorem is a positive answer of this question.

Theorem 2.1. *For a metrizable space X the following conditions are equivalent:*

(i) *X is $LC^\infty\&C^\infty$ (resp. LC^∞);*

(ii) *X is an inductively ∞-soft image of an AE (resp. ANE);*

(iii) *X is an ∞-invertible image of an AE (resp. ANE).*

By AE (resp. ANE) we denote the class of metrizable spaces which are absolute (resp. absolute neighborhood) extensors for metrizable spaces. Let $f\colon X \to Y$ be a map between metrizable spaces and $n \geqq 0$. Then:

f is n-soft [10] if for any at most n-dimensional paracompact space Z, any closed subspace A of Z and any two maps $g\colon Z \to Y$, $h\colon A \to X$ with $f.h = g|A$, there exists a map $k\colon Z \to X$ such that $f.k = g$ and $k|A = h$;

f is inductively ∞-soft if it is inductively n-soft for every natural n, i. e. for every n there is a closed subspace $A(n)$ of X such that the restriction $f|A(n): A(n) \to Y$ is n-soft;

f is n-invertible [4] if for any at most n-dimensional metrizable space Z and any map $g: Z \to Y$ there exists a map $h: Z \to X$ such that $g = f.\ h$;

f is ∞-invertible if it is n-invertible for every natural n.

Obviously, any n-soft map is n-invertible, $n \in N$. So, every inductively ∞-soft map is ∞-invertible.

1. PROOF OF THEOREM 1.1

Let Y be a metrizable $LC^{\infty}\&C^{\infty}$-space. Suppose X is a paracompact p-space, A is a closed locally finite–dimensionally embedded subset of X and f is a map from A into Y. For every $x \in X$ take an open neighborhood $O(x)$ of x in X such that $rd_X(O(x) \cap A) < \infty$ and consider the open cover $\alpha = \{O(x) : x \in X\}$ of X. Let γ be an open locally finite closure-refinement of α. Then for every $U \in \gamma$ we have $\dim(cl_X(U) \cap A) < \infty$. Put $\gamma(k) = \{U \in \gamma : \dim(cl_X(U) \cap A) \leqq k\}$ and $F(k) = \cup\{cl_X(U) : U \in \gamma(k)\}, k = 1, 2, \ldots$ Obviously $\{F(k) : k \in N\}$ is an increasing sequence of closed subsets of X and $X = \cup\{F(k) : k \in N\}$. Since for every k the family $\gamma(k)$ is locally finite, $\dim(A(k)) \leqq k$, where $A(k) = F(k) \cap A$. By the well-known factorization theorem of Pasynkov [8] there are a metrizable space Z, closed subsets $Z(k)$ of Z, $k \in N$, and maps $g: A \to Z$, $h: Z \to Y$, such that $h.g = f$, $g(A(k)) \subset Z(k)$ and $\dim(Z(k)) \leqq k$ for every $k \in N$. Without a loss of generality we can suppose that $Z(k)$ is contained in $Z(k+1)$ for each k.

Now for every $k \in N$ we construct inductively an AE-space $P(k)$ containing $Z(k)$ as a closed subset and maps $g(k): F(k) \to P(k)$, $h(k): P(k) \to Y$ such that the following conditions are fulfilled:

(1) $P(k) \cup Z(k+1)$ is a closed subspace of $P(k+1)$;

(2) $P(1)$ is attached to Z in the points of $Z(1)$ and $P(k+1)$ is attached to $Z \cup P(k)$ in the points of $P(k) \cup Z(k+1)$;

(3) $g(k)|A(k) = g|A(k)$ and $g(k+1)|F(k) = g(k)$;

(4) $h(k)|Z(k) = h|Z(k)$ and $h(k+1)|P(k) = h(k)$;

(5) $\dim(P(k)) \leqq k+1$.

Suppose we have already constructed $P(k)$, $g(k)$ and $h(k)$ for every $k \leqq n$. Consider the union $P(n) \cup Z(n+1)$. Obviously, it is metrizable and $\dim(P(n) \cup Z(n+1)) \leqq n+1$. By a result of Kodama [5] there is an AE-space $P(n+1)$ with $\dim(P(n+1)) \leqq n+2$ containing $P(n) \cup Z(n+1)$ as a closed subset. We can assume that $P(n+1)$ is attached to the space $Z \cup P(n)$ in the points of $P(n) \cup Z(n+1)$. The space $P(n+1)$, being an absolute extensor for metrizable spaces, is an absolute extensor for paracompact p-spaces [6]. Consequently, there is a map $g(n+1)$ from $F(n+1)$ to $P(n+1)$ such that $g(n+1)|A(n+1) = g|A(n+1)$ and $g(n+1)|F(n) = g(n)$. Let $h^*(n): P(n) \cup Z(n+1) \to Y$ be defined by $h^*(n)(z) = h(n)(z)$ if $z \in P(n)$, and $h^*(n)(z) = h(z)$ if $z \in Z(n+1)$. It follows from (1) — (4) that $h^*(n)$ is well defined and continuous. Now, by Theorem 2 and $Y \in LC^{\infty}\&C^{\infty}$,

there is a continuous extension $h(n+1): P(n+1) \to Y$ of $h^*(n)$. The verification of the conditions (1) — (5) is left to the reader.

