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A characterization of Dugundji spaces via set-valued maps

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Abstract

We characterize Lindelöf p-spaces which are absolute extensors for zero-dimensional perfectly normal spaces. As an application we prove that a Lindelöf Čech-complete space X is an absolute extensor for zero-dimensional spaces if and only if there exists an upper semi-continuous compact-valued map $r : X^3 \to X$ such that $r(x, y, y) = r(y, y, x) = \{x\}$ for all $x, y \in X$. This result is new even when applied to compact spaces and yields the following new characterization of Dugundji spaces: A compact Hausdorff space X is Dugundji if and only if there exists an upper semi-continuous compact-valued map $r : X^3 \to X$ such that $r(x, y, y) = r(y, y, x) = \{x\}$ for all $x, y \in X$. It is worth noting that, by a result of Uspenskij, in the above characterization of Dugundji spaces the set-valued map r cannot be replaced by a single-valued (continuous) map, the 5-dimensional sphere S^5 being a counterexample.

Keywords: Dugundji space; Paracompact p-space; Set-valued map; Upper semi-continuous; Antimixer; Regular extension operator; Absolute extensor for 0-dimensional spaces

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1. Introduction

Uspenskij [15] proved that if X is a countably compact space and there is a continuous mapping r from X^3 onto X such that

$$r(x, y, y) = r(y, y, x) = x \quad \text{for all } x, y \in X, \tag{am}$$

then the Stone–Čech compactification βX of X is a Dugundji space (a continuous mapping r satisfying the condition (am) is called an *antimixer* on X). So, if a compact space possesses an antimixer it is a Dugundji space. This is a generalization of the fact that

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compact topological groups are Dugundji spaces [8] (let us note that for every topological group G the mapping $(x, y, z) \to xy^{-1}z$ is an antimixer on G). On the other hand there is a Dugundji space which does not possess an antimixer (the 5-dimensional sphere S^5 is such a space [16]). So, Dugundji spaces can not be characterized in terms of antimixers. But if we consider *set-valued antimixers*, i.e., set-valued maps r from X^3 onto X such that

$$r(x, y, y) = r(y, y, x) = \{x\} \quad \text{for } x, y \in X, \tag{sam}$$

instead of single-valued, we get a characterization of Dugundji spaces (see Corollary 1.3 below). This characterization is obtained as a corollary from the following theorem describing Lindelöf p-spaces which are absolute extensors for 0-dimensional perfectly normal spaces:

Theorem 1.1. *Let X be a Lindelöf p-space. Then the following conditions are equivalent:*

(1) *There is a usco antimixer r from X^3 onto X;*

(2) *X is a limit space of an S-system of a length $w(X)$;*

(3) *For every closed embedding of X in a paracompact p-space Y there is a usco retraction $r: Y \to X$;*

(4) *For every closed embedding of X in a paracompact p-space Y there is a regular extension operator $u: C(X) \to C(Y)$;*

(5) *Every closed embedding of X in a normal space Y is a d-embedding;*

(6) *Every continuous mapping $f: F \to X$ from a closed G_δ-subset of a normal space Y with* $\dim(Y - F) = 0$ *is continuously extendable over Y.*

Let us introduce the notions in Theorem 1.1. All spaces considered are completely regular. A set-valued mapping r from X to Y is called *upper semi-continuous* (briefly, u.s.c.) if the set $r^{\#}(U) = \{x \in X\colon\ r(x) \subset U\}$ is open in X whenever U is open in Y. A set-valued mapping r from X to Y is called a *retraction* if Y is a subset of X and $r(x) = \{x\}$ for every $x \in Y$. By a *usco mapping* we mean an u.s.c. compact-valued mapping. An embedding $j: X \hookrightarrow Y$ is said to be *d-regular* [12] (briefly, a *d-embedding*) if to every open subset U of X an open subset $e(U)$ of Y can be assigned such that $e(\emptyset) = \emptyset$, $e(U) \cap j(X) = U$ and $e(U) \cap e(V) = e(U \cap V)$. A continuous in the sense of Scepin [11] inverse system $\{X_\alpha, q_\alpha^\beta,\ \alpha < \beta < \omega(\tau)\}$, where $\omega(\tau)$ is the initial ordinal of cardinality τ, is said to be an *S-system of a length* τ if X_1 is a separable metric space and every $q_\alpha^{\alpha+1}$ is an open and perfect mapping with a metrizable kernel, i.e., for each α there is a separable metric space M_α such that $X_{\alpha+1}$ is embedded in $X_\alpha \times M_\alpha$ and the restriction of the projection $\pi_\alpha: X_\alpha \times M_\alpha \to X_\alpha$ on $X_{\alpha+1}$ coincides with $q_\alpha^{\alpha+1}$. By $C(X)$ (respectively $C^*(X)$) we denote the real vector space of all continuous (and bounded) functions on X. Let X be a subspace of Y. A linear mapping $u: C(X) \to C(Y)$ is said to be a *regular extension operator* if u is positive, $u(1_X) = 1_Y$ and $u(f)|X = f$ for every $f \in C(X)$. A compact space X is called *Dugundji* [9] if for every embedding of X in a compact space Y there is a regular extension operator $u: C(X) \to C(Y)$. The class of Dugundji spaces has a nice topological characterization, given by Haydon [4]: X is a Dugundji space if and only if it is an absolute extensor for 0-dimensional spaces (briefly, AE(0)). The notion of AE(0) was extended for all completely regular spaces in [1]: A space X is an AE(0) if every Z-normal mapping from Y to X, where $Y \subset Z$ and $\dim Z = 0$, is continuously extendable to Z. Here dim stands for dimension defined by finite functionally open covers and f is *Z-normal* means that for every continuous function g on X the function $g \circ f$ is continuously extendable over Z.

We are now in a position to prove the following corollary of Theorem 1.1.

Corollary 1.2. *Let X be a Lindelöf p-space. Then X is an* AE(0) *if and only if X is a Čech-complete space possessing a usco antimixer.*

Proof. Let X be an AE(0)-space. Then, by [17, Theorem 1], there is a closed C-embedding of X in R^A and a usco retraction r from R^A onto X, where card $(A) = w(X)$ and R is the real line. By Lemma 2.4 below R^A possesses a usco antimixer ϕ. Then the restriction $r \circ \phi$ onto X^3 is a usco antimixer on X. Hence, every AE(0)-space has a usco antimixer. X being paracompact p-space and AE(0), is Čech-complete [17].

