ELSEVIER
Topology and its Applications 55 (1994) 1–15
TOPOLOGY AND ITS APPLICATIONS

Set-valued maps and AE(0)-spaces

A. Chigogidze
Department of Mathematics, Moscow State University, Moscow 119899, Russia

V. Valov *
Department of Mathematics, University of Sofia, A. Ivanov 5, Sofia 1126, Bulgaria

(Received 29 July 1987)
(Revised 6 April 1992)

Abstract

Characterizations of noncompact AE(0)- and AE(1)-spaces in terms of set-valued maps are given.

Key words: AE(0)-space; AE(1)-space; Usco map; Continuum-valued u.s.c. map

AMS (MOS) Subj. Class.: Primary 54C55; secondary 54C60, 54C65

1. Introduction

There are several characterizations of compact AE(0)-spaces in terms of set-valued maps (see, for example, [6,9,12]). On the other hand we know only one such result for noncompact spaces [17]. Below in Section 2 we introduce a class of set-valued maps and show that a completely regular space X is an AE(0) if and only if every map $F: Y \to X$ from this class, where $\dim Y = 0$, has a continuous selection. As a consequence of the above characterization of AE(0)-spaces we get the following result, which is new even in the compact case: A space X is an AE(0) if and only if for every countable functionally open cover $\mathscr{U}$ of X there is a countable functionally open cover $\mathscr{V}$ of X refining $\mathscr{U}$ such that for any 0-dimensional space Y, any C-embedded subset A of Y and any two maps $f: Y \to X$ and $g: A \to X$, where g is $\mathscr{V}$-close to $f \mid A$, there exists a continuous extension $\bar{g}: Y \to X$

---

* Corresponding author.

SSDI 0166-8641(93)E0029-N

of g such that $\bar{g}$ is $\mathscr{U}$-close to f. In Section 3 we extend the corresponding result of Dranishnikov [6] characterizing AE(1)-spaces as continuum-valued upper semicontinuous retracts of infinite powers of the real line. Using this result we give an external characterization of Čech-complete AE(1)-spaces. An internal characterization of Čech-complete AE(1)-spaces is also given. Let us note that the proofs in Section 3 are quite different from the proofs of the corresponding results in the compact case.

All spaces considered are completely regular and Hausdorff and all single-valued maps between topological spaces continuous. Below N, I and R denote respectively the natural number series, the closed unit interval and the real line in their natural topologies. τ always denotes an infinite cardinal. We say that the R-weight of a space X does not exceed a cardinal τ (and write $R\text{-}w(X) \leqslant \tau$) if X can be embedded in R^τ as a C-embedded subspace. Evidently, the class of spaces with a countable R-weight coincides with the class of Polish (i.e., completely metrizable and separable) spaces. For a map $f: X \to Y$ let $C(f): C(Y) \to C(X)$ be the operator induced by f, where $C(X)$ is the set of all real-valued (continuous) maps on X. If $X \subset Y$, then $C(Y) | X$ is the set of all elements of $C(X)$ extendable to the whole of Y. Clearly, the equality $C(X) = C(Y) | X$ characterizes the C-embedded subspaces. Below, dim stands for the dimension defined by finite functionally open covers.

A space X is called (see [2,3]) an *absolute extensor* in dimension n ($n = 0, 1, \ldots$) (briefly, AE(n)-space) if for any space Z of dimension $\dim Z \leqslant n$ and any subspace Z_0 of Z each map $f: Z_0 \to X$ such that $C(f)(C(X)) \subset C(Z) | Z_0$ can be extended onto Z.

It is known that each AE(0)-space is realcompact [3]. Below we need the definition of n-soft maps only between realcompact spaces [3, Proposition 1.8]: A surjection $f: X \to Y$ between realcompact spaces is said to be n-soft ($n = 0, 1, \ldots$) if for any realcompact space Z of dimension $\dim Z \leqslant n$, any closed subspace Z_0 of Z, and any two maps $g: Z_0 \to X$ and $h: Z \to Y$ such that $C(g)(C(X)) \subset C(Z) | Z_0$, there exists a map $k: Z \to X$ such that $f \cdot k = h$ and $k | Z_0 = g$.

All set-valued maps $F: X \to Y$ are closed-valued. If B is a subset of Y then $F^{-1}(B)$ stands for the set $\{x \in X\colon F(x) \cap B \neq \emptyset\}$ and $F^{\#}(B) = \{x \in X\colon F(x) \subset B\}$. Recall that a set-valued map $F: X \to Y$ is lower (respectively, upper) semi-continuous (briefly, l.s.c. respectively, u.s.c.) if for each open subset U of Y the set $F^{-1}(U)$ (respectively, $F^{\#}(U)$) is open in X. By a usco map we mean a u.s.c. compact-valued map. If Y is a subset of X then F is called a retraction provided $F(x) = \{x\}$ for every $x \in Y$.

All the undefined notions concerning inverse spectra can be found in [3,13,14].

2. Maps having the selection-factorization property and AE(0)-spaces

Nedev [11] gave a unified method for proving selection and factorization theorems for set-valued maps. The key role in Nedev's paper is played by the notion of a set-valued map having the selection-factorization property. We consider maps with a similar property.

A set-valued map $F: X \to Y$ has the *weak selection-factorization property* (briefly, F is w.s.f.p.) if for any functionally closed subset H of X and any countable family $\mathscr{U}$ consisting of functionally open subsets of Y such that $F^{-1}(\mathscr{U}) = \{F^{-1}(U): U \in \mathscr{U}\}$ covers H, there is a locally finite functionally open (in H) cover of H refining $F^{-1}(\mathscr{U})$.

The following fact was actually proved in [11]: Let $F: X \to Y$ be a l.s.c. map. Then F is w.s.f.p. if X is paracompact or X is normal and F is compact-valued.

Lemma 2.1. *If any w.s.f.p. map $F: Y \to X$, where* $\dim Y = 0$, *has a selection, then X is an* AE(0)-space.

Proof. Consider X as a C-embedded subset of R^τ for some τ. There exists a functionally closed 0-invertible map f from N^τ onto R^τ (see [3, Proposition 1.14]). The definition of a 0-invertible map is omitted here, but we will note below how the 0-invertibility of f is used. Define a set-valued map $G: N^\tau \to X$ by

$$G(y) = \begin{cases} f(y), & \text{if } y \in f^{-1}(X); \\ X, & \text{otherwise}. \end{cases}$$

Let H be a functionally closed subset of N^τ and $\mathscr{U}$ be a countable family consisting of functionally open subsets of X such that $G^{-1}(\mathscr{U})$ covers H. Since $f(H)$ is a functionally closed subset of R^τ, the set $X - f(H)$ is functionally open in X. Hence, $\mathscr{U}^* = \mathscr{U} \cup \{X - f(H)\}$ is a countable functionally open cover of X. Take a countable functionally open cover $\mathscr{U}^{**}$ of R^τ such that $\mathscr{U}^* = \mathscr{U}^{**} \mid X$. This is possible because X is C-embedded in R^τ. Obviously, $f^{-1}(\mathscr{U}^{**}) \mid H$ is a countable functionally open cover of H, refining $G^{-1}(\mathscr{U})$. Choose a locally finite functionally open cover of H, refining $f^{-1}(\mathscr{U}^{**}) \mid H$. Thus, G is w.s.f.p. By our assumption, G has a selection, say g. Consequently, $g \mid f^{-1}(X) = f \mid f^{-1}(X)$. Consider a space Z of dimension $\dim Z = 0$, a subset Z_0 of Z and a map $p: Z_0 \to X$ such that $C(p)(C(X)) \subset C(Z) \mid Z_0$. Clearly, $C(p)(C(R^\tau)) \subset C(Z) \mid Z_0$. Since R^τ is an AE(0)-space, there is a map $q: Z \to R^\tau$ such that $q \mid Z_0 = p$. By the 0-invertibility of f, there exists a map $h: Z \to N^\tau$ with $f \cdot h = q$. It remains to note that $g \cdot h \mid Z_0 = p$. Thus, X is an AE(0)-space. □

Proposition 2.2. *The following conditions are equivalent for a space X:*

(i) *X is a Polish space;*

(ii) *any w.s.f.p. map $F: Y \to X$, where* $\dim Y = 0$, *has a selection.*

Proof. (i) → (ii). Let $F: Y \to X$ be a w.s.f.p. map and $\dim Y = 0$. We use the same arguments as in the proof of [11, Lemma 3.1] (see also [5]). First we show there exist:

(a) a sequence $\{\mathscr{V}_n = \{V_\alpha^n: \alpha \in A_n\}: n \in N\}$ of open locally finite coverings of X;

(b) a sequence $\{\mathscr{U}_n = \{U_\alpha^n: \alpha \in A_n\}: n \in N\}$ of disjoint open coverings of Y;

(c) a sequence $\{\pi_n\colon n \in N\}$ of mappings $\pi_n : A_{n+1} \to A_n$ such that:

(1) $U_\alpha^n \subset F^{-1}(V_\alpha^n)$ for every $n \in N$ and $\alpha \in A_n$,

(2) $U_\alpha^n = \bigcup\{U_\beta^{n+1}\colon \beta \in \pi_n^{-1}(\alpha)\}$ for every n and $\alpha \in A_n$,

(3) $V_\alpha^n = \bigcup\{V_\beta^{n+1}\colon \beta \in \pi_n^{-1}(\alpha)\}$ for every n and $\alpha \in A_n$,

(4) $\operatorname{diam}(V_\alpha^n) < 2^{-n}$ for every n and $\alpha \in A_n$,

(5) $\operatorname{card}(A_n) \leqslant \omega$ for every n.

