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LINEAR TOPOLOGICAL CLASSIFICATIONS OF CERTAIN FUNCTION SPACES II

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Abstract: Some linear classification results for the spaces $C_p(X)$ and $C_p^*(X)$ are proved.

0. Introduction

If X is a space then $C_p(X)$ denotes the set of all continuous real-valued functions on X with the topology of pointwise convergence. We write $C_p^*(X)$ for the subspace of $C_p(X)$ consisting of all bounded functions. $\mathbb{R}$ stands for the usual space of real numbers, I for the unit segment $[0,1]$, Q is the Hilbert cube $[-1,1]^\omega$ and s is the pseudointerior $(-1,1)^\omega$ of Q. We will consider also the spaces $\sigma = \{(t_1, t_2, \dots) \in Q : : t_i = 0$ for all but finitely many $i\}$ and $\Sigma = \{(t_1, t_2, \dots) \in Q^\omega : t_i = 0$ for all but finitely many $i\}$.

In [11] some linear topological classification results of the spaces $C_p(X)$ and $C_p^*(X)$ are given. Using the ideas of [11] we prove in this paper that $C_p(X) \sim C_p(Y)$ provided Y is one of the spaces σ, Σ, $s \times \Sigma$ and X is a manifold modeled on Y. Here the symbol "$\sim$" stands for linear homeomorphism. A similar results are also proved for the spaces C_p^*.

1. Preliminaries

All spaces under discussion are Tychonoff and all mappings between topological spaces are continuous. By $L_p(X)$ is denoted the dual linear space of $C_p(X)$ with the weak (i.e., pointwise) topology. It is known that

$$L_p(X) = \left\{ \sum_{i=1}^{k} a_i \delta_{x_i} : a_i \in \mathbb{R} - \{0\} \text{ and } x_i \in X \text{ for each } i \leq k \right\}.$$

Here δ_x is the Dirac measure at the point $x \in X$. We denote

$$P_\infty(X) = \left\{ \sum_{i=1}^{k} a_i \delta_{x_i} : a_i \in (0,1) \text{ for each } i \text{ and } \sum_{i=1}^{k} a_i = 1 \right\}$$

and $\operatorname{supp}(\ell) = \{x_1, \dots, x_k\}$, where $\ell = \sum\limits_{i=1}^{k} a_i \delta_{x_i} \in L_p(X)$.

Let A be a closed subset of a space X. Consider the following conditions:

(i) there is a continuous linear extension operator $u : C_p(A) \to$ $\to C_p(X)$ (recall that $u : C_p(A) \to C_p(X)$ is an extension operator if $u(f)|A = f$ for every $f \in C_p(A)$);

(ii) there is a continuous linear extension operator $u : C_p(A) \to$ $\to C_p(X)$ and a positive constant c such that $||u(f)||| \leq c.||f||$ for every $f \in C_p^*(A)$ (here $||f||$ is the supremum norm of f);

(iii) there is a regular extension operator $u : C_p(A) \to C_p(X)$ i.e. a continuous linear extension operator u with $u(1_A) = 1_X$ and $u(f) \geq 0$ provided $f \geq 0$.

A is said to be *ℓ-embedded* (resp., *ℓ^*-embedded*) in X if the condition (i) (resp., the condition (ii)) holds. If (iii) is satisfied then A is called *strongly ℓ-embedded* in X. Dugundji [5] proved that every closed subset of a metric space X is strongly ℓ-embedded in X (he did not state this explicitly in this form). It is known (see [1], [4]) that A is ℓ-embedded (resp., strongly-embedded) in X if and only if there is a mapping $r : X \to L_p(A)$ (resp., $r : X \to P_\infty(A)$) such that $r(x) = \delta_x$ for every $x \in A$. Such a mapping will be called an *L_p-valued* (resp., a *P_∞-valued*) *retraction*. Every L_p-valued retraction $r : X \to L_p(A)$ defines a continuous linear extension operator $u_r : C_p(A) \to C_p(X)$ by setting $u_r(f)(x) = r(x)(f)$. If the operator u_r satisfies the condition (ii), r is said to be a *bounded L_p-valued retraction*.