Suppose there exists a usco antimixer on X. Then, by Theorem 1.1, X is a limit space of an S-system $\varphi = \{X_\alpha, q^\alpha_\beta,\ \beta < \alpha < \omega(\tau)\}$ of a length $w(X)$. The space X_1 is a complete separable metric space because X is Čech-complete and $q_1 : X \to X_1$ is perfect. By the same arguments, each X_α is a Lindelöf Čech-complete space. It follows from [1] that $X \in$ AE(0). □

Since the class of compact AE(0)-spaces coincides with the class of Dugundji spaces [4], we have a characterization of Dugundji spaces in terms of usco antimixers:

Corollary 1.3. *A compact Hausdorff space is Dugundji if and only if there exists a usco antimixer on it.*

2. Some lemmas

Let X be a subspace of a product $I^A \times M$, where $I = [0,1]$, and $B \subset A$. Then π_B and π_M stand for the natural projections of $I^A \times M$ onto $I^B \times M$ and M respectively. Denote $p_B = \pi_B|X$ and $p_M = \pi_M|X$. If cM is a compactification of M and cX is the closure of X in $I^A \times cM$ then $\overline{p}_B$ denotes the projection of cX into $I^B \times cM$. By a *standard open subset of* X we mean a subset of the form $V \cap X$, where V is a standard open set in $I^A \times M$. If U is a subset of $I^A \times M$, $\mathcal{K}(U)$ denotes the family $\{B \subset A\colon \pi_B^{-1}(\pi_B(U)) = U\}$. Analogously, if $U \subset X$ then $\mathcal{L}(U) = \{B \subset A\colon p_B^{-1}(p_B(U)) = U\}$.

Lemma 2.1. *Let X be a Lindelöf p-space and $r : X^3 \to X$ be a u.s.c. (not necessarily compact-valued) antimixer. Suppose X is a subset of $I^A \times M$, where M is a metric space. Then for every countable subset C of A there is a countable set $B \subset A$ containing C such that $\{p_B(z)\} = p_B(r(x,y,z)) = p_B(r(z,x,y))$ provided $x,y,z \in X$ and $p_B(x) = p_B(y)$.*

Proof. First we shall construct by induction a sequence $\{B(n)\colon\ n = 1, 2, \ldots\}$ consisting of countable subsets of A such that $B(1) \supset C$ and for every $x, y, z \in X$ and n the equality $p_{B(n+1)}(x) = p_{B(n+1)}(y)$ implies $\{p_{B(n)}(z)\} = p_{B(n)}(r(x, y, z)) = p_{B(n)}(r(z, x, y))$.

Suppose we have already constructed $B(k)$, $k = 1, 2, \ldots, n$. Take a countable base $\mathcal{B}_n$ of $p_{B(n)}(X)$ and for every $U \in \mathcal{B}_n$ consider the open subset $W(U) = r^{\#}(p_{B(n)}^{-1}(U))$ of X^3. Let Z be the space $(X \times \Delta) \cup (\Delta \times X)$, where Δ is the diagonal in X^2. Z is a Lindelöf p-space because it is closed in X^3 and X^3 is a Lindelöf p-space. Since r is an antimixer (i.e., r satisfies the condition (sam)), the restriction $h = p_{B(n)} \circ r|Z$ is a single-valued continuous mapping from Z onto $p_{B(n)}(X)$ and $W(U) \cap Z = h^{-1}(U)$. Hence, $W(U) \cap Z$ is a Lindelöf space as an F_σ-subset of Z. For every point $w = (x, y, z)$ from $W(U) \cap Z$ there are standard open neighbourhoods $O(x)$, $O(y)$ and $O(z)$ in X of x, y, and z, respectively, such that $O(w) = O(x) \times O(y) \times O(z) \subset W(U)$. Choose a finite set $B(w) \subset A$ with $B(w) \in \mathcal{L}(O(x)) \cap \mathcal{L}(O(y)) \cap \mathcal{L}(O(z))$. Then

$$\left(p_{B(w)}^3\right)^{-1}\left(p_{B(w)}^3(O(w))\right) = O(w), \tag{1}$$

where $p_{B(w)}^3$ is the mapping $p_{B(w)} \times p_{B(w)} \times p_{B(w)}$ from X^3 onto the space $(p_{B(w)}(X))^3$. Since $\gamma(U) = \{O(w)\colon\ w \in W(U) \cap Z\}$ is an open cover of $W(U) \cap Z$ and $W(U) \cap Z$ is Lindelöf, there is a countable subset $\Gamma(U)$ of $W(U) \cap Z$ such that $\{O(w)\colon\ w \in \Gamma(U)\}$ is a subcover of $\gamma(U)$. Put

$$e(U) = \bigcup\left\{O(w)\colon\ w \in \Gamma(U)\right\} \quad \text{and} \quad B(U) = \bigcup\left\{B(w)\colon\ w \in \Gamma(U)\right\}.$$

Obviously, $B(U)$ is countable and $W(U) \cap Z \subset e(U) \subset W(U)$. Finally, let $B(n+1) = \bigcup\{B(U)\colon\ U \in \mathcal{B}_n\}$. It follows from our construction and from (1) that

$$\left(p_{B(n+1)}^3\right)^{-1}\left(p_{B(n+1)}^3(e(U))\right) = e(U) \tag{2}$$

for every $U \in \mathcal{B}_n$. Suppose $x, y, z \in X$ and $p_{B(n+1)}(x) = p_{B(n+1)}(y)$. The last equality implies $p_{B(n+1)}^3(x, y, z) = p_{B(n+1)}^3(x, x, z)$ and $p_{B(n+1)}^3(z, x, y) = p_{B(n+1)}^3(z, x, x)$. Let $p_{B(n)}(z) \in U^*$, where $U^* \in \mathcal{B}_n$. Since $r(x, x, z) = r(z, x, x) = \{z\}$, we have $(z, x, x), (x, x, z) \in W(U^*)$. Hence, $(z, x, x), (x, x, z) \in W(U^*) \cap Z \subset e(U^*)$. Then, by (2), $(z, x, y), (x, y, z) \in e(U^*) \subset W(U^*)$. Therefore, $p_{B(n)}(r(x, y, z)) \subset U^*$ and $p_{B(n)}(r(z, x, y)) \subset U^*$. Consequently, we showed that every $U \in \mathcal{B}_n$ contains $p_{B(n)}(r(x, y, z))$ and $p_{B(n)}(r(z, x, y))$ provided U contains $p_{B(n)}(z)$. Thus,

$$\left\{p_{B(n)}(z)\right\} = p_{B(n)}\left(r(x, y, z)\right) = p_{B(n)}\left(r(z, x, y)\right).$$

