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GENERAL TOPOLOGY

Spaces of Bounded Functions with the Compact Open Topology

by

Vesko VALOV

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Summary. We prove that if ϕ is a continuous linear surjection from $C_k^*(X)$ onto $C_k^*(Y)$, where X and Y are metric spaces, then Y is Čech complete whenever X is. This answers a question raised by K. Yamada 1995. When, in addition, ϕ is a quotient the same conclusion is true for a class of spaces more general than metrizable ones. We also provide a positive solution to a question raised by J. Baars and J. de Groot 1992 whether psedocompactness and cofinality of the family of all compact sets are preserved by linear homeomorphisms between $C_k^*(X)$ and $C_k^*(Y)$.

1. Introduction. For a completely regular space (we consider only completely regular spaces) X let $C_k(X)$ (resp., $C_k^*(X)$) denote the space of all continuous (resp., and bounded) functions on X with the compact open topology. When the pointwise topology is considered we write $C_p(X)$ and $C_p^*(X)$, respectively.

After the result of J. Baars, J. de Groot and J. Pelant [5, Theorem 4.2] that Čech completeness is preserved by a continuous linear surjection $\phi : C_p^*(X) \to C_p^*(Y)$ for metric spaces X and Y, K. Yamada [15] raised the question of whether the same is true for the compact open topology. We are going to answer this question positively, and to show that the result remains valid even for more general spaces X and Y if ϕ is a quotient. Let us point out that when X and Y are not metrizable, then the requirement ϕ to be a quotient is essential because for each X the restriction map from $C_k(\beta X)$ into $C_k^*(X)$ is a continuous linear surjection. The methods developed for answering the question of K. Yamada allows us to show that local compactness is also preserved by quotient continuous linear surjections from

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$C_k^*(X)$ onto $C_k^*(Y)$ for some class of spaces X and Y. Similar result for linear homeomorphisms between $C_p(X)$ and $C_p(Y)$ was proved by S. Gul'ko and O. Okunev [6, Theorem 3.1], and R. McCoy and I. Ntantu [8, Corollary 4.5] (see also [4, Theorem 1.5.10] for another proof) and later on generalized in [13, Proposition 4.8].

An easy proof that real compactness, as well as the cofinality of the family of all compact sets are preserved by quotient continuous linear surjections from $C_k^*(X)$ onto $C_k^*(Y)$ is also given. The last one answers a question of J. Baars and J. de Groot [4, Question 1, p. 35]. Another question of J. Baars and J. de Groot [4, Question 1] about preservation of pseudocompactness by linear homeomorphisms between $C_k^*(X)$ and $C_k^*(Y)$ is also answered positively.

2. Some preliminary results. Suppose ϕ is a linear map from $C^*(X)$ into $C^*(Y)$. For every $y \in \beta Y$ we can define the linear functional $\mu_y : C^*(X) \to \mathbf{R}$ by $\mu_y(f) = \beta\phi(f)(y)$, where βY is the Čech-Stone compactification of Y and $\beta\phi(f) : \beta Y \to \mathbf{R}$ denotes the continuous extension of $\phi(f)$. The support of μ_y is the set $\operatorname{supp}(\mu_y)$ of all $x \in \beta X$ satisfying the condition that for every neighbourhood U of x in βX there is $f \in C^*(X)$ such that $\beta f|(\beta X - U) = 0$ and $\mu_y(f) \neq 0$. Obviously, $\operatorname{supp}(\mu_y)$ is closed in βX, so it is always compact. For $A \subset \beta Y$ we denote $\operatorname{cl}_{\beta X}(\cup\{\operatorname{supp}(\mu_y) : y \in A\})$ by $\operatorname{supp}(A)$. Our definition of support is a slight modification of that one given by Arkhangel'skiĭ [1, p. 853].

Recall that a set-valued map $\Phi : Z \to T$, possibly $\Phi(z) = \emptyset$ for some $z \in Z$, is lower semicontinuous (br., lsc) if $\Phi^{-1}(U) = \{z \in Z : \Phi(z) \cap U \neq \emptyset\}$ is open in Z for every open set $U \subset T$.

When $C^*(X)$ is endowed with the topology of uniform convergence we write $C_u^*(X)$.