Let $u : C_p(A) \to C_p(X)$ be a continuous linear extension operator. Then the mapping $v(f,g) = u(f) + g$ is a linear homeomorphism from $C_p(A) \times C_p(X;A)$ onto $C_p(X)$, where

$$C_p(X;A) = \{g \in C_p(X) : g|A = 0\}.$$

Analogously, if A is ℓ^*-embedded in X then $C_p^*(A) \times C_p^*(X;A)$ is linearly homeomorphic to $C_p^*(X)$.

Let $\mathcal{K}$ be a family of bounded subsets of a space X (i.e. $f|K$ is bounded for every $K \in \mathcal{K}$ and $f \in C_p(X)$) and E be a linear topological subset of $C_p(X)$. Then we set:

$$(\Pi E)_{\mathcal{K}} = \{(f_1, \dots, f_n, \dots) \in E^\omega : \lim_n \|f_n\|_K = 0 \text{ for every } K \in \mathcal{K}\}$$

and

$$(\Pi E)^*_{\mathcal{K}} = \{(f_1, \dots, f_n, \dots) \in (\Pi E)_{\mathcal{K}} : \sup_n \|f_n\| < \infty\}.$$

$(\Pi E)_{\mathcal{K}}$ and $(\Pi E)^*_{\mathcal{K}}$ are considered as topological linear subspaces of $C_p(X)^\omega$. We write $(\Pi E)_b$ and $(\Pi E)^*_b$ (resp., $(\Pi E)_c$ and $(\Pi E)^*_c$) if $\mathcal{K}$ is the family of all bounded (resp., of all compact) subsets of X. In the above notations $\|f\|_K$ stands for $\sup\{|f(X)| : x \in K\}$. Let us note that if X is pseudocompact and E is a linear subset of $C_p(X)$, the space

$$(\Pi E)_0 = \{(f_1, \dots, f_n, \dots) \in E^\omega : \lim_n \|f_n\| = 0\}$$

is considered in [6].

2. The spaces $C_p(X)$

Lemma 2.1. *Let X be one of the spaces σ, Σ, $s \times \Sigma$. Then $C_p(X) \sim \sim C_p(X)^\omega \sim C_p(\operatorname{cl}_X(U))$ for every open subset U of X.*

Proof. First we prove that $C_p(X) \sim C_p(X)^\omega$. Consider $X \times N$, where N is a discrete infinite countable space. Then $X \times N$ can be embedded as a closed subset of X (see [3]). Since X is metrizable, $X \times N$ is ℓ-embedded in X. Hence

$$\begin{aligned} C_p(X) &\sim C_p(X \times N) \times C_p(X; X \times N) = C_p(X)^\omega \times C_p(X; X \times N) \sim \\ &\sim C_p(X)^\omega \times C_p(X)^\omega \times C_p(X; X \times N) \sim \\ &\sim C_p(X)^\omega \times C_p(X \times N) \times C_p(X; X \times N) \sim \\ &\sim C_p(X)^\omega \times C_p(X) \sim C_p(X)^\omega . \end{aligned}$$

Now, let U be an open subset of X. Consider the open cover $\gamma = = \{U, X - \{x_0\}\}$ of X, where $x_0 \in U$, and the constant map $f : X \to x_0$.

By ([3], Corollaries 6.1, 6.2, 6.3) there is a closed embedding $h : X \to X$ such that f and h are γ-close i.e. for every $x \in X$ there is $V \in \gamma$ with $h(x), f(x) \in V$. Since $f(x) = x_0 \overline{\in} X - \{x_0\}$ for any $x \in X$ we have $h(x) \subset U$. Hence, $h(X)$ is a copy of X which is closed in $\mathrm{cl}_X(U)$. Then

$$C_p(\mathrm{cl}_X(U)) \sim C_p(h(X)) \times C_p(\mathrm{cl}_X(U); h(X)) \sim$$
$$\sim C_p(X) \times C_p(\mathrm{cl}_X(U); h(X)).$$