Let $B = \bigcup\{B(n)\colon\ n = 1, 2, \ldots\}$. Clearly, B is countable and $C \subset B$. If $x, y, z \in X$ and $p_B(x) = p_B(y)$ then $p_{B(n+1)}(x) = p_{B(n+1)}(y)$ for every n. Thus, for every n we have

$$\left\{p_{B(n)}(z)\right\} = p_{B(n)}\left(r(x, y, z)\right) = p_{B(n)}\left(r(z, x, y)\right),$$

and hence,

$$\left\{p_B(z)\right\} = p_B\left(r(x, y, z)\right) = p_B\left(r(z, x, y)\right). \qquad \square$$

Remark 1. Analyzing the proof of Lemma 2.1 one can see that the following more general proposition holds: Suppose $l(X^2) \leqslant \tau$ and there is an u.s.c. antimixer from X^3 onto X (here $l(X^2)$ is the Lindelöf number of X^2). If $X \subset I^A \times M$, where M is a metric space, then for every set $C \subset A$ of cardinality card$(C) \leqslant \tau$ there is a set $B \subset A$ containing C such that card$(B) \leqslant \tau$ and $\{p_B(z)\} = p_B(r(x,y,z)) = p_B(r(z,x,y))$ provided $x, y, z \in X$ and $p_B(x) = p_B(y)$. □

Lemma 2.2. *Let $X = \prod\{M_s\colon s \in S\}$ be a product of metric spaces. Then there exists a usco antimixer from X^3 onto X.*

Proof. First we shall prove that every M_s possesses a usco antimixer r_s. Let Z_s be the space $(M_s \times \Delta_s) \cup (\Delta_s \times M_s)$, where Δ_s is the diagonal in M_s^2. Define a continuous mapping h_s from Z_s onto M_s by $h_s(x,x,y) = h_s(y,x,x) = y$ for all $x,\ y \in M_s$. Take a usco retraction g_s from M_s^3 onto Z_s. (Here we use the well-known fact that every closed subspace of a metric space M is a usco retract of M.) Then the composition $r_s = h_s \circ g_s$ is a usco antimixer from M_s^3 onto M_s. It is easily seen that the set-valued mapping $r = \prod\{r_s\colon s \in S\}$ is a usco antimixer from X^3 onto X. □

Lemma 2.3. *Let $f: X \to Y$ be a perfect and open continuous mapping with a metrizable kernel. Then f is 0-soft, i.e., for any 0-dimensional normal space Z, a closed subset H of Z and any two continuous mappings $g: Z \to Y$ and $h: H \to X$ with $f \circ h = g|H$ there exists a continuous mapping $k: Z \to X$ such that $f \circ k = g$ and $k|H = h$.*

Proof. Let M be a separable metric space such that X is a subspace of $Y \times M$ and $\pi_Y|X = f$, where π_Y is the projection of $Y \times M$ onto Y. Denote by π_M the projection of $Y \times M$ onto M. Suppose we have a 0-dimensional normal space Z, a closed subset H of Z and two continuous mappings $g: Z \to Y$ and $h: H \to X$ with $f \circ h = g|H$. Define a set-valued mapping $\Phi: Z \to M$ by

$$\Phi(z) = \begin{cases} \{\pi_M(h(z))\} & \text{if } z \in H, \\ \pi_M(f^{-1}(g(z))) & \text{if } z \notin H. \end{cases}$$

Since f is open and perfect, Φ is compact-valued and lower semi-continuous (recall that a set-valued mapping $\Phi: Z \to M$ is *lower semi-countinuous* if $\Phi^{-1}(U) = \{z \in Z\colon \Phi(z) \cap U \neq \emptyset\}$ is open in Z whenever U is open in M). Then, by [2, Theorem 11.4], there is a continuous mapping q from Z into M with $q(z) \in \Phi(z)$ for every z in Z. Finally, define the desired mapping $k: Z \to X$ by $k(z) = (g(z), q(z))$. □

Remark 2. Let $f: X \to Y$ be as in Lemma 2.3. Then for any normal space Z, a closed G_δ-subset H of Z with $\dim(Z - H) = 0$ and any two continuous mappings $g: Z \to Y$ and $h: H \to X$ with $f \circ h = g|H$, there exists a continuous mapping $k: Z \to X$ such that $f \circ k = g$ and $k|H = h$.

Proof. Let $\Phi: Z \to M$ be the lower semi-continuous mapping defined in the proof of Lemma 2.3. Since H is G_δ in Z we have $Z - H = \bigcup\{F_n\colon n = 1, 2, \ldots\}$ with each

F_n closed in Z and $\dim F_n = 0$. By [7, Theorem 4.6], there is a continuous selection $q: Z \to M$ for Φ. Finally, put $k = (g(z), q(z))$. □

Recall that a continuous mapping $f: X \to Y$ is said to be *Milutin* if there exists a positive linear operator $u: C(X) \to C(Y)$ such that $u(1_X) = 1_Y$ and $u(h \circ f) = h$ for every h in $C(Y)$. Such an operator is called a *regular averaging operator for* f.