Let $f(k) = h(k).g(k)$ for every $k \in N$. It follows from our construction that $f(k): F(k) \to Y$ is a continuous map, $f(k)|A(k) = f|A(k)$ and $f(k+1)|F(k) = f(k)$. Therefore, we can define a map $\overline{f}: X \to Y$ by $\overline{f}(x) = f(k)(x)$ provided $x \in F(k)$. Obviously, $\overline{f}$ is an extension of f. It remains only to prove the continuity of $\overline{f}$. This can be done by the following arguments: For every $U \in \gamma$ its closure $cl_X(U)$ is contained in some $F(k)$. So, we have that $\overline{f}|U$ is continuous for each $U \in \gamma$. Thus, $\overline{f}$ is continuous.

Suppose now Y is LC^∞, X is a paracompact p-space, A is a closed locally finite–dimensionally embedded subset of X and f is a map from A into Y. Then the cone con(Y) of Y is a metrizable $LC^\infty \& C^\infty$-space. Next, by standard arguments (using the previous case), we can get an extension $\overline{f}: U \to Y$ of f, where U is a neighborhood of A in X.

The sufficiency in Theorem 1.1 follows from Theorem 2 and the obvious fact that every subset of a finite dimensional space X is locally finite–dimensionally embedded in X.

Remark 1.2. If in Theorem 1.1 Y is completely metrizable then X can be suppposed to be collectionwise normal. In this case the space Z (see the proof of Theorem 1.1) can be assumed to be complete. Then, by a result of Tsuda [11], the spaces $P(k)$ can be chosen to be also comlete. Finally, the existence of the maps $g(k)$, $k \in N$, follows from the fact that every complete AE is an absolute extensor for collectionwise normal spaces [9].

2. PROOF OF THEOREM 2.1

We shall prove only the global variant. The local one follows from the same arguments.

$(i) \to (ii)$. Let τ be the weight of Y. Then for every $n \in N$ there exist an n-dimensional metrizable space $A(n,\tau)$ of weight τ and an n-soft map $f(n)$ from $A(n,\tau)$ onto Y [1, Corollary 2.3]. Consider the discrete sum A of the spaces $A(n,\tau)$, $n \in N$, and the map $f: A \to Y$, defined by $f(x) = f(n)(x)$ if $x \in A(n,\tau)$. Embed A into an AE-space X as a closed subset. Since A is locally finite–dimensionally embedded in X, by Theorem 1.1 there is an extension $\overline{f}: X \to Y$ of f. Clearly, $\overline{f}$ is inductively ∞-soft.

$(ii) \to (iii)$. This implication is trivial, because any inductively ∞-soft map is ∞-invertible.

$(iii) \to (i)$. Let X be an AE-space and f be an ∞-invertible map from X onto Y. Suppose B is an n-dimensional closed subset of a metrizable space Z and $g: B \to Y$ is a map, where $n \in N$. Since f is n-invertible, there exists a map $h: B \to X$ such that $f.h = g$. Take any extension $k: Z \to X$ of h (the existence of k follows from $X \in AE$). Then the map $f.k$ is an extension of g. Hence, by Theorem 3, Y is $LC^n \& C^n$. Thus, we prove that $Y \in LC^\infty \& C^\infty$.

Let us consider the proof of Theorem 1.1, implication $(i) \to (ii)$. If Y is a complete metrizable space of weight τ, then by [1, Corollary 2.3] the spaces $A(n,\tau)$

are also complete. Consequently, A can be embedded as a closed subset in the Hilbert space $l_2(\tau)$. So, we can suppose that X is the space $l_2(\tau)$. Thus, the following theorem is true.

Theorem 2.2. *Let Y be a completely metrizable space of weight $\tau \geqq \omega$. Then the following conditions are equivalent:*

(i) *X is $LC^{\infty}\&C^{\infty}$ (resp. LC^{∞});*

(ii) *X is an inductively ∞-soft image of $l_2(\tau)$ (resp. of an open subspace of $l_2(\tau)$);*

(iii) *X is an ∞-invertible image of $l_2(\tau)$ (resp. of an open subset of $l_2(\tau)$).*

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