On the other hand $\mathrm{cl}_X(U)$ is closed in X, so

$$C_p(X) \sim C_p(\mathrm{cl}_X(U)) \times C_p(X; \mathrm{cl}_X(U)).$$

Hence,

$$C_p(\mathrm{cl}_X(U)) \sim C_p(X) \times C_p(\mathrm{cl}_X(U); h(X)) \sim$$
$$\sim C_p(X)^\omega \times C_p(\mathrm{cl}_X(U); h(X)) \sim$$
$$\sim C_p(X)^\omega \times C_p(X) \times C_p(\mathrm{cl}_X(U); h(X)) \sim$$
$$\sim C_p(X)^\omega \times C_p(\mathrm{cl}_X(U)) \sim$$
$$\sim (C_p(\mathrm{cl}_X(U)) \times C_p(X; \mathrm{cl}_X(U)))^\omega \times C_p(\mathrm{cl}_X(U)) \sim$$
$$\sim C_p(\mathrm{cl}_X(U))^\omega \times C_p(X; \mathrm{cl}_X(U))^\omega \sim$$
$$\sim (C_p(\mathrm{cl}_X(U)) \times C_p(X; \mathrm{cl}_X(U)))^\omega \sim C_p(X)^\omega \sim C_p(X). \ \Diamond$$

Remark 2.2. By similar arguments one can prove that *if* $X = \ell_2(\tau)$ *and* U *is open in* $\ell_2(\tau)$ *then* $C_p(X) \sim C_p(X)^\tau \sim C_p(\mathrm{cl}_X(U))$. Here $\ell_2(\tau)$ is the Hilbert space of weight $\tau \geq \omega$.

Let f be a mapping from a space X onto a space Y. Recall that a continuous linear operator $C_p(X) \to C_p(Y)$ is said to be an *averaging operator* for f if $u(h.f) = h$ for every $h \in C_p(Y)$. If f admits a regular averaging operator $u : C_p(X) \to C_p(Y)$ we can define a mapping $r : Y \to P_\infty(X)$ by the formula $r(y)(g) = u(g)(y)$. The mapping r has the following property [4]: $\mathrm{supp}(r(y))$ is contained in $f^{-1}(y)$ for each $y \in Y$. Conversely, if there is a mapping $r : Y \to P_\infty(X)$ such that $\mathrm{supp}(r(y)) \subset f^{-1}(y)$ for every $y \in Y$, the formula $u(g)(y) = r(y)(g)$ defines a regular averaging operator u for f. It is easily seen that if u is a regular averaging operator for f, the mapping $v(g) = (u(g), g - u(g).f)$ is a linear homeomorphism from $C_p(X)$ onto $C_p(Y) \times E$, where

$$E = \{g - u(g).f : g \in C_p(X)\}.$$

Proposition 2.3. *Let* $\gamma = \{U_\alpha : \alpha < \tau\}$ *be an infinite locally finite functionally open cover of a space* X *of cardinality* τ. *Suppose there is* *a space* Y *with* $C_p(\operatorname{cl}_X(U_\alpha)) \sim C_p(Y)$ *for each* $\alpha < \tau$. *Then* $C_p(X) \sim \sim C_p(Y)^\tau$ *provided* X *contains an* ℓ*-embedded copy of a topological sum* $\sum_{\alpha<\tau} F_\alpha$ *such that* $C_p(F_\alpha) \sim C_p(Y)$ *for every* $\alpha < \tau$.

Proof. For every $\alpha < \tau$ take an $f_\alpha \in C_p(X)$ such that $f_\alpha^{-1}(0) = X - U_\alpha$ and $f_\alpha \geq 0$. Without loss of generality we can suppose that $\sum_{\alpha<\tau} f_\alpha = 1$ because γ is a locally finite cover of X. Define $f \in C_p\left(\sum_{\alpha<\tau} \operatorname{cl}_X(U_\alpha)\right)$ such that $f|\operatorname{cl}_X(U_\alpha) = f_\alpha|\operatorname{cl}_X(U_\alpha)$ and consider the natural mapping $p : \sum_{\alpha<\tau} \operatorname{cl}_X(U_\alpha) \to X$ with all finite preimages. Let $r : X \to \to P_\infty\left(\sum_{\alpha<\tau} \operatorname{cl}_X(U_\alpha)\right)$ be defined by $r(x) = \sum\{f(y)\delta_y : y \in p^{-1}(x)\}$. It is easily seen that r is continuous and $\operatorname{supp}(r(x)) \subset p^{-1}(x)$ for every $x \in \in X$. Thus there is a regular averaging operator $u : C_p\left(\sum_{\alpha<\tau} \operatorname{cl}_X(U_\alpha)\right) \to \to C_p(X)$ for p. Hence, $C_p\left(\sum_{\alpha<\tau} \operatorname{cl}_X(U_\alpha)\right) \sim C_p(X) \times E$ where E is a linear subspace of $C_p\left(\sum_{\alpha<\tau} \operatorname{cl}_X(U_\alpha)\right)$. Since $\sum_{\alpha<\tau} F_\alpha$ is ℓ-embedded in X we have $C_p(X) \sim C_p\left(\sum_{\alpha<\tau} F_\alpha\right) \times C_p\left(X; \sum_{\alpha<\tau} F_\alpha\right)$. Observe that $C_p\left(\sum_{\alpha<\tau} \operatorname{cl}_X(U_\alpha)\right) \sim \prod_{\alpha<\tau} C_p(\operatorname{cl}_X(U_\alpha)) \sim C_p(Y)^\tau \sim C_p\left(\sum_{\alpha<\tau} F_\alpha\right)$. Now, using the technique of Pelczynski [9] and Bessega [2] we have