Lemma 2.4. *Let A be an infinite set and M be a metric space. Then there exist a 0-dimensional metric space M_0 and perfect Milutin mappings $f: D^A \to I^A$ and $g: M_0 \to M$ such that $f \times g$ is also a Milutin mapping, where D is the discrete two-point set.*

Proof. Let $f: D^A \to I^A$ and $g: M_0 \to M$ be perfect Milutin mappings where M_0 is a 0-dimensional metric space (the existance of such mappings follows, respectively, from [9, Theorem 5.6] and from [3]). Then using the same arguments as in the proof of Proposition 1.14 from [1], one can see that $f \times g$ is a Milutin mapping. □

For a compact space X by $P(X)$ is denoted the space of all regular probability measures on X endowed with the weak-star topology. $P(X)$ can be considered as the space of all continuous (with respect to the uniform norm) positive linear functionals μ on $C(X)$ with $\mu(1_X) = 1$. There is natural embedding $i: X \hookrightarrow P(X)$ defined by $i(x) = \delta_x$, where δ_x is Dirac's measure at the point x. For $\mu \in P(X)$ the support of μ is denoted by $\operatorname{supp}(\mu)$. For a continuous mapping f between compact spaces X and Y there is a continuous mapping $P(f): P(X) \to P(Y)$ defined by $P(f)(\mu)(h) = \mu(h \circ f)$ for every $\mu \in P(X)$ and $h \in C(Y)$. If $\mu \in P(X)$ then $\operatorname{supp}(P(f)(\mu)) \subset f(\operatorname{supp}(\mu))$.

Lemma 2.5. *Let cM be a metric compactification of M. Suppose X is a subspace of $Y = I^A \times M$ and there is a usco mapping $r: Y \to P(cX)$, where cX is the closure of X in $I^A \times cM$, such that $r(x) = \delta_x$ for every $x \in X$. Then for any countable $C \subset A$ there exists a countable set $B \subset A$ containing C such that $x \in X$, $y \in Y$ and $\pi_B(x) = \pi_B(y)$ implies $\{p_B(x)\} = \overline{p}_B(\operatorname{supp}(\mu))$ for every $\mu \in r(y)$.*

Proof. It follows from Kuratowski–Zorn lemma that r can be supposed to be minimal, i.e., every usco selection of r coincides with r. Then for any open subset U of $P(cX)$ we have [17]:

$$r(y) \subset \operatorname{cl}_{P(cX)}(U) \text{ provided } y \in \operatorname{Int}_Y\big(\operatorname{cl}_Y\big(r^{\#}(U)\big)\big) \text{ and}$$
$$\operatorname{cl}_Y\big(r^{-1}(U)\big) = \operatorname{cl}_Y\big(r^{\#}(U)\big). \tag{3}$$

We construct an increasing sequence $\{B(n): n = 1, 2, \ldots\}$ of countable subsets of A such that $B(1) \supset C$ and for every n we have $\mathbb{P}_n(\delta_x) = \mathbb{P}_n(r(y))$ provided $x \in X$, $y \in Y$ and $\pi_{B(n+1)}(x) = \pi_{B(n+1)}(y)$. Here $\mathbb{P}_n$ is the mapping $P(\overline{p}_{B(n)})$ from $P(cX)$ to $P(\overline{p}_{B(n)}(cX))$, $n = 1, 2, \ldots$. Assume we have already constructed $B(k)$, $k = 1, 2, \ldots, n$. Take a countable base $\mathcal{B}_n$ for the space $P(\overline{p}_{B(n)}(cX))$ (this is possible because $\overline{p}_{B(n)}(cX)$ is a compact metric space) and for each $U \in \mathcal{B}_n$ consider the open subset $W(U) = r^{\#}(\mathbb{P}_n^{-1}(U))$ of Y. By a result from [10] there exists a countable set $B(U) \subset A$ with

$$B(U) \in \mathcal{K}(\mathrm{cl}_Y(W(U))) \cap \mathcal{K}(\mathrm{Int}_Y(\mathrm{cl}_Y(W(U)))). \tag{4}$$

Now, let $B(n+1) = B(n) \cup (\bigcup\{B(U)\colon U \in \mathcal{B}_n\})$. Suppose $x \in X$, $y \in Y$ and $\pi_{B(n+1)}(x) = \pi_{B(n+1)}(y)$. If $\mathbb{P}_n(\delta_x) \in U^*$, where $U^* \in \mathcal{B}_n$, then $x \in W(U^*)$. Since, by (4), $B(n+1) \in \mathcal{K}(\mathrm{Int}_Y(\mathrm{cl}_Y(W(U^*))))$, we have $y \in \mathrm{Int}_Y(\mathrm{cl}_Y(W(U^*)))$. Hence, by (3), $\mathbb{P}_n(r(y))$ is contained in the closure of U^* in $P(\overline{p}_{B(n)}(cX))$. Consequently, $\mathbb{P}_n(\delta_x) = \mathbb{P}_n(r(y))$.

Put $B = \bigcup\{B(n)\colon n = 1, 2, \ldots\}$. Assume $x \in X$, $y \in Y$, $\mu \in r(y)$ and $\pi_B(x) = \pi_B(y)$. Then $\pi_{B(n+1)}(x) = \pi_{B(n+1)}(y)$ for every n. Thus, for every n we have $\mathbb{P}_n(\delta_x) = \mathbb{P}_n(\mu)$. Since $\mathrm{supp}(\mathbb{P}_n(\delta_x))$ is the one-point set $\{p_{B(n)}(x)\}$, the last equality implies $\{p_{B(n)}(x)\} = \overline{p}_{B(n)}(\mathrm{supp}(\mu))$. Therefore, $\{p_B(x)\} = \overline{p}_B(\mathrm{supp}(\mu))$. □

Let M be a separable metric space and cM its metric compactification. Suppose X is a closed subspace of $I^A \times M$ and r is a usco antimixer on X. Then a subset B of A is said to be *r-admissible* if for every $x, y, z \in X$ the equality $p_B(x) = p_B(y)$ implies $\{p_B(z)\} = p_B(r(x,y,z)) = p_B(r(z,x,y))$.

Lemma 2.6. *Let X be a closed subset of $I^A \times M$ and r be a usco antimixer on X. Then:*

(i) *every union of r-admissible sets is r-admissible too;*

(ii) *for every r-admissible set B the mapping p_B is open.*

Proof. (i) Suppose $\{B(s)\colon s \in S\}$ is a family of r-admissible subsets of A and $B = \bigcup\{B(s)\colon s \in S\}$. Let $x, y, z \in X$ and $p_B(x) = p_B(y)$. Then $p_{B(s)}(x) = p_{B(s)}(y)$ for each $s \in S$. Hence, $\{p_{B(s)}(z)\} = p_{B(s)}(r(x,y,z)) = p_{B(s)}(r(z,x,y))$ for every $s \in S$. This implies $\{p_B(z)\} = p_B(r(x,y,z)) = p_B(r(z,x,y))$.