Fix a sequence $\{\mathscr{W}_n = \{W_\beta^n\colon \beta \in B_n\}\colon n \in N\}$ of countable locally finite open covers of X, such that $\operatorname{diam}(W_\beta^n) < 2^{-n}$ for every n and $\beta \in B_n$. We can choose a locally finite functionally open cover $\{G_\beta^1\colon \beta \in B_1\}$ of Y which is an index-refinement of $F^{-1}(\mathscr{W}_1)$. Since $\dim Y = 0$, there is a disjoint open cover $\{U_\beta^1\colon \beta \in B_1\}$ of Y such that $U_\beta^1 \subset G_\beta^1$ for each $\beta \in B_1$ [8]. Now put $A_1 = B_1$ and $V_\alpha^1 = W_\alpha^1$ for every $\alpha \in A_1$. Let $\alpha \in A_1$. Consider the family $\gamma_\alpha = \{V_\alpha^1 \cap W_\beta^2\colon \beta \in B_2\}$. Obviously, $U_\alpha^1 \subset F^{-1}(\gamma_\alpha)$. Since U_α^1 is functionally closed in Y and F is w.s.f.p., there exists a locally finite functionally open cover $\{G_{\alpha\beta}^2\colon \beta \in B_2\}$ of U_α^1 which is an index-refinement of $F^{-1}(\gamma_\alpha)$. Take a disjoint open cover $\{U_{\alpha\beta}^2\colon \beta \in B_2\}$ of U_α^1 such that $U_{\alpha\beta}^2 \subset G_{\alpha\beta}^2$ for every $\beta \in B_2$. This is possible because $\dim U_\alpha^1 = 0$. Put $A_2 = A_1 \times B_2$, $V_{(\alpha,\beta)}^2 = V_\alpha^1 \cap W_\beta^2$ and $U_{(\alpha,\beta)}^2 = U_{\alpha\beta}^2$. Define $\pi_1\colon A_2 \to A_1$ to be the natural projection. The next steps are obvious.

Now for every n and $\alpha \in A_n$ choose a point $x_\alpha^n \in V_\alpha^n$. Define maps $g_n : Y \to X$, $n \in N$, by $g_n(y) = x_\alpha^n$ if $y \in U_\alpha^n$. Clearly, g_n are continuous, and for every n and $y \in Y$ there is $\alpha \in A_n$ such that $g_n(y)$, $g_{n+1}(y) \in V_\alpha^n$. This allows us to define a continuous map $g : Y \to X$ by $g(y) = \lim g_n(y)$. Let us note that g is a selection for F.

(ii) → (i). By Lemma 2.1, X is an AE(0)-space. It should be observed that each closed subspace of any space satisfying condition (ii) also satisfies this condition. Consequently, each closed subspace of X is an AE(0)-space. The last implies X is a Polish space. Indeed, suppose the contrary. Then X is not metrizable [4]. By [4, Theorem 3], X contains a copy of the Cantor cube D^{ω_1}. Hence, each closed subspace of D^{ω_1}, being an AE(0)-space, satisfies the countable chain condition [3, Theorem 4]. Contradiction. Thus, X is a Polish space. □

A closed-valued map $F : X \to Y$ is said to have a countable singularity if the collection of all images of F, containing at least two points, is countable and consists of functionally closed subsets of Y.

Theorem 2.3. *The following conditions are equivalent for any space X:*

(i) *X is an* AE(0)*-space;*

(ii) *any w.s.f.p. map $F : Y \to X$ with a countable singularity, where* $\dim Y = 0$, *has a selection.*

Proof. (i) → (ii). Let $F : Y \to X$ be a w.s.f.p. map with a countable singularity and $\dim Y = 0$. By Proposition 2.2, we can assume that X is not a Polish space. Then, by [3], X is represented as a limit space of a factorizing sigma-spectrum $\mathscr{S} = \{X_\alpha, p_\alpha^\beta, A\}$ consisting of Polish spaces and 0-soft limit projections p_α. Let $\{F_n\colon n \in N\}$ be the countable collection of nontrivial images of F. Since $\mathscr{S}$ is a factorizing sigma-spectrum and each F_n is functionally closed in X, there is an index $\alpha \in A$ and a collection $\{Z_n\colon n \in N\}$ of closed subsets of X_α such that $F_n = p_\alpha^{-1}(Z_n)$ for each n. Consider the w.s.f.p. map $F_\alpha : Y \to X_\alpha$ defined by $F_\alpha(y) = p_\alpha(F(y))$, $y \in Y$. By Proposition 2.2, the map F_α has a selection $g : Y \to X_\alpha$. Put $Y_0 = \{y \in Y\colon \operatorname{card}(F(y)) = 1\}$ and define a map $f : Y_0 \to X$ by $f(y) = F(y)$ for each $y \in Y_0$. Clearly, f is continuous and $p_\alpha \cdot f = g \mid Y_0$.

Claim. $C(f)(C(X)) \subset C(Y) \mid Y_0$.

Proof. Let $\varphi \in C(X)$. Define a set-valued map $G : Y \to R$, $G(y) = \operatorname{cl}(\varphi(F(y)))$, $y \in Y$. Obviously, G is w.s.f.p. Consequently, by Proposition 2.2, G has a selection ψ. It is easy to see that ψ extends $\varphi \cdot f$ and hence $C(f)(C(X)) \subset C(Y) \mid Y_0$. The claim is proved.

Now, the 0-softness of p_α implies the existence of a map $h : Y \to X$ such that $h \mid Y_0 = f$ and $p_\alpha \cdot h = g$. One can easily verify that h is the desired selection for F.

(ii) → (i). This implication is actually proved in Lemma 2.1. □

Let $\mathscr{U}$ be a cover of a space X and f and g be two maps from a space Y to X. We say that f and g are *$\mathscr{U}$-close* if for every $y \in Y$ there is $U \in \mathscr{U}$ such that $f(y)$, $g(y) \in U$.

Corollary 2.4. *A space X is an* AE(0) *if and only if for every countable functionally open cover $\mathscr{U}$ of X there is a countable functionally open and locally finite cover $\mathscr{V}$ of X refining $\mathscr{U}$ such that for any* 0*-dimensional space Y, any subset A of Y and any two maps $f : Y \to X$ and $g : A \to X$ with $C(g)(C(X)) \subset C(Y) \mid A$ and g is $\mathscr{V}$-close to $f \mid A$, there exists a continuous extension $\bar{g} : Y \to X$ of g such that $\bar{g}$ is $\mathscr{U}$-close to f.*

Proof. Let $X \in$ AE(0) and $\mathscr{U}$ be a countable functionally open cover of X. Choose countable functionally open and locally finite covers $\mathscr{V} = \{V_i\colon i = 1, 2, \ldots\}$ and $\mathscr{A}$ of X such that $\mathscr{V}$ is a closure-refinement of $\mathscr{A}$ and $\mathscr{A}$ is a closure-refinement of $\mathscr{U}$. Suppose Y is a 0-dimensional space, $A \subset Y$ and $f : Y \to X$, $g : A \to X$ are two maps such that g is $\mathscr{V}$-close to $f \mid A$ and $C(g)(C(X)) \subset C(Y) \mid A$. Since $X \in$ AE(0), there is a continuous extension $g_1 : Y \to X$ of g. Put $W = \bigcup\{V_i \times V_i\colon i \in N\}$. Then W is open in X^2 and the set $P = \operatorname{cl}_{X^2}(W)$ is functionally closed in X^2 because X^2 is k-metrizable as an AE(0)-space [18]. Hence, the set $B = (f \vartriangle g_1)^{-1}(P)$ is also functionally closed in Y and $A \subset B$, where $f \vartriangle g_1 : Y \to X^2$ is the diagonal product of f and g_1. Let $g_2 = g_1 \mid B$. Define a set-valued map $F : Y \to X$ by

$$F(y) = \begin{cases} g_2(y), & y \in B; \\ \operatorname{cl}_X(\operatorname{st}_{\mathscr{A}}(f(y))), & y \notin B. \end{cases}$$

