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Function spaces

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Abstract

For a completely regular space X and a normed space E let $C_k(X,E)$ (respectively $C_p(X,E)$) be the set of all E-valued continuous maps on X endowed with the compact-open (respectively pointwise convergence) topology. We prove that some topological properties $\mathcal{P}$ satisfy the following conditions: (1) if $C_k(X,E)$ and $C_k(Y,F)$ (respectively $C_p(X,E)$ and $C_p(Y,F)$) are linearly homeomorphic, then $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$; (2) if there is a continuous linear surjection from $C_k(X,E)$ onto $C_p(Y,F)$, then $Y \in \mathcal{P}$ provided $X \in \mathcal{P}$; (3) if there is a continuous linear injection from $C_k(X,E)$ into $C_p(Y,F)$, then X has a dense subset with the property $\mathcal{P}$ provided Y has a dense subset with the same property. © 1997 Elsevier Science B.V.

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1. Introduction

Throughout this paper E and F are always normed linear spaces; a topological space means a completely regular space; $C_k(X,E)$ (respectively $C_p(X,E)$) denotes the set of all E-valued continuous maps on X with the compact-open (respectively pointwise convergence) topology; when E is the real line $\mathbb{R}$ we simply write $C_k(X)$ or $C_p(X)$; a subset K of X is bounded in X if $f(K)$ is a bounded subset of $\mathbb{R}$ for every $f \in C(X)$. The phrase "*a property $\mathcal{P}$ is preserved by linear continuous map from $C_k(X,E)$ onto $C_p(Y,F)$*" means that if there exists a continuous linear map from $C_k(X,E)$ onto $C_p(Y,F)$ and X has property $\mathcal{P}$, then Y has the same property.

Many results treating the following general question were proved during the last decade (see [2,4]): let $C_p(X)$ and $C_p(Y)$ be linearly homeomorphic (in such a case X and Y

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are called *l_p-equivalent*). Which topological properties $\mathcal{P}$ satisfy: X has property $\mathcal{P}$ if and only if Y has property $\mathcal{P}$? The same question is considered in [6] for *l_k-equivalent spaces* (i.e., when $C_k(X)$ and $C_k(Y)$ are linearly homeomorphic).

The purpose of this paper is to give further generalizations to some of the above group of results, as well as to present new ones in the more general situation when E-valued maps are considered instead of real-valued functions. More precisely, the paper is organized as follows. Section 2 is devoted to some preliminary facts about F-valued linear continuous maps on $C_k(X,E)$ and their supports in X.

Section 3 contains the basic technical results. The most helpful one is Proposition 3.14 stating that if u is a continuous linear map from $C_k(X,E)$ into $C_p(Y,F)$, where X is a μ-space and Y is a wq-space, then there is an u.s.c.o. map $\psi : Y \to 2^X$ such that $\operatorname{supp}(y) \subset \psi(y)$ for every $y \in Y$. Here $\operatorname{supp}(y)$ stands for the support of the linear map $\mu_y : C(X,E) \to F$ defined by $\mu_y(f) = u(f)(y)$ (see Section 2 for the definition of $\operatorname{supp}(\mu)$, μ is a linear map from $C(X,E)$ into F). Recall that X is said to be a μ-space (or μ-complete) if every closed and bounded subset of X is compact. The notion of a wq-space is related to that one of a q-space [16]. We say that X is a wq-space if for every point $x \in X$ there is a countable family $\{U_n\colon n \in \mathbb{N}\}$ of neighborhoods of x in X such that whenever $x_n \in U_n$ for each $n \in \mathbb{N}$, then $\{x_n\} \subset X$ is bounded. The class of wq-spaces contains all pseudocompact and all q-spaces. Section 3 contains also generalizations of some results of Baars [5].

In the last section (Section 4) we consider properties which are preserved by linear continuous surjections from $C_k(X,E)$ onto $C_p(Y,F)$ or linear homeomorphisms between $C_k(X,E)$ and $C_k(Y,F)$, as well as between $C_p(X,E)$ and $C_p(Y,F)$. This group of properties includes (under some restrictions on X and Y) compactness, Čech completeness, locally compactness, analyticity, K-analyticity, Lindelöf Σ-space property and τ-compactness. It was known that most of them are preserved by l_p-equivalence or l_k-equivalence. Let us explicitly mention Corollary 4.16 which is new even in the case of real-valued function spaces: in the class of μ-complete wq-spaces Lindelöf degree is preserved by linear homeomorphisms between $C_k(X,E)$ and $C_k(Y,F)$, as well as between $C_p(X,E)$ and $C_p(Y,F)$. It should be compared with the following result of Velicko [2]: Lindelöfness is preserved by l_p-equivalence. One of the starting points of the present paper is the following result of Baars, de Groot and Pelant [7]: Čech completeness is preserved by continuous linear surjections from $C_p(X)$ onto $C_p(Y)$ in the class of metric spaces. Proposition 4.3 generalizes this result and is applied to give a partial answer to a question of Arkhangel'skiĭ [4, Problem 20]. Another application of the main results is Proposition 4.20 answering a question of Baars and de Groot [6, Question 1]. In the case when there is a continuous linear injection from $C_k(X,E)$ into $C_p(Y,F)$ assertions of the following type are proved: if Y contains a dense subset having property $\mathcal{P}$, then X has a dense subset with the same property.