(ii) We use the same arguments as in [15, Maltsev's Theorem]. Suppose B is r-admissible and U is open in X. Consider the set $V = p_B^{-1}(p_B(U))$. Since p_B is a perfect mapping, it is enough to show that V is open in X. If $y \in V$ there is $x \in U$ such that $p_B(x) = p_B(y)$. Then $(x,y,y) \in r^{\#}(U)$ because $r(x,y,y) = \{x\}$. Take an open neighborhood $O(y)$ of y in X with $\{x\} \times \{y\} \times O(y) \subset r^{\#}(U)$. We show that $O(y) \subset V$. Let $z \in O(y)$. Since $\{p_B(z)\} = p_B(r(x,y,z))$ and $p_B(z) \in p_B(U)$, we have $z \in V$. Hence, $O(y) \subset V$. Therefore V is open. □

The notion of r-admissibility will also be used in a slightly different situation. Suppose X is a closed subset of $Y = I^A \times M$ and cM is a metric compactification of M. Assume r is a usco mapping from Y to $P(cX)$, where $cX = \mathrm{cl}_{I^A \times cM}(X)$, such that $r(x) = \delta_x$ for every $x \in X$. Then a subset B of A is called *r-admissible* if for every $x \in X$, $y \in Y$ and $\mu \in r(y)$ the equality $\pi_B(x) = \pi_B(y)$ implies $\{p_B(x)\} = \overline{p}_B(\mathrm{supp}(\mu))$.

Lemma 2.7. *Let X be a closed subset of $Y = I^A \times M$ and cM be a metric compactification of M. Assume r is a usco mapping from Y to $P(cX)$, where $cX = \mathrm{cl}_{I^A \times cM}(X)$, such that $r(x) = \delta_x$ for every $x \in X$. Then:*

(i) *every union of r-admissible sets is r-admissible too;*
(ii) *for every r-admissible set B the mapping p_B is open.*

Proof. The proof of (i) is similar to the proof of Lemma 2.6(i), so it is omitted.

(ii) We use some arguments from [18]. Let U be open in X. Choose an open subset $W(U)$ of cX with $W(U) \cap X = U$. Without loss of generality we can suppose that there exists a continuous function $f : cX \to [0,1]$ such that $W(U) = f^{-1}(0,1]$. Consider the continuous extension $f_1 : P(cX) \to [0,1]$ of f, defined by $f_1(\mu) = \mu(f)$. Put $U_1 = f_1^{-1}(0,1]$ and $V = r^{\#}(U_1)$. Since V is open in Y and π_B is an open mapping, to prove that $p_B(U)$ is open in $p_B(X)$ it is enough to show the equality $p_B(U) = p_B(X) \cap \pi_B(V)$. We have $U = X \cap V$ because $r(x) = \delta_x$ for each $x \in X$. Thus, $p_B(U) \subset p_B(X) \cap \pi_B(V)$. Suppose $z \in p_B(X) \cap \pi_B(V)$. Then there are two points $x \in X$ and $y \in V$ such that $z = p_B(x) = \pi_B(y)$. Let $\mu^* \in r(y)$. It follows from the r-admissibility of B that

$$p_B(x) = \overline{p}_B(\mathrm{supp}(\mu^*)). \tag{5}$$

We will show that $\mathrm{supp}(\mu^*) \cap W(U) \neq \emptyset$. Indeed, otherwise we would have $f_1(\mu^*) = \mu^*(f) = 0$ because $f|(cX - W(U)) \equiv 0$. But this is in contradiction with $\mu^* \in r(y) \subset U_1$. Since p_B is a perfect mapping, we have $(\overline{p}_B)^{-1}(p_B(X)) = X$. Therefore, by (5), $\mathrm{supp}(\mu^*) \subset X$. This implies

$$\mathrm{supp}(\mu^*) \cap U = \mathrm{supp}(\mu^*) \cap W(U) \neq \emptyset.$$

So, $z = \overline{p}_B(\mathrm{supp}(\mu^*)) \in p_B(U)$. Hence, we proved the desired equality $p_B(U) = p_B(X) \cap \pi_B(V)$. □

3. Proof of Theorem 1.1.

We use the notations from the previous two sections.

(1) implies (2). Let r be a usco antimixer on X. Consider X as a closed subspace of $I^A \times M$, where $A = \{\alpha\colon \alpha < \omega(\tau)\}$, $\tau = w(X)$ and M is a separable metric space. By Lemma 2.1, for every $\alpha < \omega(\tau)$ there exists a countable r-admissible set $B(\alpha)$ containing α. Next, denote $A(\alpha) = \bigcup\{B(\beta)\colon \beta < \alpha\}$, $q_\alpha = p_{A(\alpha)}$ and $X_\alpha = q_\alpha(X)$ for each $\alpha < \omega(\tau)$. If $\alpha > \beta$ we put $q^\alpha_\beta = q_\beta \circ q_\alpha^{-1}$. According to Lemma 2.6, every q_α is open and perfect, hence $q^{\alpha+1}_\alpha$ is also open and perfect. Obviously, each $q^{\alpha+1}_\alpha$ has a metrizable kernel. Thus, we have constructed an S-system $\varphi = \{X_\alpha, q^\alpha_\beta,\ \beta < \alpha < \omega(\tau)\}$ of a length $w(X)$ such that $X = \varprojlim \varphi$.