Here $\operatorname{st}_{\mathscr{A}}(f(y))$ is the star of $f(y)$ with respect to $\mathscr{A}$. Since $\mathscr{A}$ is locally finite, for every $y \notin B$ the set $F(y)$ is a finite union of closures $\operatorname{cl}_X(V)$, $V \in \mathscr{A}$. Thus, the family $\{F(y)\colon y \notin B\}$ is countable and consists of functionally closed subsets of X. So, F has a countable singularity. We prove that F is w.s.f.p. Let H be a functionally closed subset of Y and $\mathscr{U}^* = \{U_i\colon i \in N\}$ be a countable family of functionally open subsets of X such that $\{F^{-1}(U_i)\colon i\in N\}$ covers H. Obviously, $F^{-1}(U_i)=g_2^{-1}(U_i)\cup(f^{-1}(\mathrm{st}_{\mathscr{A}}(U_i))-B)$, $i\in N$. Since each U_i is functionally open and $\mathscr{A}$ consists of functionally open subsets of X, $\mathrm{st}_{\mathscr{A}}(U_i)$ is also functionally open. Thus, $f^{-1}(\mathrm{st}_{\mathscr{A}}(U_i))-B$ is a functionally open subset of Y. Let $W_i=g_1^{-1}(U_i)$, $i\in N$. It is easily seen that for every $y\in B$ there is $V_i\in\mathscr{V}$ with $f(y)$, $g_2(y)\in \mathrm{cl}_X(V_i)$. This implies that g_2 is $\mathscr{A}$-close to $f\,|\,B$ and $g_2^{-1}(U_i)\subset f^{-1}(\mathrm{st}_{\mathscr{A}}(U_i))$. Then $\sigma=\{f^{-1}(\mathrm{st}_{\mathscr{A}}(U_i))-B,\ W_i\cap f^{-1}(\mathrm{st}_{\mathscr{A}}(U_i)),\ i\in N\}$ is a countable family of functionally open subsets of Y such that σ refines $F^{-1}(\mathscr{U}^*)$ and σ covers H. Choose a locally finite functionally open cover of H refining σ. Hence F is w.s.f.p. Now, by Theorem 2.3, there is a continuous selection $\bar{g}$ for F. Clearly, $\bar{g}$ is an extension of g and $\bar{g}$ is $\mathscr{U}$-close to f.

To prove the sufficiency consider X as a C-embedded subset of R^τ for some τ. Take a 0-invertible map k from N^τ onto R^τ [3]. Using the 0-invertibility of k one can see that it is enough to show the map $k_1=k\,|\,k^{-1}(X)$ has a continuous extension $\bar{k}: N^\tau\to X$ (see the proof of Lemma 2.1). Let $\mathscr{U}=\{X\}$. By our assumption, there is a countable functionally open cover $\mathscr{V}$ of X such that for any 0-dimensional space Y, any subset A of Y and any two maps $f: Y\to X$ and $g: A\to X$ with $C(g)(C(X))\subset C(Y)\,|\,A$ and g is $\mathscr{V}$-close to $f\,|\,A$, there exists a continuous extension $\bar{g}: Y\to X$ of g so that $\bar{g}$ is $\mathscr{U}$-close to f. Let $\mathscr{V}^*$ be a countable functionally open cover of R^τ such that $\mathscr{V}^*\,|\,X=\mathscr{V}$. Since $\dim N^\tau=0$ and $k^{-1}(\mathscr{V}^*)$ is a countable functionally open cover of N^τ, there is a countable open, locally finite and disjoint cover $\{V_i\colon i\in N\}$ of N^τ refining $k^{-1}(\mathscr{V}^*)$. Put $Z=k^{-1}(X)$ and for every i choose a point $x(i)\in k(V_i\cap Z)$ (if $V_i\cap Z=\emptyset$, then $x(i)$ is an arbitrary point in X). Define a continuous map $f: N^\tau\to X$ by $f(y)=x(i)$ provided $y\in V_i$. Clearly, k_1 and $f\,|\,Z$ are $\mathscr{V}$-close. We have $C(k_1)(C(X))\subset C(N^\tau)\,|\,Z$ because X is C-embedded in R^τ. Therefore, there exists a continuous extension $\bar{k}: N^\tau\to X$ of k_1. $\square$

3. Upper semi-continuous maps and AE(1)-spaces

Lemma 3.1. *Let $\mathscr{S}=\{Y_i, g_i^{i+1}, N\}$ be an inverse sequence and $r_i: X\to Y_i$ be a usco map, $i\in N$. Suppose $q_i^{i+1}\cdot r_{i+1}=r_i$ for each $i\in N$. Then there is a usco map $r: X\to \varprojlim\mathscr{S}$ such that $q_i\cdot r=r_i$ for every i.*

Proof. For any point $x\in X$ consider the inverse sequence $\mathscr{S}(x)=\{r_i(x), t_i^{i+1}, N\}$, where $t_i^{i+1}=q_i^{i+1}\,|\,r_{i+1}(x)$. Define a compact-valued map $r: X\to\varprojlim\mathscr{S}$ by $r(x)=\varprojlim\mathscr{S}(x)$, $x\in X$. Clearly, $q_i\cdot r=r_i$ for each i. An easy verification of the upper semi-continuity of r is left to the reader. $\square$

A u.s.c. map $r: X\to Y$ is said to be *strongly u.s.c.* if for any functionally open subset U of Y there is a functionally open subset $W(U)$ of X such that $r^{\#}(U)\subset W(U)\subset r^{\#}(\mathrm{cl}_Y(U))\cap G_\delta(r^{\#}(U))$. Here $G_\delta(r^{\#}(U))$ is the G_δ-closure of $r^{\#}(U)$ in X, i.e., the set $\{x\in X\colon$ every G_δ-subset of X containing x intersects $r^{\#}(U)\}$.

Remark 3.2. If $r: X \to Y$ is a strongly u.s.c. retraction then $W(U) \subset r^{-1}(U)$ for every functionally open subset U of Y (in particular $U \subset W(U)$).

Proof. Let $x \in W(U)$. Suppose $r(x) \subset Y - U$. Then $x \in r^{\#}(Y-U) \cap G_{\delta}(r^{\#}(U))$. But $r^{\#}(Y-U)$ is a G_{δ}-set in X because $Y-U$ is functionally closed in Y. Hence, $r^{\#}(Y-U) \cap r^{\#}(U) \neq \emptyset$, which is impossible. Thus, $r(x) \cap U \neq \emptyset$. So, $W(U) \subset r^{-1}(U)$. □

Lemma 3.3. Let $\mathscr{S}_X = \{X_{\alpha}, p_{\alpha}^{\beta}, A\}$ *and* $\mathscr{S}_Y = \{Y_{\alpha}, q_{\alpha}^{\beta}, A\}$ *be two factorizing sigma-spectra with surjective limit projections and* $r: \varprojlim \mathscr{S}_X \to \varprojlim \mathscr{S}_Y$ *be a strongly usco map. Then the set* $A^* = \{\alpha \in A$: *there is a usco map* r_{α}: $X_{\alpha} \to Y_{\alpha}$ *with* $q_{\alpha} \cdot r = r_{\alpha} \cdot p_{\alpha}\}$ *is cofinal and sigma-complete* [14] *in* A.

Proof. Let $X = \varprojlim \mathscr{S}_X$ and $Y = \varprojlim \mathscr{S}_Y$. Consider the following relation $L \subset A^2$: $(\alpha, \beta) \in L$ if $\beta \geqslant \alpha$ and there is a usco map $r_{\alpha,\beta}: X_{\beta} \to Y_{\alpha}$ such that $q_{\alpha} \cdot r = r_{\alpha,\beta} \cdot p_{\beta}$.