LEMMA 2.1. *Let ϕ be a continuous linear map from $C_k^*(X)$ onto $C_k^*(Y)$. Then we have:*

(i) *μ_y is continuous on $C_u^*(X)$ and* $\operatorname{supp}(\mu_y) \neq \emptyset$ *for every $y \in \beta Y$,*

(ii) $\operatorname{supp}(\mu_y) : \beta Y \to \beta X$ *is* lsc *and* $\operatorname{supp}(\mu_y) \subset X$ *for any $y \in Y$.*

Proof. (i) By the Closed Graph Theorem, ϕ considered as map from $C_u^*(X)$ into $C_u^*(Y)$ is also continuous. Consequently, every μ_y, $y \in \beta Y$, is continuous on $C_u^*(X)$. It can be shown that if μ is a continuous linear functional on $C_u^*(X)$ and $\beta f | \operatorname{supp}(\mu) = 0$ for some $f \in C^*(X)$ then $\mu(f) = 0$ (see [14, Lemma 2.1]). On the other hand $\mu_y \neq 0$ for all $y \in \beta Y$ because ϕ is surjective. Combining the last two facts we obtain that $\operatorname{supp}(\mu_y) \neq \emptyset$, $y \in \beta Y$.

(ii) Using the same arguments as in Lemma 1.2.7 from [4] we can show that $\operatorname{supp}(\mu_y) : \beta Y \to \beta X$ is lsc. It remains only to prove that $\operatorname{supp}(\mu_y) \subset X$ for any $y \in Y$. That follows from the continuity of μ_y, $y \in Y$, on $C_k^*(X)$ (see [4, Lemma 1.2.13] or [13, Proposition 2.2]). □

For a space X and a cardinal τ let $\nu_\tau X$ be the set of all $x \in \beta X$ such that every G_τ-set in βX containing x meets X. If $X = \nu_\tau X$, we say that X is τ-real compact. Obviously, $\nu_\tau X \subset \nu_\omega X = \nu X$ for every infinite τ, where νX is the Hewitt real compactification of X and ω is the first infinite cardinal.

COROLLARY 2.2. *In the hypotheses of Lemma* 2.1, $\operatorname{supp}(\mu_y) \subset \nu_\tau X$ *for every* $y \in \nu_\tau Y$.

Proof. Suppose there is $x \in \operatorname{supp}(\mu_y) \cap (\beta X - \nu_\tau X)$ for some $y \in \nu_\tau Y$. Let γ be a family of cardinality at most τ consisting of open in βX sets containing x such that γ is closed under finite intersections and $\bigcap\{\operatorname{cl}_{\beta X}(U) : U \in \gamma\} \subset \beta X - \nu_\tau X$. Then, by Lemma 2.1, $V(U) = \{z \in \beta Y : \operatorname{supp}(\mu_z) \cap U \neq \emptyset\}$, $U \in \gamma$, are open in βY and contain y. Because $\operatorname{supp}(\mu_z) \subset X$ for every $z \in Y$ we have $G = \bigcap\{V(U) : U \in \gamma\} \subset \beta Y - Y$. On the other hand, since $y \in \nu_\tau Y$, $G \cap Y \neq \emptyset$, which is a contradiction. □

LEMMA 2.3. *Let* ϕ *be a quotient linear continuous surjection from* $C_k^*(X)$ *onto* $C_k^*(Y)$. *Then* $H^\# = \{y \in \beta Y : \operatorname{supp}(\mu_y) \subset H\}$ *is a compact subset of* Y *for every compact* $H \subset X$.

Proof. By Lemma 2.1 $\operatorname{supp}(\mu_y) : \beta Y \to \beta X$ is lsc, so $H^\#$ is closed in βY. It remains to show that $H^\# \subset Y$. To this end fix $y \in H^\#$.

CLAIM. μ_y *is continuous on* $C_k^*(X)$.

Since the restriction map $\pi : C_k^*(X) \to C_k(H)$ is an open and continuous surjection (see [10, Lemma 3]), and $\pi(f) = \pi(g)$ yields $\mu_y(f) = \mu_y(g)$, it is enough to show that the map $\psi_y : C_k(H) \to \mathbf{R}$, $\psi_y(\pi(f)) = \mu_y(f)$, is continuous. Moreover, it is certainly true because μ_y is continuous on $C_u^*(X)$ (Lemma 2.1) and π, considered as a map from $C_u^*(X)$ onto $C_k(H)$, is continuous and open (by the Open Mapping Theorem).