$$C_p(X) \sim C_p\left(\sum_{\alpha<\tau} F_\alpha\right) \times C_p\left(X; \sum_{\alpha<\tau} F_\alpha\right) \sim$$

$$\sim C_p(Y)^\tau \times C_p\left(X; \sum_{\alpha<\tau} F_\alpha\right) \sim$$

$$\sim (C_p(Y)^\tau \times \ldots C_p(Y)^\tau \times \ldots) \times C_p(Y)^\tau \times C_p\left(X; \sum_{\alpha<\tau} F_\alpha\right) \sim$$

$$\sim (C_p(Y)^\tau \times \ldots C_p(Y)^\tau \times \ldots) \times C_p(X) \sim$$

$$\sim (C_p(X) \times E \times \ldots C_p(X) \times E \ldots) \times C_p(X) \sim C_p(X)^\omega \times E^\omega \sim$$

$$\sim (C_p(X) \times E)^{\omega} \sim C_p\left(\sum_{\alpha<\tau} \operatorname{cl}_X(U_\alpha)\right)^{\omega} \sim C_p(Y)^{\omega.\tau} = C_p(Y)^{\tau}\,.\ \Diamond$$

Theorem 2.4. *Let X be a metrizable space of weight $\tau \geq \omega$. Suppose X admits an open cover by sets homeomorphic to open subsets of Y, where Y is one of the spaces σ, Σ, $s \in \Sigma$, $\ell_2(\tau)$. Then $C_p(X) \sim C_p(Y)^{\tau}$.*

Proof. Since every point of Y does not contain a compact neighbourhood in Y the space X can not be compact. So there is a locally finite open cover $\{U_\alpha : \alpha < \tau\}$ of X of cardinality τ such that $\operatorname{cl}_X(U_\alpha)$ is a regularly closed subset of Y for every $\alpha < \tau$. On the other hand X contains as a closed subset a topological sum $\sum\limits_{\alpha<\tau} F_\alpha$ of regularly closed subsets F_α of Y. Then, by Lemma 2.1 and Remark 2.2, $C_p(\operatorname{cl}_X(U_\alpha)) \sim C_p(F_\alpha) \sim C_p(Y)$ for every $\alpha < \tau$. Hence, by Prop. 2.3 $C_p(X) \sim C_p(Y)^{\tau}$. $\Diamond$

Remark 2.5. *$C_p(\Sigma)$ is not homeomorphic to $C_p(s \times \Sigma)$ and $C_p(\sigma)$ is not linearly homeomorphic to $C_p(\Sigma)$.*

The first assertion follows from the observation that Σ is σ-compact and $s \times \Sigma$ is not σ-compact and the following result of Okunev [7]: if $C_p(X)$ and $C_p(Y)$ are homeomorphic and X is σ-compact then Y is also σ-compact.

Assume $C_p(\Sigma) \sim C_p(\sigma)$. By a result of Pestov [10] we have $\Sigma = \bigcup\limits_{i=1}^{\infty} Y_i$ such that:

(i) each Y_i is closed in Σ;

(ii) for any i and any $y \in Y_i$ there is an open neighbourhood V of y in Y_i such that V is a union of finitely many its closed subspaces A_k which can be embedded in σ.