Claim 1. *For each* $\alpha \in A$ *there exists an index* $\beta \in A$ *such that* $(\alpha, \beta) \in L$.

Proof. Fix a finitely additive countable open base $\mathscr{U}(\alpha) = \{U_n$: $n \in N\}$ of Y_{α}. For every n let $W(n) = W(q_{\alpha}^{-1}(U_n))$ be a functionally open subset of X such that

$$r^{\#}\left(q_{\alpha}^{-1}(U_n)\right) \subset W(n) \subset r^{\#}\left(\mathrm{cl}_Y\left(q_{\alpha}^{-1}(U_n)\right)\right) \cap G_{\delta}\left(r^{\#}\left(q_{\alpha}^{-1}(U_n)\right)\right). \qquad (1)$$

Since $\mathscr{S}_X$ is a factorizing sigma-spectrum, there is $\beta \in A$ and open sets V_n, $n \in N$, of X_{β} with $p_{\beta}^{-1}(V_n) = W(n)$ for each $n \in N$. We can assume $\beta \geqslant \alpha$. Let us show that $q_{\alpha}(r(x_1)) = q_{\alpha}(r(x_2))$ for any two points $x_1, x_2 \in p_{\beta}^{-1}(b)$, where b is an arbitrary point of X_{β}. Since $\mathscr{U}(\alpha)$ is a finitely additive base and $q_{\alpha}(r(x_2))$ is a compact subset of Y_{α}, there is $U_n \in \mathscr{U}(\alpha)$ such that $q_{\alpha}(r(x_2)) \subset U_n$. This implies, by (1), $x_2 \in W(n)$. Consequently, by the choice of β, $x_1 \in W(n)$. It follows from (1) that $q_{\alpha}(r(x_1))$ is contained in the closure of U_n. Thus, $q_{\alpha}(r(x_2))$ contains $q_{\alpha}(r(x_1))$. Analogously, $q_{\alpha}(r(x_2)) \subset q_{\alpha}(r(x_1))$. Hence, $q_{\alpha}(r(x_2)) = q_{\alpha}(r(x_1))$. Therefore, we can define a compact-valued map $r_{\alpha,\beta}: X_{\beta} \to Y_{\alpha}$ by $r_{\alpha,\beta}(b) = q_{\alpha}(r(p_{\beta}^{-1}(b)))$, $b \in X_{\beta}$. Obviously, $q_{\alpha} \cdot r = r_{\alpha,\beta} \cdot p_{\beta}$. It remains to prove that $r_{\alpha,\beta}$ is u.s.c. Let $r_{\alpha,\beta}(b) \subset U$, where $b \in X_{\beta}$ and U is open in Y_{α}. Take $U_n \in \mathscr{U}(\alpha)$ such that $r_{\alpha,\beta}(b) \subset U_n \subset \mathrm{cl}(U_n) \subset U$. Then $p_{\beta}^{-1}(b) \subset W(n)$, i.e., $b \in V_n$, and for every $z \in V_n$ we have $r_{\alpha,\beta}(z) \subset \mathrm{cl}(U_n)$. Thus, $r_{\alpha,\beta}$ is u.s.c. So, $(\alpha, \beta) \in L$.

Claim 2. *If* $(\alpha, \beta) \in L$ *and* $\gamma \geqslant \beta$, *then* $(\alpha, \gamma) \in L$.

Proof. Define a usco map $r_{\alpha,\gamma}: X_{\gamma} \to Y_{\alpha}$ by $r_{\alpha,\gamma} = r_{\alpha,\beta} \cdot p_{\beta}^{\gamma}$.

Claim 3. *Let* $\{\alpha_i$: $i \in N\} \subset A$ *be a countable chain in* A *and* $(\alpha_i, \beta) \in L$ *for each* i. *Then* $(\alpha, \beta) \in L$, *where* $\alpha = \sup \alpha_i$.

Proof. The validity of this claim follows from Lemma 3.1. and continuity [14] of the spectrum $\mathscr{S}_Y$.

Thus, the relation L has the properties formulated in the above claims. Consequently, by [14, Proposition 1.3], the set $A^* = \{\alpha \in A$: $(\alpha, \alpha) \in L\}$ is cofinal and sigma-complete in A. □

Lemma 3.4. *Let $f : X \to Y$ be a functionally open surjection between* AE(0)*-spaces. Suppose that X is C-embedded in the product $Y \times R^{\omega}$ and $f = \pi \mid X$, where $\pi : Y \times R^{\omega} \to Y$ is the natural projection. If there is a continuum-valued strongly u.s.c. retraction $r : Y \times R^{\omega} \to X$ such that $f \cdot r = \pi$, then f is* 1*-soft.*

Proof. First suppose that Y is a Polish space. Since the projection π is a trivial fibration with the fiber R^{ω}, each fiber of π is connected and the family $\{\pi^{-1}(y)\colon y \in Y\}$ is equi-LC^0 in the sense of Michael [10]. One can easily verify that the existence of a continuum-valued u.s.c. retraction r with $f \cdot r = \pi$, implies that all fibers of f are connected and the family $\{f^{-1}(y)\colon y \in Y\}$ is also equi-LC^0. Hence, by [3, Proposition 1.12], f is 1-soft.

Now, consider the case $R\text{-}w(Y) > \omega$. Represent Y as a limit space of a factorizing sigma-spectrum $\mathscr{S}_Y = \{Y_\alpha, q_\alpha^\beta, A\}$ consisting of Polish spaces and 0-soft limit projections q_α, $\alpha \in A$ [3]. Then $\mathscr{S} = \{Y_\alpha \times R^{\omega}, q_\alpha^\beta \times \mathrm{id}, A\}$ is also a factorizing sigma-spectrum such that $Y \times R^{\omega} = \varprojlim \mathscr{S}$ and $\pi = \varprojlim \pi_\alpha$, where $\pi_\alpha : Y_\alpha \times R^{\omega} \to Y_\alpha$ denotes the natural projections. Put $X_\alpha = \mathrm{cl}((q_\alpha \times \mathrm{id})(X))$ and $p_\alpha^\beta = (q_\alpha^\beta \times \mathrm{id}) \mid X_\beta$ for each $\alpha, \beta \in A$, $\alpha \leqslant \beta$. Since X is closed and C-embedded in $Y \times R^{\omega}$, $\mathscr{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ is a factorizing sigma-spectrum with $X = \varprojlim \mathscr{S}_X$. By [3] and the spectral theorem for sigma-spectra [14], without loss of generality we can assume that all limit projections p_α of $\mathscr{S}_X$ are 0-soft (in particular, $X_\alpha = (q_\alpha \times \mathrm{id})(X)$, $\alpha \in A$). [3, Theorem 1.19] implies that all $f_\alpha = \pi_\alpha \mid X_\alpha$ and all limit square diagrams (formed by the limit projections p_α, q_α and the maps f and f_α) can be supposed to be, respectively, open surjections and pullback squares. By Lemma 3.3, there exists at least one index $\alpha \in A$ and a continuum-valued u.s.c. map $r_\alpha : Y_\alpha \times R^{\omega} \to X_\alpha$ such that $p_\alpha \cdot r = r_\alpha \cdot (q_\alpha \times \mathrm{id})$. It follows from our construction that r_α is a continuum-valued u.s.c. retraction and $f_\alpha \cdot r_\alpha = \pi_\alpha$. Then, by the previous case, f_α is 1-soft. Finally, since the corresponding limit square diagram is a pullback square, we can conclude that f is 1-soft. □

Let $r : R^A \to X$ be a u.s.c. retraction. Below we use the following notations: If B is a subset of A, π_B is the natural projection from R^A onto R^B, $X_B = \pi_B(X)$ and $X(B) = \pi_B^{-1}(X_B)$. If $C \subset B$, π_C^B denotes the projection from R^B onto R^C. The restrictions $\pi_B \mid X$ and $\pi_C^B \mid X_B$ are denoted respectively by p_B and p_C^B.

Let $r : R^A \to X$ be a u.s.c. retraction. A subset B of A is called *r-admissible* if $p_B(r(x)) = \pi_B(x)$ for every $x \in X(B)$.

Lemma 3.5. *Let $r : R^A \to X$ be a strongly u.s.c. retraction. Then we have*:

(i) *A union of r-admissible sets is r-admissible*;

(ii) *every countable set $C \subset A$ is contained in a countable r-admissible set B.*

Proof. (i) Suppose $\{B(s)\colon s \in S\}$ is a family of r-admissible sets and $B = \bigcup\{B(s)\colon s \in S\}$. Let $x \in X(B)$. Since $X(B) \subset X(B(s))$, $s \in S$, we have $P_{B(s)}(r(x)) = \pi_{B(s)}(x)$ for every $s \in S$. This implies $p_B(r(x)) = \pi_B(x)$. So, B is r-admissible.