Let us finish the proof of Lemma 2.3. Because ϕ is quotient and for any two $f, g \in C^*(X)$ we have $\mu_y(f) = \mu_y(g)$ provided $\phi(f) = \phi(g)$, the map $\xi_y : C_k^*(Y) \to \mathbf{R}$, defined by $\xi_y(\phi(h)) = \mu_y(h)$, is continuous. Hence there is a compact set $P \subset Y$ and $\eta > 0$ such that $|\xi_y(g)| < 1$ for every $g \in C^*(Y)$ with $|g(z)| < \eta$ for all $z \in P$. It is easily seen that the last yields $y \in P$. Thus $H^\# \subset Y$. □

COROLLARY 2.4. *Let* $\phi : C_k^*(X) \to C_k^*(Y)$ *be a quotient continuous linear surjection. If* X *is compact (resp.,* τ*-real compact), then* Y *is.*

Proof. It follows from Lemma 2.3 that Y is compact provided X is. Suppose X is τ-real compact and there is $y \in \nu_\tau Y - Y$. Then, $\mathrm{supp}(\mu_y) \subset X$ (Corollary 2.2) and, by Lemma 2.3, $y \in Y$, which is a contradiction. □

Let us note that the proof of the first part of Corollary 2.4 can be done directly, without using Lemma 2.3. Indeed, by the Closed Graph Theorem, ϕ considered as a map from $C_k(X)$ onto $C_u^*(Y)$ is continuous. Then the Open Mapping Theorem together with the fact that ϕ (as a map between $C_k(X)$ and $C_k^*(Y)$) is quotient give us that the identity map from $C_u^*(Y)$ onto $C_k^*(Y)$ is a homeomorphism. This yields Y is compact.

For a space X let $\mathcal{K}(X)$ be the family of all compact subsets of X. A subset $\mathcal{B} \subset \mathcal{K}$ is cofinal in $\mathcal{K}(X)$ if for any $K \in \mathcal{K}(X)$ there is $B \in \mathcal{B}$ such that $K \subset B$. The cofinality of $\mathcal{K}(X)$ is defined by

$$\mathrm{cof}\,\mathcal{K}(X) = \min\{\mathrm{card}\,\mathcal{B} : \ \mathcal{B} \text{ is cofinal in } \mathcal{K}(X)\}.$$

Now we can give a positive answer to the following question of J. Baars and J. de Groot [4, Question 1]: Is it true that $\mathrm{cof}\,\mathcal{K}(X) = \mathrm{cof}\,\mathcal{K}(Y)$ if $C_k^*(X)$ is linearly homeomorphic to $C_k^*(Y)$?

COROLLARY 2.5. *Let $C_k^*(Y)$ be a quotient image of $C_k^*(X)$ under a continuous linear map. Then* $\mathrm{cof}\,\mathcal{K}(Y) \leqslant \mathrm{cof}\,\mathcal{K}(X)$.

Proof. We follow the proof of Theorem 1.5.2 from [4]. Suppose $\{H_j : j \in J\}$ is cofinal in $\mathcal{K}(X)$. For each $j \in J$ let $K_j = \{y \in \beta Y : \mathrm{supp}(\mu_y) \subset H_j\}$. By Lemma 2.3, all $K_j \subset Y$ are compact. So, it remains to show that $\{K_j : j \in J\}$ is cofinal in $\mathcal{K}(Y)$. To this end, let $B \subset Y$ be compact. Then, by Lemma 1.5.6 from [4], $\mathrm{supp}(B) \subset X$ is compact. Hence $\mathrm{supp}(B) \subset H_j$ for some $j \in J$, which implies $B \subset K_j$. □

Lemma 2.3 and all consequent corollaries are, in general, not true if the requirement for ϕ to be quotient is omitted. For example, the restriction map from $C_k(\beta X)$ onto $C_k^*(X)$ is always a linear continuous surjection, but X is not always compact (resp., τ-real compact). Sometimes this requirement can be avoided.