(2) implies (3) and (4). Let X be a limit space of an S-system $\varphi = \{X_\alpha, q^\alpha_\beta,\ \beta < \alpha < \omega(\tau)\}$ of a length $\tau = w(X)$. Suppose X is a closed subspace of a paracompact p-space Y. Consider Y as a subset of a product $I^A \times M$, where M is a metric space and A is uncountable. It is enough to prove that there is a usco retraction r from $I^A \times M$ onto X (respectively, there is a regular extension operator $u : C(X) \to C(I^A \times M)$). Since X_1 is a separable metric space, by results from [13] and [19], there exists a countable set $B \subset A$ such that $q_1(x) = q_1(y)$ whenever $x, y \in X$ and $p_B(x) = p_B(y)$. Here q_1 is the

projection from X onto X_1. Because π_B is perfect the space $M_1 = p_B(X)$ is closed in $I^B \times M$ and there is a continuous mapping h from M_1 onto X_1 such that $h \circ p_B = q_1$. By Lemma 2.4, there exist a 0-dimensional metric space M_0 and perfect Milutin mappings $l : D^{A-B} \to I^{A-B}$ and $g : M_0 \to I^B \times M$ such that $k = l \times g : D^{A-B} \times M_0 \to I^A \times M$ is also a Milutin mapping, where D is the two-point discrete space. Obviously, X is contained in $I^{A-B} \times M_1$ as a closed subset and $\pi_1(X) = M_1$ (π_1 is the projection from $I^{A-B} \times M_1$ onto M_1). Put $M_0^* = g^{-1}(M_1)$, $Z = D^{A-B} \times M_0^*$ and $H = k^{-1}(X)$. Clearly, Z is a 0-dimensional normal space and $k^{-1}(I^{A-B} \times M_1) = Z$. We will show there is a continuous extension $\overline{k} : Z \to X$ of the mapping $k|H$. This will be done if for every α a continuous mapping $\overline{k}_\alpha : Z \to X_\alpha$ is constructed such that $q_\alpha \circ k|H = \overline{k}_\alpha|H$ and $q_\alpha^{\alpha+1} \circ \overline{k}_{\alpha+1} = \overline{k}_\alpha$. Put $\overline{k}_1 = h \circ \pi_1 \circ k|Z$. Clearly, $q_1 \circ k|H = \overline{k}_1|H$. Suppose we have already constructed $\overline{k}_\beta$ for every $\beta < \alpha$. Assume α is a limit ordinal. Since φ is continuous, $X_\alpha = \varprojlim\{X_\beta, q_\beta^\gamma, \beta < \gamma < \alpha\}$, so we can define $\overline{k}_\alpha = \varprojlim \overline{k}_\beta$. If α is isolated, by Lemma 2.3, there is a continuous mapping $\overline{k}_\alpha : Z \to X_\alpha$ such that $q_\alpha \circ k|H = \overline{k}_\alpha|H$ and $q_{\alpha-1}^\alpha \circ \overline{k}_\alpha = \overline{k}_{\alpha-1}$. Now, $\overline{k} = \varprojlim \overline{k}_\alpha : Z \to X$ is a continuous extension of $k|H$. Since M_0 is a 0-dimensional metric space and M_0^* its closed subset, there exists a continuous retraction r_1 from M_0 onto M_0^*. Then $r_2 = \mathrm{id} \times r_1$ is a continuous retraction from $D^{A-B} \times M_0$ onto Z. Next, set-valued mapping $r : I^A \times M \to X$, defined by $r(y) = \overline{k}(r_2(k^{-1}(y)))$, is a usco retraction because k is perfect and $r(x) = \{x\}$ for every $x \in X$. Thus, (2) implies (3). To show that (2) implies (4) take a regular averaging operator $u_1 : C(D^{A-B} \times M_0) \to C(I^A \times M)$ for k (such an operator exists because k is a Milutin mapping) and define a regular extension operator $u : C(X) \to C(I^A \times M)$ by $u(f) = u_1(f \circ \overline{k} \circ r_2)$.

(3) implies (1). Let X be a closed subset of $I^A \times M$, where M is a metric space. By (3), there is a usco retraction $r_1 : I^A \times M \to X$. By Lemma 2.2, there exists a usco antimixer $r_2 : (I^A \times M)^3 \to I^A \times M$. Then the map $r : X^3 \to X$ defined by $r(x, y, z) = r_1(r_2(x, y, z))$, is a usco antimixer.

(4) implies (2) and (5) implies (2). Consider X as a closed subspace of $I^A \times M$, where $A = \{\alpha\colon \alpha < \omega(\tau)\}$, $\tau = w(X)$ and M is a separable metric space. Let cM be a metric compactification of M and cX be the closure of X in $I^A \times cM$.

Suppose there is a regular extension operator $u : C(X) \to C(I^A \times M)$. Define a continuous mapping r from $I^A \times M$ to $P(cX)$ by $r(y)(f) = u(f|X)(y)$ for every $y \in I^A \times M$ and $f \in C(cX)$. Since u is an extension operator, we have $r(x) = \delta_x$ for each $x \in X$.

If X is d-embedded in $I^A \times M$, we can define a usco mapping $r : I^A \times M \to cX$ by putting $r(y) = cX$ if $y \notin \bigcup\{e(U)\colon U$ is open in $X\}$ and $r(y) = \bigcap\{\mathrm{cl}_{cX}(U)\colon y \in e(U)\}$ otherwise. Clearly, $r(x) = \{x\}$ for $x \in X$.

Hence, both cases can be combined as follows: there is a usco mapping $r : I^A \times M \to P(cX)$ such that $r(x) = \delta_x$ for every $x \in X$. Next, we use the same arguments as in the proof of the implication (1) $\to$ (2) with Lemmas 2.1 and 2.6 replaced by Lemmas 2.5 and 2.7, respectively.

(2) implies (6). Let X be a limit space of an S-system $\varphi = \{X_\alpha, q^\alpha_\beta,\ \beta < \alpha < \omega(\tau)\}$ of a length $\tau = w(X)$. Suppose $f: F \to X$ is a continuous mapping from a closed G_δ-subset of a normal space Y with $\dim(Y - F) = 0$. Consider X_1 as a subset of the Hilbert cube I^ω and take a continuous extension $g: Y \to I^\omega$ of the mapping $q_1 \circ f$. Since F is G_δ in Y, $Y - F$ is also normal and $Y - F = \bigcup\{F_n\colon n = 1, 2, \ldots\}$, where each F_n is closed in Y. Without loss of generality we can suppose that every F_n is G_δ in Y. Then, by [6, Theorem 2.1], there are a separable metric space Z and continuous mappings $h: Y \to Z$, $g_1: Z \to I^\omega$ such that $g = g_1 \circ h$, $h(F)$ is closed in Z, $h^{-1}(h(F)) = F$ and $\dim(Z - h(F)) = 0$. Let $r: Z \to h(F)$ be a continuous retraction. Then $g_1 \circ r \circ h: Y \to X_1$ is a continuous extension of $q_1 \circ f$. Now, using Remark 2 we can get a continuous extension $\overline{f}: Y \to X$ of f (see the construction of the mapping $\overline{k}$ in the proof of implications (2) → (3) and (2) → (4)).