(ii) First we construct by induction an increasing sequence $\{B(n)\colon n \in N\}$ of countable subsets of A such that $B(1) = C$ and $P_{B(n)}(r(x)) = \pi_{B(n)}(x)$ for any

$x \in X(B(n+1))$, $n \in N$. Assume we have already constructed $B(i)$ for $i \leqslant n$. Take a countable base $\mathscr{B}$ of $X_{B(n)}$ and for every $U \in \mathscr{B}$ let $W(U)$ be a functionally open subset of R^A such that $(p_{B(n)})^{-1}(U) \subset W(U) \subset r^{\#}((p_{B(n)})^{-1}(\mathrm{cl}(U)))$ (see Remark 3.2). By a result from [19], there is a countable set $B(U) \subset A$ with $(\pi_{B(U)})^{-1}(\pi_{B(U)}(W(U))) = W(U)$. Put $B(n+1) = B(n) \cup \bigcup\{B(U)\colon U \in \mathscr{B}\}$. If $x \in X(B(n+1))$ then there is $y \in X$ such that $\pi_{B(n+1)}(x) = p_{B(n+1)}(y)$. Hence, $\pi_{B(n)}(x) = p_{B(n)}(y) = z$. Let $z \in U^*$, where $U^* \in \mathscr{B}$. Then $y \in W(U^*)$, and since $B(U^*) \subset B(n+1)$, x is contained in $W(U^*)$. The last implies $p_{B(n)}(r(x)) \subset \mathrm{cl}(U^*)$. Consequently, we prove for every $U \in \mathscr{B}$ the set $\mathrm{cl}_{X_{B(n)}}(U)$ contains $p_{B(n)}(r(x))$ provided U contains $\pi_{B(n)}(x)$. Thus, $P_{B(n)}(r(x)) = \pi_{B(n)}(x)$ for each $x \in X(B(n+1))$. So, the inductive step is finished. Finally, let $B = \bigcup\{B(n)\colon n \in N\}$. Obviously, B is countable. It remains to prove B is r-admissible. Suppose $x \in X(B)$. Since $X(B) \subset \bigcap\{X(B(n))\colon n \in N\}$, x is contained in each $X(B(n))$. Consequently, $p_{B(n)}(r(x)) = \pi_{B(n)}(x)$ for every n. Hence, $p_B(r(x)) = \pi_B(x)$. Therefore, B is r-admissible. □

Lemma 3.6. *Let $r\colon R^A \to X$ be a strongly usco retraction. Suppose B is an r-admissible subset of A. Then:*

(i) *P_B is a functionally open map;*

(ii) *X_B is a C-embedded* AE(0)*-subspace of R^B and there is a usco retraction r_B from R^B onto X_B;*

(iii) *for every r-admissible set C with $C \subset B$ there is a usco retraction $r_C^B\colon X_C \times R^{B-C} \to X_B$ such that $p_C^B \cdot r_C^B = \pi_C^B | (X_C \times R^{B-C})$.*

Proof. (i) Let U be a functionally open subset of X. Take a functionally open subset $W(U)$ of R^A such that $U \subset W(U) \subset r^{\#}(\mathrm{cl}_X(U)) \cap G_\delta(r^{\#}(U))$. Since the projection π_B is functionally open, it suffices to show that $p_B(U) = \pi_B(W(U)) \cap X_B$. Suppose $\pi_B(x) \in X_B \cap \pi_B(W(U))$, where $x \in W(U)$. Then $x \in X(B)$ and by the r-admissibility of B, we have $\pi_B(x) = p_B(r(x))$. By Remark 3.2, $W(U) \subset r^{-1}(U)$. Therefore, $r(x) \cap U \neq \emptyset$. This implies $\pi_B(x) = P_B(r(x)) \in P_B(U)$. Thus, $\pi_B(W(U)) \cap X_B \subset P_B(U)$. The inverse inclusion is obvious.

(ii) Let $i_B\colon R^B \to R^A$ be an embedding with $\pi_B \cdot i_B = \mathrm{id}_{R^B}$. Then $r_B\colon R^B \to X_B$, defined by $r_B = p_B \cdot r \cdot i_B$, is a usco retraction. Hence, by [17, Lemma 7 and Theorem 1], X_B is a C-embedded AE(0)-subspace of R^B.

(iii) Put $r_C^B = r_B | (X_C \times R^{B-C})$, where r_B is the usco retraction defined in the proof of (ii). □

A continuum-valued u.s.c. map F from X to Y is said to be *minimal continuum-valued* if every continuum-valued u.s.c. map $F^*\colon X \to Y$ such that $F^*(x) \subset F(x)$ for each $x \in X$, coincides with F. It follows from the Kuratowski–Zorn lemma that every continuum-valued u.s.c. map has a minimal continuum-valued u.s.c. selection.

Lemma 3.7. *Let X be perfectly k-normal in the sense of Ščepin* [13].

(i) *If M is G_δ-dense in X, i.e., $G_\delta(M)=X$, then every continuum-valued u.s.c. map from M to a compact space Y can be extended to a continuum-valued u.s.c. map from X to Y.*

(ii) *Suppose $G_\delta(U)$ is open for every open set U in X. Then any minimal continuum-valued u.s.c. map $F: X \to Y$ has the following property: $G_\delta(F^{\#}(V)) \subset F^{\#}(\mathrm{cl}_Y(V))$ for each open V in Y.*

Proof. (i) Let $F: M \to Y$ be a continuum-valued u.s.c. map. For every $x \in X$ denote by $\mathscr{U}(x)$ a local base at x in X. Then the usco map F_1 defined by $F_1(x) = \bigcap\{\mathrm{cl}_Y(F(U \cap M)): U \in \mathscr{U}(x)\}$ is an extension of F (see [17, Lemma 8]). Suppose there is $x_0 \in X - M$ such that $F_1(x_0)$ is not connected. Then $F_1(x_0) = H_1 \cup H_2$, where H_1 and H_2 are nonempty disjoint closed subsets of Y. Take disjoint open neighborhoods V_1 and V_2 in Y of H_1 and H_2, respectively. Since F_1 is u.s.c., $U = F_1^{\#}(V_1 \cup V_2)$ is an open neighborhood of x_0 in X. Let $U_i = F^{\#}(V_i)$, $i = 1, 2$. Obviously, U_1 and U_2 are open disjoint subsets of M and $U \cap M = U_1 \cup U_2$. It follows from $G_\delta(M) = X$ that $x_0 \in G_\delta(U \cap M) = G_\delta(U_1) \cup G_\delta(U_2)$. Assume $x_0 \in G_\delta(U_1)$. The set $W = \mathrm{Int}_X(\mathrm{cl}_X(U_1))$ is functionally open in X because X is perfectly k-normal. So, $G_\delta(U_1) \subset W$. Hence, $x_0 \in W$. Then $W \cap U \in \mathscr{U}(x_0)$ and $W \cap U \cap M = U_1$. Therefore, $F_1(x_0) \subset \mathrm{cl}_Y(F(W \cap U \cap M)) \subset \mathrm{cl}_Y(V_1)$. This implies $H_2 = \emptyset$. Thus, $F_1(x)$ is connected for each $x \in X$.