LEMMA 2.6. *Let ϕ be a continuous linear map from $C_k^*(X)$ onto $C_k^*(Y)$ and $H \subset X$ be compact and metrizable such that $L = H^{\#} \cap Y$ is C^*-embedded in Y. Then L is compact and metrizable.*

Proof. The restrictions $\pi : C_u^*(X) \to C_u(H)$, $q : C_u^*(Y) \to C_u^*(L)$ are open and continuous surjections. By the Closed Graph Theorem ϕ, considered as map from $C_u^*(X)$ onto $C_u^*(Y)$ is also continuous. Then the formula $\theta(\pi(f)) = q(\phi(f))$ defines a continuous linear surjection $\theta : C_u(H) \to C_u^*(L)$. Now we use an idea from the proof of Lemma 5.4 in [5]. Since we have for a space Z that $C_u^*(Z)$ is separable if and only if Z is compact and metrizable [11, Proposition 7.6.2], L is compact and metrizable.

3. The main results. A point $x \in X$ is said to be a strongly q-point (br. sq-point) in X if there is a countable family $\{U_n\}$ of neighbourhoods of x in X such that any sequence $\{x_n\}$ with $x \in U_n$, $n \in \mathbf{N}$, has a compact closure in X. The set of all sq-points of X is called the sq-derivative and is denoted by $X^{(\mathrm{sq})}$. When $X^{(\mathrm{sq})} = X$ we say that X is an sq-space. The class of sq-spaces contains all spaces of pointwise countable type in sense of Arkhangel'skiĭ [3], in particular first countable spaces, and it is contained in the class of q-spaces in the sense of Michael [9].

Recall that a set-valued map $\psi : Y \to X$ is upper semicontinuous (br., usc) if the set $\{y \in Y : \psi(y) \subset U\}$ is open in Y whenever U is open in X. Upper semicontinuous compact-valued maps are called usco maps. A subset F of a space X is called bounded in X if $f(F)$ is bounded in $\mathbf{R}$ for every $f \in C(X)$. When every closed and bounded set in X is compact, then we say that X is a μ-space.

LEMMA 3.1. *Let X be a μ-space and $\varphi : Y \to X$ be a set-valued map such that $\varphi(K)$ is bounded in X whenever $K \subset Y$ is compact. Then for every $F \subset Y^{(\mathrm{sq})}$ there is an* usco *map $\psi : F \to X$ such that $\varphi(y) \subset \psi(y)$ for every $y \in F$.*

Proof. We define $\psi : F \to \beta X$ by $\psi(y) = \cap\{\mathrm{cl}_{\beta X}(\varphi(U)) : U \in \mathcal{U}(y)\}$, where $\mathcal{U}(y)$ is the family of all neighbourhoods of y in F. It is trivially seen that ψ is usco and $\varphi(y) \subset \psi(y)$ for every $y \in F$. So, it remains to show that $\psi(y) \subset X$, $y \in F$. To this end for each $y \in F$ fix a countable family $\gamma(y) = \{U_n(y)\}$ of neighbourhoods of y in F such that any sequence $\{y_n\}$ with $y_n \in U_n(y)$ has a compact closure in Y. Let

$$\Phi(y) = \cap\{\mathrm{cl}_X(\varphi(U_n(y))) : n \in \mathbf{N}\}, \quad y \in F.$$

CLAIM 1. *$\Phi(y)$ is compact for each $y \in F$.*

Because X is a μ-space, it suffices to show that $\Phi(y)$ is bounded in X. Assuming the contrary we can find a discrete open in X family $\{V_n\}$ and $h \in C(X)$ such that $V_n \cap \Phi(y) \neq \emptyset$ for each n and $\{h(V_n) : n \in \mathbf{N}\}$ is discrete in $\mathbf{R}$. Then for every n there are $y_n \in U_n(y)$ and $x_n \in \varphi(y_n) \cap V_n$. Since $\{y_n\}$ has a compact closure in Y, $\varphi\{y_n\}$ is bounded in X. This is a contradiction because $\{h(x_n)\} \subset \mathbf{R}$ is not bounded.