Since Σ contains a copy of Q, there is an m such that Y_m also contains a copy of Q i.e. $Q \subset Y_m$. It follows from (ii) that for every $y \in Q \subset Y_m$ there exists an open neighbourhood V of y in Q with $V = \bigcup\limits_{k=1}^{n} A_k$, where each A_k is closed in V and can be embedded in σ. But V is a complete metric space, so $\operatorname{Int}_V(A_{k'}) \neq \emptyset$ for some k'. Thus $A_{k'}$, contains a copy of Q. Consequently σ contains also a copy of Q. Hence Q is a union of countably many finite-dimensional compacta because σ is a such space. It is well known that this is not possible. Therefore $C_p(\Sigma)$ is not linearly homeomorphic to $C_p(\sigma)$. $\Diamond$

3. The spaces $C_p^*(X)$

Lemma 3.1. [11]. *Suppose X is a metric space. Then $C_p^*(X;\times I) \sim \sim (\Pi C_p^*(X \times I))_c^*$.*

Corollary 3.2. *Let X be one of the spaces σ, Σ, $s \times \Sigma$, $\ell_2(\tau)$. If $Y = \sum\limits_{\alpha<\lambda} X$ is a topological sum of λ many copies of X, where $\lambda \geq \omega$, then $C_p^*(Y) \sim (\Pi C_p^*(Y))_c^*$*

Proof. Since $X \times I$ is homeomorphic to X we have that $Y \times I$ is homeomorphic to Y. Thus, by Lemma 3.1,

$$C_p^*(Y) \sim C_p^*(Y \times I) \sim (\Pi C_p^*(Y \times I))_c^* \sim (\Pi C_p^*(Y))_c^*. \quad \diamond$$

Lemma 3.3. [11]. *Let $\{X_\alpha : \alpha < \tau\}$ be an infinite family of spaces such that each X_α is closed in a metric space Y and contains a closed copy Y_α of Y. Then $C_p^*\left(\sum\limits_{\alpha<\tau} Y_\alpha\right) \sim \left(\Pi C_p^*\left(\sum\limits_{\alpha<\tau} X_\alpha\right)\right)_c^* \sim C_p^*\left(\sum\limits_{\alpha<\tau} X_\alpha\right)$ if $C_p^*\left(\sum\limits_{\alpha<\tau} Y\right) \sim \left(\Pi C_p^*\left(\sum\limits_{\alpha<\tau} Y\right)\right)_c^*$.*

Proposition 3.4. [11]. *Let $\{U_\alpha : \alpha < \tau\}$ be an infinite locally finite functionally open cover of a space X. Suppose there is a space Y such that $C_p^*(Y) \sim C_p^*\left(\sum\limits_{\alpha<\tau} \mathrm{cl}_X(U_\alpha)\right) \sim \left(\Pi C_p^*\left(\sum\limits_{\alpha<\tau} \mathrm{cl}_X(U_\alpha)\right)\right)_c^*$. Then $C_p^*(X) \sim C_p^*(Y)$ if X contains an ℓ^*-embedded copy of Y.*

Theorem 3.5. *Let X be a metrizable space of weight $\tau \geq \omega$. Suppose X admits an open cover by sets homeomorphic to open subsets of Y, where Y is one of the spaces σ, Σ, $s \times \Sigma$, $\ell_2(\tau)$. Then $C_p^*(X) \sim \sim C_p^*\left(\sum\limits_{\alpha<\tau} Y\right)$.*

Proof. Let $\{U_\alpha : \alpha < \tau\}$ be a locally finite open cover of X of cardinality τ such that $\mathrm{cl}_X(U_\alpha)$ is a regularly closed subset of Y for every $\alpha < \tau$. By Cor. 3.2 we have $C_p^*\left(\sum\limits_{\alpha<\tau} Y\right) \sim \left(\Pi C_p^*\left(\sum\limits_{\alpha<\tau} Y\right)\right)_c^*$. Since each set $\mathrm{cl}_X(U_\alpha)$ is closed in Y and contains a closed copy of Y, it follows from Lemma 3.3 that

$$C_p^*\left(\sum_{\alpha<\tau} \mathrm{cl}_X(U_\alpha)\right) \sim \left(\Pi C_p^*\left(\sum_{\alpha<\tau} \mathrm{cl}_X(U_\alpha)\right)\right)_c^* \sim C_p^*\left(\sum_{\alpha<\tau} Y\right).$$

Obviously X contains a closed copy of $\sum\limits_{\alpha<\tau} Y$. Thus, by Prop. 3.4,

$C_p^*(X) \sim C_p^*\left(\sum_{\alpha<\tau} Y\right)$. ◊

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