(ii) Let F be a minimal continuum-valued u.s.c. map from X to a space Y. Take an open set V in Y and a point $x^* \in G_\delta(F^{\#}(V)) - F^{\#}(V)$. Put $M = X - (G_\delta(F^{\#}(V)) - F^{\#}(V))$. Clearly, $G_\delta(M) = X$. By (i), the map $F_1: X \to \beta Y$, $F_1(x) = \bigcap\{\mathrm{cl}_{\beta Y}(F(U \cap M)): U \in \mathscr{U}(x)\}$ is a continuum-valued u.s.c. extension of $F \mid M$. It is easily seen that $F_1(x) \subset F(x)$ for every x from X. Hence, F_1 coincides with F. Since $G_\delta(F^{\#}(V))$ is open in X, there is an $U \in \mathscr{U}(x^*)$ with $U \subset G_\delta(F^{\#}(V))$. Then $F(x^*) = F_1(x^*)$ is contained in $\mathrm{cl}_{\beta Y}(F(U \cap M))$. But $U \cap M = U \cap F^{\#}(V)$. Thus, $F(x^*) \subset \mathrm{cl}_{\beta Y}(V) \cap Y = \mathrm{cl}_Y(V)$. □

Corollary 3.8. *Let X be an* AE(0)*-space. Then any minimal continuum-valued u.s.c. map, defined on X, is strongly u.s.c.*

Proof. Let $F: X \to Y$ be a minimal continuum-valued u.s.c. map. X being an AE(0)-space is k-metrizable [18]. Thus, X is perfectly k-normal. We show that $G_\delta(U)$ is functionally open for every open subset U of X. By [17], there are a closed embedding of X into R^A, where $A = w(X)$, and a usco retraction r from R^A onto X. Since, by [19], the set $G_\delta(r^{\#}(U))$ is functionally open in R^A, it suffices to prove that $G_\delta(r^{\#}(U)) \cap X = G_\delta(U)$. Obviously, $G_\delta(U)$ is contained in $G_\delta(r^{\#}(U)) \cap X$. Let $x \in G_\delta(r^{\#}(U)) \cap X$ and H be a G_δ-subset of X containing x. Then $r^{\#}(H)$ is G_δ in R^A and $x \in r^{\#}(H)$. Thus, $r^{\#}(U) \cap r^{\#}(H) \neq \emptyset$. This implies $U \cap H \neq \emptyset$. Hence, $G_\delta(r^{\#}(U)) \cap X \subset G_\delta(U)$. So, $G_\delta(U)$ is functionally open. Finally, if V is open in Y, put $W(V) = G_\delta(F^{\#}(V))$. It follows from Lemma 3.7 that $F^{\#}(V) \subset W(V) \subset F^{\#}(\mathrm{cl}_Y(V)) \cap G_\delta(F^{\#}(V))$. Therefore F is strongly u.s.c. □

Theorem 3.9. *The following conditions are equivalent for any space X:*

(i) *X is an* AE(1)*-space;*

(ii) *for any C-embedding of X into R^τ there is a continuum-valued u.s.c. retraction from R^τ onto X;*

(iii) *there exists a C-embedding of X into R^τ (for $\tau = R\text{-}w(X)$) and a continuum-valued u.s.c. retraction from R^τ onto X.*

Proof. (i) → (ii). Let X be a C-embedded subset of R^τ. Embed R^τ into the cube I^τ as a dense subspace and consider an open monotone surjection $f: T \to I^\tau$, where T is an AE(0)-compactum with dim $T = 1$ [7, Theorem 9]. Put $K = f^{-1}(R^\tau)$ and $g = f \mid K$. Since f is open, K is dense in T. Consequently, by [1, Corollary 7], dim $K = 1$. Denote $K_0 = g^{-1}(X)$ and $g_0 = g \mid K_0$. Since X is C-embedded in R^τ, we can conclude that $C(g_0)(C(X)) \subset C(K) \mid K_0$. Thus, there is a map $h: K \to X$ such that $h \mid K_0 = g_0$. Then $r = h \cdot g^{-1}: R^\tau \to X$ is a continuum-valued u.s.c. retraction because g is a perfect and monotone map.

(ii) → (iii). This implication is obvious.

(iii) → (i). First consider the case $\tau = \omega$. Then X is a Polish space. The existence of a continuum-valued u.s.c. retraction from R^ω onto X implies that $X \in \mathrm{LC}^0$ and C^0. Hence, by [3, Proposition 1.11], X is an AE(1)-space.

Now consider the case $\omega < \tau = R\text{-}w(X)$. Let $A = \{\alpha\colon \alpha < \tau\}$. Fix a continuum-valued u.s.c. retraction $r: R^A \to X$. Without loss of generality we can suppose r is a minimal continuum-valued u.s.c. retraction. Since R^A is an AE(0)-space, by Corollary 3.8, r is strongly u.s.c. By Lemma 3.5(ii), for every $\alpha < \tau$ choose a countable r-admissible set $B(\alpha) \subset A$ containing α. Let $A(\alpha) = \bigcup\{B(\beta)\colon \beta < \alpha\}$, $\alpha < \tau$. Denote $X_\alpha = X_{A(\alpha)}$ and $p_\alpha^{\alpha+1} = p_{A(\alpha)}^{A(\alpha+1)}$. In this way we obtain a transfinite inverse spectrum $\mathscr{S} = \{X_\alpha, p_\alpha^{\alpha+1}, \tau\}$ of length τ. It is easy to see that $X = \underleftarrow{\lim}\, \mathscr{S}$. By Lemma 3.5(i), $\mathscr{S}$ is a continuous spectrum and every set $A(\alpha)$ is r-admissible. According to Lemma 3.6, all X_α are AE(0)-spaces and all limit projections p_α are functionally open. Hence, the maps $p_\alpha^{\alpha+1}$ are also functionally open. The retraction $r_{A(1)}: R^{A(1)} \to X_1$ from Lemma 3.6(ii) is continuum-valued (see the proof of Lemma 3.6(ii)). Thus, by the case $\tau = \omega$, X_1 is an AE(1)-space. According to [3], it remains to prove that $p_\alpha^{\alpha+1}$ is 1-soft for every $\alpha < \tau$. Fix an α. By Lemma 3.6(iii), there is a usco retraction $r_\alpha^{\alpha+1} = r_{A(\alpha)}^{A(\alpha+1)}: X_\alpha \times R^{A(\alpha+1)-A(\alpha)} \to X_{\alpha+1}$ with

$$p_\alpha^{\alpha+1} \cdot r_\alpha^{\alpha+1} = \pi_{A(\alpha)}^{A(\alpha+1)} \Big| X_a \times R^{A(\alpha+1)-A(\alpha)}. \tag{2}$$

It follows from the proof of Lemma 3.6(iii) that $r_\alpha^{\alpha+1}$ is continuum-valued. So, $r_\alpha^{\alpha+1}$ can be supposed to be minimal continuum-valued. Then (2) remains true. Since X_α and $R^{A(\alpha+1)-A(\alpha)}$ are AE(0)-spaces, their product is also an AE(0). Consequently, by Corollary 3.8, $r_\alpha^{\alpha+1}$ is strongly u.s.c. Let us note that $X_{\alpha+1}$ is C-embedded in $X_\alpha \times R^{A(\alpha+1)-A(\alpha)}$ (see Lemma 3.6(ii)). Finally, by Lemma 3.4, $p_\alpha^{\alpha+1}$ is 1-soft. □

Corollary 3.10. *An* AE(0)*-subspace X of R^A is an* AE(1)*-space if and only if for any usco retraction $r: R^A \to X$ there is a continuum-valued u.s.c. retraction $\Phi: R^A \to X$ such that $r(x) \subset \Phi(x)$ for every $x \in R^A$.*

Proof. Let $X \in \mathrm{AE}(1)$ and $r : R^A \to X$ be a usco retraction. Then, by [17, Lemma 7], X is C-embedded in R^A. Hence, according to Theorem 3.9, there exists a continuum-valued u.s.c. retraction $r_1 : R^A \to X$. Define a set-valued map $F : R^A \to R^A$ by $F(x) = \mathrm{cl}_{R^A}(\mathrm{conv}(r(x)))$. $F(x)$ is a compact convex subset of R^A for each $x \in R^A$ because $r(x)$ is compact. We show that F is u.s.c. Let $x_0 \in R^A$ and U be an open neighborhood of $F(x_0)$ in R^A. There are a finite set $B \subset A$ and a convex open subset V of R^B such that $\pi_B(F(x_0)) \subset V \subset \mathrm{cl}_{R^B}(V) \subset \pi_B(U)$ and the closure of $W = \pi_B^{-1}(V)$ in R^A is contained in U. Take a neighborhood $O(x_0)$ of x_0 in R^A with $r(x) \subset W \cap X$ for every $x \in O(x_0)$. This implies $F(x) \subset \mathrm{cl}_{R^A}(W) \subset U$, $x \in O(x_0)$. Thus, F is u.s.c. Now, let $\Phi(x) = r_1(F(x))$, $x \in R^A$. It is easily seen that Φ is a continuum-valued u.s.c. retraction from R^A onto X and $r(x) \subset \Phi(x)$ for each $x \in R^A$. The "only if" part of the corollary follows directly from Theorem 3.9. □

An embedding j of X in Y is said to be a *g-embedding* if for every open subset U of $j(X)$ there exists an open subset $e(U)$ of Y such that the following conditions are fulfilled:

(1) $e(\emptyset) = \emptyset$ and $e(j(X)) = Y$;
(2) $e(U) \cap j(X) = U$;
(3) $e(U) \cap e(V) = e(U \cap V)$;
(4) if $U \cap V = \emptyset$ then $e(U \cup V) = e(U) \cup e(V)$.