CLAIM 2. *$\psi(y) \subset \Phi(y)$ for every $y \in F$.*

Suppose there is $x \in \psi(y) - \Phi(y)$ for some $y \in F$. Let W be open in βX such that $x \in W$ and $\mathrm{cl}_{\beta X}(W) \cap \Phi(y) = \emptyset$. Then for every n we can find

$y_n \in \cap\{U_k(y) : k \leqslant n\}$ with $\varphi(y_n) \cap W \neq \emptyset$. Let $P_n = \mathrm{cl}_Y(\{y_k : k \geqslant n\})$. All P_n are compact, so $H_n = \mathrm{cl}_X(\varphi(P_n))$ are also compact. Because $\gamma = \{H_n \cap cl_{\beta X}(W) : n \in \mathbf{N}\}$ is a decreasing family of non-empty compact sets, we have $\cap\gamma \neq \emptyset$. However, $\cap\gamma \subset \cap\{H_n\} \subset \Phi(y)$. Hence, $\mathrm{cl}_{\beta X}(W) \cap \Phi(y) \neq \emptyset$, which is a contradiction.

The proof of Lemma 3.1 follows now from the claims above.

THEOREM 3.2. *Let ϕ be a continuous linear map from $C_k^*(X)$ onto $C_k^*(Y)$ and X be a Čech complete μ-space. Then every closed* sq*-subspace of Y is Čech complete provided either ϕ is quotient or Y is normal and each compact set in X is metrizable.*

Proof. Let F be a closed sq-subspace of Y and $P = \mathrm{cl}_{\beta Y}(F)$. Because $\mathrm{supp}(\mu_y) : Y \to X$ is *lsc* such that $\mathrm{supp}(K)$ is compact in X for every compact $K \subset Y$ [4, Lemma 1.5.6], by Lemma 3.1, there is an usco map $\psi : F \to X$ with $\mathrm{supp}(\mu_y) \subset \psi(y)$ for every $y \in F$. Define $\varphi : P \to \beta X$ by $\varphi(y) = \cap\{\mathrm{cl}_{\beta X}(\psi(U \cap F)) : U \in \mathcal{U}(y)\}$, where $\mathcal{U}(y)$ is a local base at y in P. It is easily seen that φ is usco, $\varphi|F = \psi$ and $\mathrm{supp}(\mu_y) \subset \varphi(y)$ for every $y \in P$. Since X is Čech complete, $\varphi^*(X) = \{y \in P : \varphi(y) \subset X\}$ is G_δ in P and contains F. It suffices to show that $\varphi^*(X) = F$. Fix $y \in \varphi^*(X)$. Then $\mathrm{supp}(\mu_y) \subset X$ is compact and, by Lemma 2.3, $y \in F$ if ϕ is quotient. Now, let consider the case when ϕ is only surjection but Y is normal and every compact set in X is metrizable. Suppose $y \notin F$ and take $f \in C(P)$ such that $f^{-1}(f(y)) \subset \varphi^*(X)$. Because $H = \varphi(f^{-1}(f(y)))$ is a compact set in X and $\mathrm{supp}(\mu_z) \subset H$ for every $z \in f^{-1}(f(y))$, by Lemma 2.6, $H^{\#} \cap Y$ is compact and contains $L = f^{-1}(f(y)) \cap F$. Thus, L is also compact and we can find $g \in C(P)$ with $g(L) = 1$ and $g(y) = 0$. Let h be the diagonal product of g and f. Then $h^{-1}(h(y)) \subset f^{-1}(f(y))$ and h separates y from F. Since $h(F)$ is dense in $h(P)$, there is $\{y_n\} \subset F$ such that $\{h(y_n)\}$ converges to $h(y)$. Hence $B = \{y_n\} \cup h^{-1}(h(y))$ is a compact subset of $\varphi^*(X)$ and $B \cap F = \{y_n\}$. Again, applying Lemma 2.6 we obtain that $\{y_n\}$ is compact. Consequently, $h(y) \in \{h(y_n)\}$, which is a contradiction because $h(y) \notin h(F)$.

COROLLARY 3.3. *Let X and Y be metric spaces and there exists a continuous linear surjection from $C_k^*(X)$ onto $C_k^*(Y)$. Then Y is completely metrizable whenever X is.*

S. Gul'ko and O. Okunev [6, Theorem 3.1], and R. McCoy and I. Ntantu [8, Corollary 4.5] proved that local compactness is preserved by linear homeomorphisms between $C_p(X)$ and $C_p(Y)$ in the class of paracompact spaces of pointwise countable type (see also [4, Theorem 1.5.10] for another proof of the same fact). This result was generalized in [13, Proposition 4.8]. Now, we can prove a similar result for bounded function spaces.

