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Cantor set selectors*

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Abstract

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If every l.s.c. mapping from the Cantor set $\mathscr{C}$ to the closed subsets of a metric space X admits a u.s.c. selection, then X is a Baire space and either X is scattered or X contains a copy of $\mathscr{C}$.

Keywords: Lower (upper) semi-continuous set-valued mapping, selection, selector, Baire space.

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1. Introduction

Michael has shown [4] that if X is a Čech-complete metric space and Y is a paracompact space, then every l.s.c. closed-valued mapping $\phi : Y \to 2^X$ has a u.s.c. (compact-valued) selection $\psi : Y \to 2^X$ (i.e., $\psi(y) \subset \phi(y)$ for each y in Y). Adopting terminology from [6] we say that complete metric spaces are selectors with respect to paracompact spaces. It is known as well [6] that if a metric space X is a selector with respect to paracompact spaces, then X is Čech-complete. We do not know whether the last conclusion remains valid when X is supposed to be only a selector with respect to metric spaces, and in the present paper we quote all we know concerning this question. More precisely, the main question treated in the present paper reads as follows: Is a metric space X Čech-complete provided X is a selector with respect to the *Cantor set* $\mathscr{C}$?

* Dedicated to the memory of Z. Frolík.

Here *X is a selector with respect to* $\mathscr{C}$ means that every l.s.c. closed-valued mapping $\phi:\mathscr{C}\to 2^X$ admits a u.s.c. selection $\psi:\mathscr{C}\to 2^X$; 2^X stands for the set of all nonvoid subsets of X. A mapping $\theta: Y\to 2^X$ is *lower semi-continuous* (l.s.c.) (or *upper semi-continuous* (u.s.c.)) provided the set $\theta^{-1}(M)=\{y\in Y: \theta(y)\cap M\neq\emptyset\}$ is open (closed) in Y for every open (closed) subset M of X.

2. Results

We shall say that X is a *Z-selector* (X and Z being topological spaces), provided every l.s.c. closed-valued mapping $\phi: Z\to 2^X$ has a u.s.c. closed-valued selection $\psi: Z\to 2^X$.

Note that if a metric space X is a $\mathscr{C}$-selector, then X is a Y-selector for every locally-compact metrizable space Y. This follows easily since every compact metric space is a continuous (and hence perfect) image of $\mathscr{C}$ and every locally-compact metric space is a perfect image of a discrete union (topological sum) of metric compacta.

Open question. Is every metrizable $\mathscr{C}$-selector a selector with respect to metric spaces?

Theorem 1. *Every closed subset F of a metric $\mathscr{C}$-selector X is a Baire space (i.e., every countable family of dense open subsets of F has an intersection which is dense).*

Proof. Since every closed subset of X is a $\mathscr{C}$-selector, it suffices to show that every $\mathscr{C}$-selector is not meagre (i.e., the intersection of any countable family of dense open subsets of a $\mathscr{C}$-selector is not empty). To this end, we follow Michael's construction from [5, Example 9.4]. Suppose Y is any nonempty meagre space, i.e., $Y=\bigcup\{Y_n: n=1,2,\dots\}$ with each Y_n a closed nowhere dense set (i.e., with empty interior in Y). Let $\{C_n\}$ be a disjoint sequence of countable dense subsets of $\mathscr{C}$. Define a set-valued mapping $\varphi:\mathscr{C}\to 2^Y$ by letting $\varphi(x)=Y\setminus Y_n$ if $x\in C_n$ and $\varphi(x)=Y$ if $x\in\mathscr{C}\setminus\bigcup\{C_n: n=1,2,\dots\}$. By [5, Proposition 2.4] there is an l.s.c. map $\theta:\mathscr{C}\to 2^Y$ with $\theta(x)\subset\varphi(x)$ and $\theta(x)$ is closed in Y for every $x\in\mathscr{C}$. Suppose θ had a u.s.c. selection ψ. Let $F_n=\psi^{-1}(Y_n)=\{x\in\mathscr{C}: \psi(x)\cap Y_n\neq\emptyset\}$. Then $\mathscr{C}=\bigcup\{F_n: n=1,2,\dots\}$ and the F_n are closed in $\mathscr{C}$, so $\operatorname{Int}(F_m)\neq\emptyset$ for some m. Pick $x\in F_m\cap C_m$. Then $\psi(x)\cap Y_m\neq\emptyset$ and $\psi(x)\subset\theta(x)\subset\phi(x)=Y\setminus Y_m$; a contradiction. □

Theorem 2. *If X is a metric $\mathscr{C}$-selector, then either X is scattered (i.e., every closed subset of X has an isolated point) or X contains a copy of $\mathscr{C}$.*

Proof. We may suppose, without loss of generality, X has no isolated point. Let $\{X_i\}$ be a disjoint sequence of dense subsets of X [2] and let $C_0=\{q_i\}$ be a countable dense subset of $\mathscr{C}$. Define a set-valued mapping $\phi:\mathscr{C}\to 2^X$ by letting $\phi(q_i)=X_i$ and