(6) implies (5). Let Y be a normal space containing X as a closed subset. Denote by Z the space obtained from Y by making the points of $Y - X$ isolated. Let $Z(X)$ be the set $(X \times \{0\}) \cup \bigcup\{(Y - X) \times \{1/n\}\colon n = 1, 2, \ldots\}$ with the subspace topology inherited from the product $Z \times I$. It is easily seen that $Z(X)$ is normal, $Z(X) - X$ is 0-dimensional and X is a closed G_δ-subset of $Z(X)$. So, there is a continuous retraction $r_1: Z(X) \to X$. Clearly, $Z(X)$ is dense in $(Z \times \{0\}) \cup Z(X)$. Then r_1 can be extended to a usco mapping $r_2: (Z \times \{0\}) \cup Z(X) \to \beta X$ (see [17, Lemma 8]). The restriction $r = r_2|(Z \times \{0\})$ is a usco mapping from Z to βX such that $r(x) = \{x\}$ for every $x \in X$. For any open set U in X put $W(U) = \bigcup\{W\colon W$ is open in βX and $W \cap X = U\}$ and $e(U) = \mathrm{Int}_Y(r^\#(W(U)))$. Obviously, $e(U)$ is open in Y, $e(U) \cap X = U$ and $e(U) \cap e(V) = e(U \cap V)$. Hence, X is d-embedded in Y. □

Remark 3. Let us note that the condition "Y is a paracompact p-space" in (3) or (4) cannot be weakened to "Y is paracompact". The first observation follows from the following fact [17, Theorem 3]: Let X be a paracompact p-space and for every closed embedding of X in a paracompact space Y there is a usco retraction from Y to X. Then X is Čech-complete. The second one follows from [5], where it is proved that the space Q of rational numbers is closed in the Michael line M and there is no regular extension operator $u: C(Q) \to C(M)$. On the other hand Q is metrizable and hence, Q admits a usco antimixer. □

4. Appendix

For a given space Z let $\mathrm{nw}(Z)$ denote the net weight of Z. We write $X \in \mathrm{Nag}(\mathrm{nw}(\tau))$ if X is a continuous image of a space Y which admits a continuous and perfect mapping onto a space Z with $\mathrm{nw}(Z) \leqslant \tau$. It is easily seen that the class $\mathrm{Nag}(\mathrm{nw}(\tau))$ is finitely multiplicative and $l(X) \leqslant \tau$ for every $X \in \mathrm{Nag}(\mathrm{nw}(\tau))$. The aim of this section is to prove the following generalization of Theorem 1.5 from [14] (Tkachenko proved a particular case of Proposition 4.1 when X is a topological group which is a continuous image of a space Y admitting a continuous and perfect mapping onto a space of weight $\leqslant \tau$).

Proposition 4.1. *Let $r: X^3 \to X$ be a u.s.c. antimixer on X and $X \in \mathrm{Nag}(\mathrm{nw}(\tau))$. Then for any family $\mathcal{F}$ consisting of G_τ-subsets of X there exists a subfamily $\theta \subset \mathcal{F}$ of cardinality $\mathrm{card}(\theta) \leqslant \tau$ such that $\bigcup \theta$ is dense in $\bigcup \mathcal{F}$.*

Proof. Let f be a continuous mapping from a space Y onto X and Y admit a perfect continuous mapping g onto a space Z with $\mathrm{nw}(Z) \leqslant \tau$. Consider X as a subset of $I^A \times M$, where A is infinite and M is a separable metric space. Since $l(X^2) \leqslant \tau$, by Remark 1, for every set $C \subset A$ of cardinality $\mathrm{card}(C) \leqslant \tau$ there exists a r-admissible set $B \subset A$ containing C such that $\mathrm{card}(B) \leqslant \tau$. So, without loss of generality we can suppose that for any family $\mathcal{F}$ of G_τ-subsets of X and each $F \in \mathcal{F}$ there is a r-admissible set $B(F)$ such that $\mathrm{card}(B(F)) \leqslant \tau$ and $B(F) \in \mathcal{L}(F)$, i.e., $p_{B(F)}^{-1}(p_{B(F)}(F)) = F$. We shall construct by induction an increasing sequence $\{B(n)\colon n = 1,2,\ldots\}$ of r-admissible subsets of A and a sequence $\{\mathcal{F}(n)\colon n = 1,2,\ldots\}$ of subfamilies of $\mathcal{F}(f) = \{f^{-1}(F)\colon F \in \mathcal{F}\}$ such that for every n the following conditions are fulfilled:

(a) $\mathrm{card}(\mathcal{F}(n)) \leqslant \tau$ and $\mathrm{card}(B(n)) \leqslant \tau$;

(b) $h(n)(\bigcup \mathcal{F}(n))$ is dense in $h(n)(\bigcup \mathcal{F}(f))$, where $h(n)$ is the diagonal product $h(n) = g\triangle(p_{B(n)} \circ f)$;

(c) $B(F) \subset B(n+1)$ for each $F \in \theta(n) = f(\mathcal{F}(n))$.

Suppose we have already constructed $B(k)$ and $\mathcal{F}(k)$ for $k = 1,2,\ldots,n$. Let

$$B(n+1) = B(n) \cup \bigcup \{B(F)\colon F \in \theta(n)\}.$$

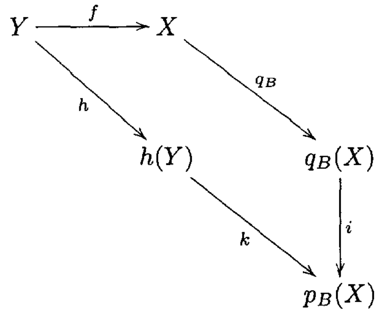
According to Lemma 2.6(i), $B(n+1)$ is r-admissible and obviously, $\mathrm{card}(B(n+1)) \leqslant \tau$. Consider the mapping