THEOREM 3.4. *Let X be a locally compact μ-space and there exist a continuous quotient linear surjection from $C_k^*(X)$ onto $C_k^*(Y)$. Then:*

(i) *every closed* sq-*subset of Y is locally compact,*

(ii) $Y^{(\mathrm{sq})}$ *is locally compact and open in Y.*

Proof. (i) Let F be a closed sq-subset of Y. By Lemma 3.1, there is an usco map $\psi : F \to X$ such that $\mathrm{supp}(\mu_y) \subset \psi(y)$ for each $y \in F$. Fix $y^* \in F$ and a compact neighbourhood U of $\psi(y^*)$ in X. Take a closed neighbourhood V of y^* in F with $\psi(y) \subset U$ for any $y \in V$. By Lemma 2.3, $U^{\#} \subset Y$ is compact and, obviously, it contains V. Hence V is compact.

(ii) For every $y \in Y^{(\mathrm{sq})}$ fix a countable decreasing family $\{U_n(y)\}$ of neighbourhoods of y in X such that any sequence $\{y_n\}$ with $y_n \in U_n(y)$ has a compact closure in Y. Define $\Phi(y) = \cap\{\mathrm{supp}(U_n(y)) \cap X : n \in \mathbf{N}\}, y \in Y^{(sq)}$. As in Lemma 3.1 (Claim 1) we can show that each $\Phi(y)$ is compact. So, there is a compact neighbourhood $W(y)$ of $\Phi(y)$ in X. If $\mathrm{supp}(U_n(y)) \cap X$ meets $X - W(y)$ for every n, using the arguments of Claim 2 from Lemma 3.1 we get a contradiction. Thus, $\mathrm{supp}(U_i(y)) \cap X \subset W(y)$ for some i. This implies $\cup\{\mathrm{supp}(\mu_z) : z \in U_i(y)\} \subset W(y)$. Finally, by Lemma 2.3, $U_i(y)$ has a compact closure in Y, so $U_i(y) \subset Y^{(\mathrm{sq})}$. Therefore, $Y^{(\mathrm{sq})}$ is locally compact and open in Y. □

It is well known that if f is a map from a space X onto a set Y, then the quotient topology on Y (with respect to f) is not always completely regular. However, the strongest topology between all completely regular topologies on Y with respect to which f is continuous exists and it is called R-quotient [2]. The set Y with the R-quotient topology is denoted by Y^q and $p(f)$ is the corresponding to f continuous map from X onto Y^q. We also adopt the following notations. For a space X, let $\mathcal{F}_X$ denote the family of all continuous maps from X onto spaces of countable weight, and direct this family by the relation $g \succ f$ if there is a continuous surjection $p(g, f)$ from $g(X)$ onto $f(X)$ such that $p(g, f) \circ g = f$. For $f \in \mathcal{F}_X$ let $X_f = f(X)$. We also consider the spaces

$$C_k(f) = f_k^*(C_k^*(X_f)) \quad \text{and} \quad C_k^q(f) = p_k^*(f)(C_k^*(X_f^q)),$$

where

$$f_k^* : C_k^*(X_f) \to C_k^*(X) \quad \text{and} \quad p_k^*(f) : C_k^*(X_f^q) \to C_k^*(X)$$

are the continuous duals of f and $p(f)$, respectively. More precisely, we consider the duals $f^* : C^*(X_f) \to C^*(X)$, $p^*(f) : C^*(X_f^q) \to C^*(X)$ of f and $p(f)$, respectively, defined by

$$f^*(h) = h \circ f \quad \text{for } h \in C^*(X_f) \quad \text{and} \quad p^*(f)(h) = h \circ p(f) \quad \text{for } h \in C^*(X_f^q).$$

When the corresponding function spaces are equipped with the compact open topology, the above duals become continuous and we denote them by f_k^* and $p_k^*(f)$.