If there is a continuum-valued u.s.c. retraction r from Y to a space X, then X is g-embedded in Y (in this case $e(U) = r^{\#}(U)$). On the other hand, every g-embedded subset of a connected space is also connected. Now we give an external characterization of locally connected Čech-complete AE(1)-spaces. Another external characterization of compact AE(1)-spaces was given by Shirokov [15,16].

Theorem 3.11. *Let X be a locally connected Čech-complete space. Then the following conditions are equivalent*:

(i) *The Hewitt-realcompactification νX of X is a Lindelöf Čech-complete* AE(1)-*space*;

(ii) *every C-embedding of X in any space is a g-embedding*;

(iii) *X is a g-embedded subset of R^A, for some A.*

Proof. (i) → (ii). Suppose X is a C-embedded subset of a space Y. Consider an embedding j of X in $R^{C(X)}$ such that $\mathrm{cl}_{R^{C(X)}}(j(X)) = \nu X$. Let $h : Y \to R^{C(X)}$ be a continuous extension of j. By Theorem 3.9, there is a continuum-valued u.s.c. retraction r from $R^{C(X)}$ onto νX. Then for every open U in X put $e(U) = h^{-1}(r^{\#}(W(U)))$, where $W(U) = \bigcup\{W : W$ is open in νX and $W \cap j(X) \subset j(U)\}$. Clearly, $e(U)$ are open in Y and the conditions (1)–(3) are satisfied. We show that (4) is also true. It suffices to prove that $W(U \cup V) = W(U) \cup W(V)$ provided $U \cap V = \emptyset$. Let $x \in W(U \cup V)$. Then there is an open set W in νX containing x such that $W \cap j(X) \subset j(U \cup V)$. Assume $\mathrm{cl}_{\nu X}(j(U)) \cap \mathrm{cl}_{\nu X}(j(V)) \cap W \neq \emptyset$. Clearly, $\mathrm{cl}_{\nu X}(j(U)) = \mathrm{cl}_{\nu X}(W(U))$ and $\mathrm{cl}_{\nu X}(j(V)) = \mathrm{cl}_{\nu X}(W(V))$. Since νX is perfectly k-normal (as an AE(0)-space), each of the sets $H_1 = \mathrm{cl}_{\nu X}(j(U))$ and $H_2 = \mathrm{cl}_{\nu X}(j(V))$

is G_δ in νX. So, the set $H = H_1 \cap H_2 \cap W$ is also G_δ in νX. But $j(X)$ is G_δ-dense in νX. Consequently, $H \cap j(X) \neq \emptyset$, which is impossible because $U \cap V = \emptyset$. Thus, $x \notin H_1 \cap H_2$. Let $x \notin H_2$. Then $x \in W - H_2$. Obviously, $W - H_2$ is open in νX and $(W - H_2) \cap j(X) \subset j(U)$, i.e., $x \in W(U)$. Therefore, $W(U \cup V) \subset W(U) \cup W(V)$. The inverse inclusion is obvious. Hence, $W(U \cup V) = W(U) \cup W(V)$. As it was noted, this implies that (4) is true. So, X is g-embedded in Y.

(ii) → (iii). This implication is obvious.

(iii) → (i). Let X be a g-embedded subset of R^A for some A and e be the corresponding g-operator from the topology of X to the topology of R^A. If A is countable, then X is a locally connected and connected Polish space. Thus, X is LC^0 and C^0. Hence, by [3], $X \in \mathrm{AE}(1)$. So, we can assume that A is uncountable. By [17, Theorem 4], the G_δ-closure $G_\delta(X)$ of X in R^A is a Lindelöf Čech-complete AE(0)-space and $G_\delta(X)$ is homeomorphic to νX. Consider the natural dense embedding of R^A in I^A and put $Y = \mathrm{cl}_{I^A}(X)$. Define a usco mapping r from R^A to Y by $r(x) = \bigcap\{\mathrm{cl}_Y(U)\colon x \in e(U)\}$. Clearly, $r(x) = \{x\}$ for every $x \in X$. Using the g-embeddability of X in R^A (precisely, property (4) of the operator e), one can easily show that r is continuum-valued. Without loss of generality we can suppose r is minimal continuum-valued. Then, by Corollary 3.8, r is strongly u.s.c.

Claim 1. $r(x) = \{x\}$ *for every* $x \in \nu X$.

Proof. We have already noted that $r(x) = \{x\}$ for every $x \in X$. Let $x \in \nu X - X$ and U be an open neighborhood of x in Y. Then $x \in G_\delta(r^\#(U))$. Indeed, if H is a G_δ-subset of R^A containing x, then $x \in H \cap U$, which is G_δ in νX. Hence, $H \cap U \cap X \neq \emptyset$. But $U \cap H \cap X \subset r^\#(U)$. So $H \cap r^\#(U) \neq \emptyset$. Now, by Lemma 3.7(ii), $x \in G_\delta(r^\#(U)) \subset r^\#(\mathrm{cl}_Y(U))$. Thus, $r(x) \subset \mathrm{cl}_y(U)$. This implies $r(x) = x$.

Claim 2. $G_\delta(r^\#(\nu X)) = r^\#(\nu X)$.

Proof. Since νX is Lindelöf and Čech-complete, there is a countable family $\{H_i\colon i \in N\}$ of compact G_δ-subsets of Y such that $Y - \bigcup\{H_i\colon i \in N\} = \nu X$. For every i take a countable finitely additive open cover $\gamma(i)$ of $Y - H_i$ with $\mathrm{cl}_Y(U) \subset Y - H_i$ for each $U \in \gamma(i)$ and put $W(i) = \bigcup\{r^\#(U)\colon U \in \gamma(i)\}$. The sets $\bigcup\{G_\delta(r^\#(U))\colon U \in \gamma(i)\}$ are functionally open in R^A because any set $G_\delta(r^\#(U))$, $U \in \gamma(i)$, is functionally open [19]. Hence, $G_\delta(W(i)) = \bigcup\{G_\delta(r^\#(U))\colon U \in \gamma(i)\}$. Since $r^\#(\nu X) \subset W(i)$, $i \in N$, we have $G_\delta(r^\#(\nu X)) \subset G_\delta(W(i))$, $i \in N$. Let $x \in G_\delta(r^\#(\nu X))$. Then for every $i \in N$ there is $U(i) \in \gamma(i)$ such that $x \in G_\delta(r^\#(U(i)))$. Consequently, by Lemma 3.7(ii), $r(x) \subset \bigcap\{\mathrm{cl}_Y(U(i))\colon i \in N\} \subset \bigcap\{Y - H_i\colon i \in N\} = \nu X$.

By, Claim 1, r is a continuum-valued u.s.c. retraction from $r^\#(\nu X)$ onto νX. Since $r^\#(\nu X)$ is G_δ in R^A and $G_\delta(r^\#(\nu X)) = r^\#(\nu X)$, there is a countable set $B \subset A$ such that $r^\#(\nu X) = \pi_B^{-1}(Z)$, where Z is a G_δ-subset of R^B (see [17, Lemma 1]). According to [17], there exists a countable subset C of A containing B such that $\pi_C \mid \nu X$ is functionally open and $\pi_C(\nu X)$ is a Polish space. Then $\pi_C(\nu X) \times R^{A-C}$ is contained in $r^\#(\nu X)$. Thus, r is a continuum-valued retraction from $\pi_C(\nu X) \times R^{A-C}$ onto νX. But $\pi_C(\nu X) = \pi_C(X)$ and since X is connected (as a g-embedded subset of R^A), $\pi_C(X)$ is also connected. It follows from the locally connectedness of X and the openness of $\pi_C \mid \nu X$ that $\pi_C(X)$ is locally connected. So, $\pi_C(X)$ is an AE(1)-space. Consequently, $\pi_C(X) \times R^{A-C}$ is also an AE(1)-space. it follows from Theorem 3.9 that every continuum-valued u.s.c. retract of an AE(1)-space is an AE(1)-space. Therefore, $\nu X \in \mathrm{AE}(1)$. □

A family $\mathscr{U}$ of open subsets of a space X is said to be a *1-base* if for every open set U in X there is a subfamily $\gamma(U)$ of $\mathscr{U}$ such that the following conditions are satisfied:

(1) $X \in \gamma(X)$;

(2) $\bigcup \gamma(U) = U$;

(3) $\gamma(U_1) \wedge \gamma(U_2) = \{W_1 \cap W_2 \colon W_i \in \gamma(U_i),\ i = 1, 2\} \subset \gamma(U_1 \cap U_2)$;

(4) for any $W \in \gamma(U)$ and any two points $x,\ y \in W$ there is a connected set $C(x, y)$ with $x,\ y \in C(x, y) \subset U$.