$$h(n+1) = g\triangle(p_{B(n+1)} \circ f).$$

Since $\mathrm{nw}(Z) \leqslant \tau$ and $w(p_{B(n+1)}(X)) \leqslant \tau$, we have $\mathrm{nw}(Z \times p_{B(n+1)}(X)) \leqslant \tau$. Hence, $\mathrm{nw}(h(n+1)(Y)) \leqslant \tau$. The last implies there is a subfamily $\mathcal{F}(n+1)$ of $\mathcal{F}(f)$ such that $\mathrm{card}(\mathcal{F}(n+1)) \leqslant \tau$ and $h(n+1)(\bigcup \mathcal{F}(n+1))$ is dense in $h(n+1)(\bigcup \mathcal{F}(f))$. Now, let

$$B = \bigcup \{B(n)\colon n = 1,2,\ldots\},$$
$$\mathcal{F}^* = \bigcup \{\mathcal{F}(n)\colon n = 1,2,\ldots\}, \qquad \theta = \bigcup \{\theta(n)\colon n = 1,2,\ldots\} = f(\mathcal{F}^*)$$

and $h = g\triangle(p_B \circ f)$. The mappings $h(n)$, $n = 1,2,\ldots$ and h are perfect because g is perfect. Since $B(n) \subset B(n+1)$ for each n, there exists a continuous (and perfect) mapping $h(n+1,n): h(n+1)(Y) \to h(n)(Y)$ such that $h(n+1,n) \circ h(n+1) = h(n)$. Hence, $h(Y)$ is a limit space of the inverse sequence $\{h(n)(Y), h(n+1,n),\ n = 1,2,\ldots\}$. Using this fact and (b) we can show that $h(\bigcup \mathcal{F}^*)$ is dense in $h(\bigcup \mathcal{F}(f))$. Denote by q_B the quotient mapping from X onto the set $p_B(X)$ equipped with the quotient topology. There is a continuous one-to-one mapping i from $q_B(X)$ onto $p_B(X)$. Then the following diagram is commutative,



where k is the projection from $h(Y)$ onto $p_B(X)$, defined by $k(h(y)) = p_B(f(y))$. Since h is a quotient mapping, there is a continuous mapping $\overline{k}: h(Y) \to q_B(X)$ such that $\overline{k} \circ h = q_B \circ f$ and $i \circ \overline{k} = k$. Then $\overline{k}(h(\bigcup \mathcal{F}^*))$ is dense in $\overline{k}(h(\bigcup \mathcal{F}(f)))$. But $\overline{k}(h(\bigcup \mathcal{F}^*)) = q_B(f(\bigcup \mathcal{F}^*)) = q_B(\bigcup \theta)$ and $\overline{k}(h(\bigcup \mathcal{F}(f))) = q_B(\bigcup \mathcal{F})$. Thus, $q_B(\bigcup \theta)$ is dense in $q_B(\bigcup \mathcal{F})$. By Lemma 2.6(i), B is r-admissible. It follows from the proof of Lemma 2.6(ii) that q_B is open. Using this observation we prove that $\bigcup \theta$ is dense in $\bigcup \mathcal{F}$. Suppose $x \in \bigcup \mathcal{F}$ and $O(x)$ is an open neighborhood of x in X. Then $q_B(O(x))$ is open in $q_B(X)$ and since $q_B(\bigcup \theta)$ is dense in $q_B(\bigcup \mathcal{F})$, $q_B(O(x))$ meets $q_B(\bigcup \theta)$. It follows from (c) that $B \in \mathcal{L}(F)$ for every $F \in \theta$. Hence,

$$q_B^{-1}\Big(q_B\Big(\bigcup \theta\Big)\Big) = \bigcup \theta.$$

So, $O(x) \cap (\bigcup \theta) \neq \emptyset$. □

References

[1] A. Chigogidze, Noncompact absolute extensors in dimension n, n-soft mappings, and their applications, Math. USSR Izv. 28 (1987) 151–174.

[2] M. Čoban, Many-valued mappings and Borel sets II, Trans. Moscow Math. Soc. 23 (1970) 276–310.

[3] M. Čoban, Topological structure of subsets of topological groups and their factor-spaces, Topological Structures and Algebraic Systems (Kishinev) 44 (1977) 117–163 (in Russian).

[4] R. Haydon, On a problem of Pelczynski: Milutin spaces, Dugundji spaces and AE(0-dim), Studia Math. 52 (1974) 23–31.

[5] R. Heath and D. Lutzer, Dugundji extension theorems for linearly ordered spaces, Pacific J. Math. 55 (1974) 419–425.

[6] W. Kulpa, Factorization theorems and properties of the covering type, Universytet Slaski, Katovice (1980).

[7] St. Nedev, Selection and factorization theorems for set-valued mappings, Serdica 6 (1980) 291–317.

[8] B. Pasynkov, On spaces with a compact group actions, Dokl. Akad. Nauk SSSR 231 (1976) 39–42 (in Russian).

[9] A. Pelczynski, Linear Extensions, Linear Averagings, and Their Applications to Linear Classification of Spaces of Continuous Functions (Warszawa, 1968).

[10] E. Pol and R. Pol, Remarks on Cartesian products, Fund. Math. 98 (1978) 61–73.
[11] E. Scepin, Topology of limit spaces of uncountable inverse spectra, Uspekhi Mat. Nauk 31 (1976) 191–226 (in Russian).
[12] L. Shirokov, An external characterization of Dugundji spaces and k-metrizable compacta, Dokl. Akad. Nauk SSSR 263 (1982) 1073–1077 (in Russian).
[13] M. Tkachenko, Some results on inverse spectra I, Comment. Math. Univ. Carolinae 22 (1981) 621–633.
[14] M. Tkachenko, Factorization theorems for topological groups and their applications, Topol. and Appl. 38 (1991) 21–37.
[15] V. Uspenskij, The Mal'tsev operation on countably compact spaces, Comment. Math. Univ. Carolinae 30 (1989) 395–402.
[16] V. Uspenskij, Topological groups and Dugundji compacta, Mat. Sb. 180 (1989) 1092–1118 (in Russian).
[17] V. Valov, Another characterization of AE(0)-spaces, Pacific J. Math. 127 (1987) 199–208.
[18] V. Valov, Yet another characterization of AE(0)-spaces, Proceedings of the Seventeenth Spring Conference of the Union of Bulgarian Mathematicians. Sunny Beach, April 6–9 (1988) 147–150.
[19] Y. Yajima, Products of a compact space and a metric space, Preprint.