When the uniform convergence topology is considered, the corresponding notations are similar, the only difference is that the subscript k is replaced by u. Note that f_k^* and $p_k^*(f)$ are not, in general, embeddings, while f_u^* and $p_u^*(f)$ are always embeddings. We need the following observations:

FACT 1. *$C_k^*(X) = \cup\{C_k(f) : f \in \mathcal{F}_X\}$ and $C_k(f) \subset C_k^q(f)$, $f \in \mathcal{F}_X$.*

FACT 2. *For every countable $A \subset C^*(X)$ there exists $f \in \mathcal{F}_X$ such that $A \subset C_k(f)$.*

FACT 3. *$C_k^q(f)$ is a closed separable subset of $C_k^*(X)$, $f \in \mathcal{F}_X$.*

Proof. $C_k^q(f)$ is closed in $C_k^*(X)$ because $p(f)$ is R-quotient [2], and $C_k^q(f)$ is separable because $C_k^*(X_f^q)$ is separable [7, Corollary 4.2.2].

THEOREM 3.5. *Let $C_k^*(X)$ be linearly homeomorphic to $C_k^*(Y)$. Then X is pseudocompact if and only if Y is.*

Proof. Suppose X is pseudocompact and $\phi : C_k^*(X) \to C_k^*(Y)$ is a linear homeomorphism. To show that Y is pseudocompact it is enough to prove that $g(Y)$ is compact for every $g \in \mathcal{F}_Y$. Thus, let $g_1 \in \mathcal{F}_Y$ be fixed. Since, by Fact 3, $C_k^q(g_1) \subset C_k^*(Y)$ is closed and separable, Facts 1 and 2 imply that $\phi^{-1}(C_k^q(g_1)) \subset C_k^q(f_1)$ for some $f_1 \in \mathcal{F}_X$. Again, using Facts 1–3 we can find $g_2 \in \mathcal{F}_Y$ such that $\phi(C_k^q(f_1)) \subset C_k^q(g_2)$. We can assume $g_2 \succ g_1$, otherwise instead of g_2 we take the diagonal product of g_1 and g_2. Proceeding in this way two sequences $\{f_n\} \subset \mathcal{F}_X$, $\{g_n\} \subset \mathcal{F}_Y$ will be constructed such that:

(1) $f_{n+1} \succ f_n$ and $g_{n+1} \succ g_n$

(2) $\phi(C_k^q(f_n)) \subset C_k^q(g_{n+1})$ and $\phi^{-1}(C_k^q(g_n)) \subset C_k^q(f_n)$.

Let f (resp., g) be the diagonal product of all f_n (resp., g_n). It is easily seen that $\cup\{p_k^*(f,f_n)(C_k^*(f_n(X))) : n \in \mathbf{N}\}$ is dense in $C_k^*(X_f)$, where $p_k^*(f,f_n) : C_k^*(f_n(X)) \to C_k^*(X_f)$ is the continuous dual of the map $p(f,f_n) : X_f \to f_n(X)$. Because $C_k(f)$ is dense in $C_k^q(f)$, we have that $\cup\{C_k^q(f_n) : n \in \mathbf{N}\}$ is dense in $C_k^q(f)$ and, similarly, $\cup\{C_k^q(g_n) : n \in \mathbf{N}\}$ is dense in $C_k^q(g)$. Then, by (2), $\phi(C_k^q(f)) = C_k^q(g)$. Because ϕ, considered as a map between $C_u^*(X)$ and $C_u^*(Y)$ is a homeomorphism, we finally obtain that $C_u^q(f)$ and $C_u^q(g)$ are linearly homeomorphic. This yields that $C_u^*(X_f^q)$ and $C_u^*(Y_g^q)$ are also linearly homeomorphic (recall that $p_u^*(f)$ and $p_u^*(g)$ are homeomorphisms). However, X is pseudocompact which implies X_f^q is homeomorphic to X_f for all $f \in \mathcal{F}_X$ ([2], see also [12] for regular spaces). Hence, $C_u^*(X_f)$ is homeomorphic to $C_u^*(Y_g^q)$. So, $C_u^*(Y_g^q)$ is separable (recall that X_f is compact, hence $C_u^*(X_f)$ s separable). Therefore, Y_g^q is compact [11, Proposition 7.6.2], so is Y_g . Then $g_1(Y)$ is also compact because $g \succ g_1$.

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UNIVERSITY OF SWAZILAND, PRIVATE BAG 4, KWALUSENI, SWAZILAND
E-mail: valov@realnet.co.sz

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