$\phi(q)=X$ for $q\in\mathscr{C}\setminus C_0$. Since $\overline{\phi(q)}=X$ for every $q\in\mathscr{C}$, the map ϕ is l.s.c. Next, let $\varphi:\mathscr{C}\to 2^X$ be a closed-valued l.s.c. selection of ϕ, existing by [5, Proposition 2.4]. Denote now by $\psi:\mathscr{C}\to 2^X$ a u.s.c. closed-valued selection of φ (X is a $\mathscr{C}$-selector). From now on we are going to construct a copy C' of $\mathscr{C}$ in $\mathscr{C}$ such that the distinct points of C' have disjoint images under ψ. Let ρ be a metric on $\mathscr{C}$ and let a_0 and a_1 be two distinct points in C_0 with $\rho(a_0,a_1)\leq 1$. Since $\psi(a_0)$ and $\psi(a_1)$ are disjoint closed subsets of X, there are disjoint open sets V_0 and V_1 in X with $V_0\supset\psi(a_0)$ and $V_1\supset\psi(a_1)$. Next, let U_0 and U_1 be open nonvoid subsets of $\mathscr{C}$ with $\operatorname{diam}(U_i)\leq 2^{-1}$ and such that $a_i\in\bar{U}_i\subset\psi^{\#}(V_i)=\{q\in\mathscr{C}:\psi(q)\subset V_i\}$ $(i=0,1)$. Choose in $U_i\cap C_0$ two distinct points a_{i0} and a_{i1} $(i=0,1)$ and repeat the above procedure, i.e., take disjoint open sets V_{i0} and V_{i1} in X with $V_i\supset\bar{V}_{i0}\supset V_{i0}\supset\psi(a_{i0})$ and $V_i\supset\bar{V}_{i1}\supset V_{i1}\supset\psi(a_{i1})$, then choose nonvoid open subsets U_{i0} and U_{i1} of $\mathscr{C}$ with $\operatorname{diam}(U_{ij})\leq 2^{-2}$, $a_{ij}\in\bar{U}_{ij}\subset\psi^{\#}(V_{ij})$ $(j=0,1)$ and so on, by induction. Next, let $F_n=\bigcup\{\bar{U}_{i_0\dots i_n}: i_j$ is either 0 or 1 for $j=0,1,\dots,n\}$ and $C'=\bigcap\{F_n: n=1,2,\dots\}$. It is easy to realize that C' has the required properties. First, C' is homeomorphic to $\mathscr{C}$ by the construction. Next, if a and b are distinct points of C' then, for some n, they are in different sets of the kind $\bar{U}_{i_0\dots i_n}$, so $\psi(a)$ and $\psi(b)$ are disjoint. Thus (see [3]) the restriction $\psi|_{C'}$ admits a u.s.c. compact-valued selection $\theta: C'\to 2^X$, so that $\theta(C')=\bigcup\{\theta(q): q\in C'\}$ is a compact subset of X, of cardinality $\geq 2^{\aleph_0}$. Hence X contains a copy of $\mathscr{C}$. □

3. Examples

(1) The space of rational numbers $\mathbb{Q}$ is not a $\mathscr{C}$-selector (which answers a question of Frolík). Simply, $\mathbb{Q}$ is not a Baire space.

(2) There is a separable metric space X each closed subset of which is a Baire space but X is not a $\mathscr{C}$-selector. Indeed, by Theorem 2, every Bernstein space is of this kind. By a Bernstein space we mean a space which is homeomorphic to a subspace X of a nonvoid, complete separable metric space Y with no isolated point, such that if D is a compact subset of Y with $\operatorname{Card}(D)\geq 2^{\aleph_0}$, then $X\cap D\neq\emptyset\neq(Y\setminus X)\cap D$. One easily constructs Bernstein spaces using the fact that the cardinality of the family of all compact subsets in a given separable metric space is not greater than $2^{\aleph_0}$.

(3) Let us mention other open problems:

(a) In [1] (under an additional set-theoretical assumption, $V=L$) a subspace X of the unit interval I is constructed, such that $I\setminus X$ is coanalytic and contains no uncountable compact set. X is Baire (see [7]) and contains copies of $\mathscr{C}$. Nevertheless, we are unable to decide whether X is a $\mathscr{C}$-selector.

(b) Is every G_δ-subset of a metrizable $\mathscr{C}$-selector (a metrizable selector with respect to metric spaces) also such a selector?

(c) Let X and Y be metrizable $\mathscr{C}$-selectors (selectors with respect to metric spaces). Is $X\times Y$ also such a selector?

Note that if $X \times \mathbb{R}^{\aleph_0}$ is a $\mathscr{C}$-selector for every metrizable $\mathscr{C}$-selector X, then the answer to the question (b) is affirmative.

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