The above definition is a simple modification of the definition of an n-regular base for $n = 1$, given by Shirokov [16] (in [16] a k-metrizable compact space is shown to be an AE(n) if and only if it has an n-regular base).

Corollary 3.12. *A Čech-complete* AE(0)*-space X is an* AE(1) *if and only if X has a 1-base.*

Proof. First we prove that every AE(1)-space X has a 1-base. Let r be a continuum-valued u.s.c. retraction from R^A onto X for some A (see Theorem 3.9) and $\mathscr{B}$ be a standard open base for R^A. We can assume $R^A \in \mathscr{B}$. Let $\mathscr{U} = \{W \cap X\colon W \in \mathscr{B}\}$. For every open subset U of X put $\gamma(U) = \{W \cap X\colon W \in \mathscr{B}$ and $W \subset r^{\#}(U)\}$. Obviously, the conditions (1)–(3) in the definition of a 1-base are fulfilled. Suppose $W \cap X \in \gamma(U)$ and $x,\ y \in W \cap X$. Since W is connected and $W \subset r^{\#}(U)$, the set $r(W)$ is also connected and $x,\ y \in r(W) \subset U$. Hence, $\mathscr{U}$ is a 1-base.

Suppose X is a Čech-complete AE(0)-space with a 1-base $\mathscr{U}$. Let X be C-embedded in R^A, for some A. Then there is a usco retraction r from R^A onto X [17]. It follows from the conditions (1) and (4) that X is connected and the components of all open subsets of X are open. So, by (4), if $W \in \gamma(U)$ then there is a unique component $\mathrm{supp}_U(W)$ of U containing W. Now, for every open subset U of X we denote

$$\gamma_1(U) = \{W \in \mathscr{U}\colon \text{there is an open subset } V \text{ of } X \text{ such that } W \in \gamma(V),\ \mathrm{supp}_V(W) \subset U \text{ and for any component } C \text{ of } V \text{ either } C \subset U \text{ or } C \cap U = \emptyset\},$$

$$\gamma_2(U) = \{W \in \mathscr{U}\colon \text{there is an open subset } V \text{ of } U \text{ with } W \in \gamma_1(V)\}$$

and

$$e(U) = \bigcup\{r^{\#}(W)\colon W \in \gamma_2(U)\}.$$

Let U, V be open subsets of X and $U \subset V$. Then $\gamma(U) \subset \gamma_1(U)$ and $\gamma_2(U) \subset \gamma_2(V)$. This implies $e(X) = R^A$, $e(U) \cap X = U$ and $e(U) \subset e(V)$, i.e., e is a monotone operator.

Claim 1. $e(U_1) \cap e(U_2) = e(U_1 \cap U_2)$ *for any open subsets U_1 and U_2 of X.*

Proof. Let $x \in e(U_1) \cap e(U_2)$. Then $x \in r^{\#}(W_1 \cap W_2)$, where $W_i \in \gamma_2(U_i)$, $i = 1, 2$. It follows from the definition of γ_1 and γ_2 that there are open sets $V_i \subset U_i$ and $P(i)$ such that $W_i \in \gamma(P(i))$, $\text{supp}_{P(i)}(W_i) \subset V_i$ and for every component C of $P(i)$ either $C \subset V_i$ or $C \cap V_i = \emptyset$, $i = 1, 2$. Let $W = W_1 \cap W_2$, $V = V_1 \cap V_2$ and $P = P(1) \cap P(2)$. Then $W \in \gamma(P)$ because $\gamma(P(1)) \wedge \gamma(P(2)) \subset \gamma(P)$. It is easily seen that $\text{supp}_P(W) \subset V$ and any component of P is contained in V or it does not meet V. Hence, $W \in \gamma_1(V)$. Since $V \subset U_1 \cap U_2$, $W \in \gamma_2(U_1 \cap U_2)$. Thus, $x \in e(U_1 \cap U_2)$. Therefore $e(U_1) \cap e(U_2) \subset e(U_1 \cap U_2)$. The inverse inclusion is also true because e is monotone.

Claim 2. *$e(U_1 \cup U_2) = e(U_1) \cup e(U_2)$ provided $U_1 \cap U_2 = \emptyset$.*

Proof. It suffices to prove $e(U_1 \cup U_2) \subset e(U_1) \cup e(U_2)$. Let $x \in e(U_1 \cup U_2)$. Then $x \in r^{\#}(W)$ for some $W \in \gamma_2(U_1 \cup U_2)$. Thus, there are two open subsets V and P of X such that $V \subset U_1 \cup U_2$, $W \in \gamma(P)$, $\text{supp}_P(W) \subset V$ and any component of P is contained in V or it does not meet V. Put $V_i = V \cap U_i$, $i = 1, 2$. Then $V = V_1 \cup V_2$ and since $\text{supp}_P(W)$ is connected, we have $\text{supp}_P(W) \subset V_1$ or $\text{supp}_P(W) \subset V_2$. Suppose $\text{supp}_P(W) \subset V_1$. Obviously, every component C of P is contained in V_1 or $C \cap V_1 = \emptyset$. Therefore, $W \in \gamma_1(V_1)$. This implies $W \in \gamma_2(U_1)$. So, $x \in e(U_1)$.

It follows from Claims 1 and 2 that e is a g-operator. Hence X is g-embedded in R^A. Since X is realcompact as a closed subset of R^A, by Theorem 3.11, $X \in \text{AE}(1)$. □

References

[1] A. Chigogidze, On a generalization of perfectly normal spaces, Topology Appl. 13 (1982) 15–20.

[2] A. Chigogidze, Uncountable powers of the real line and the natural series and n-soft maps, Dokl. Akad. Nauk SSSR 278 (1984) 50–53.

[3] A. Chigogidze, Noncompact absolute extensors in dimension n, n-soft maps and their applications, Izv. Akad. Nauk SSSR 50 (1986) 156–180.

[4] A. Chigogidze, On the structure of non-metrizable AE(0)-spaces, Mat. Zametki 41 (1987) 406–411.

[5] M. Čoban and V. Valov, On a Michael's selection theorem, Comptes Rendus Bulgare Sci. 28 (1975) 871–873.

[6] A. Dranishnikov, Many-valued absolute retracts and absolute extensors in dimension 0 and 1, Uspekhi Mat. Nauk 39 (1984) 241–242.

[7] A. Dranishnikov, Universal Menger compacta and universal maps, Mat. Sb. 129 (1986) 121–139.

[8] R. Engelking, Dimension Theory (PWN, Warsaw, 1978).

[9] V. Fedorchuk, On open maps, Uspekhi Mat. Nauk 37 (1982) 187–188.

[10] E. Michael, Continuous selections II, Ann. of Math. 64 (1956) 562–580.

[11] St. Nedev, Selection and factorization theorems for set-valued mappings, Serdica 6 (1980) 291–317.

[12] G. Nepomnjashchii, On the structure of many-valued absolute retracts of uncountable weight, Trudi Moskov. Mat. Obshch. 47 (1984) 146–157.

[13] E. Ščepin, On topological products, groups, and a new class of spaces more general than metric spaces, Soviet Math. Dokl. 17 (1976) 152–155.

[14] E. Ščepin, Functors and uncountable powers of compacta, Uspekhi Mat Nauk 36 (1981) 3–62.

[15] L. Shirokov, On some embeddings of topological spaces, Uspekhi Mat. Nauk 42 (1987) 253–254.

[16] L. Shirokov, On AE(n)-compacta and n-soft maps, Siberian Mat. J. 33 (2) (1992) 151–156.

[17] V. Valov, Another characterization of AE(0)-spaces, Pacific J. Math. 127 (1987) 199–208.

[18] V. Valov, Milutin mappings and AE(0)-spaces, Comptes Rendus Bulgare Sci. 40 (11) (1987) 9–12.

[19] R. Pol and E. Pol, Remarks on Cartesian products, Fund. Math. 93 (1976) 57–69.