Finally, let us briefly discuss the possibility to extend the results presented here to function spaces $C(X,E)$ with E a more general linear space. Probably something can be proved but it can not be expected so much. For example, $C_p(X,\mathbb{R})$ and $C_p(X\times\mathbb{N},\mathbb{R}^\omega)$ are linearly homeomorphic for every space X [3], but X is compact does not imply $X\times\mathbb{N}$

is compact or pseudocompact. The main reason for this gap is the following fact: if E and F are arbitrary locally convex linear spaces and $\mu : C_k(X, E) \to F$ is a continuous linear map, then supp(μ) is not compact, in general; it contains a dense subset which is a union of τ many compact sets, where $\tau = \min\{\text{card}(\mathcal{A})$: $\mathcal{A}$ is a family of seminorms on F generating the topology of $F\}$.

2. Preliminaries

We consider mainly three topologies on $C(X, E)$: the pointwise convergence topology, the compact-open topology and the topology of uniform convergence on all bounded subsets of X, and the corresponding notations are $C_p(X, E)$, $C_k(X, E)$ and $C_b(X, E)$. Sometimes, they are called p-, k- or b -topology, respectively. The neighborhood base at a given $f \in C(X, E)$ consists of all sets of the form

$$A(f, K, \varepsilon) = \{g \in C(X, E)\colon \ \|g(x) - f(x)\| < \varepsilon \text{ for all } x \in K\},$$

where $\varepsilon > 0$ and K is, finite, compact or bounded in X. An easy check shows that $C_p(X, E)$, $C_k(X, E)$ and $C_b(X, E)$ are locally convex linear topological spaces. Note that if K is finite (respectively compact), then $A(f, K, \varepsilon)$ are open in $C_p(X, E)$ (respectively $C_k(X, E)$). If K is bounded in X the sets $A(f, K, \varepsilon)$ are not open in $C_b(X, E)$, in general, but f belongs to the b-interior of $A(f, K, \varepsilon)$. Since the closure of any bounded set in X is also bounded, in the definition of b-topology we can consider only the sets $A(f, K, \varepsilon)$ with $K \subset X$ closed and bounded. If X is a μ-space, then obviously $C_k(X, E) = C_b(X, E)$. In particular, $C_k(\nu_E X, E) = C_b(\nu_E X, E)$, where $\nu_E X$ is the E-completion of X (the biggest subset Z of the Hewitt realcompactification νX of X such that every $f \in C(X, E)$ can be continuously extended over Z) because $\nu_E X$ is a μ-space as a closed set in $E^{C(X,E)}$. Observe that the b-topology is stronger than the k-topology and the last one is stronger than the p-topology. So we have the following diagram

$$C_k(\nu_E X, E) = C_b(\nu_E X, E) \to C_b(X, E) \to C_k(X, E) \to C_p(X, E),$$

where the first map $\pi_X : C_b(\nu_E X, E) \to C_b(X, E)$ is the restriction $f \to f|X$, $f \in C(\nu_E X, E)$, and next two $i_X : C_b(X, E) \to C_k(X, E)$ and $j_X : C_k(X, E) \to C_p(X, E)$ are the identity maps. Let F be a normed space (in general different from E). Then $M_b(X, E, F)$ (respectively $M_k(X, E, F)$, $M_p(X, E, F)$) stands for the set of all continuous linear maps from $C_b(X, E)$ (respectively $C_k(X, E)$, $C_p(X, E)$) into F with the pointwise topology. It is easily seen that the above diagram generates the following one

$$\begin{aligned} M_p(X, E, F) &\to M_k(X, E, F) \to M_b(X, E, F) \\ &\to M_b(\nu_E X, E, F) = M_k(\nu_E X, E, F), \end{aligned}$$

where all bonding maps j_X^*, i_X^* and π_X^* are the linear duals of j_X, i_X and π_X, respectively. Since j_X^*, i_X^* and π_X^* are linear embeddings, we consider all of them as inclusions. We also consider the space $M(X, E, F)$ of all linear maps (not necessarily continuous) from $C(X,E)$ into F with the pointwise convergence topology. In the sequel, E and F are fixed normed spaces.

Following Arkhangel'skiĭ [1], for any $\mu \in M(X,E,F)$ the support of μ in X is the set supp(μ) of all points $x \in X$ satisfying the condition that for every neighborhood U of x in X here is $f \in C(X,E)$ such that $f(X-U)=0$ and $\mu(f) \neq 0$. We also consider the family $\mathcal{S}(\mu)$ of all subsets B of X such that for every $f,g \in C(X,E)$ with $f|B = g|B$ we have $\mu(f)=\mu(g)$. Observe that supp(μ) is closed in X and it is contained in $\mathrm{cl}_X(B)$ for every $B \in \mathcal{S}(\mu)$.

Proposition 2.1. *If* $\mu \in M(X,E,F)$, *then* $\operatorname{supp}(\mu) = X \cap \operatorname{supp}(\pi_X^*(\mu))$.

Proof. First, let us check that $\operatorname{supp}(\mu) \subset X \cap \operatorname{supp}(\pi_X^*(\mu))$. Suppose there is

$$x \in \operatorname{supp}(\mu) - \operatorname{supp}\bigl(\pi_X^*(\mu)\bigr),$$

and let $U \subset \nu_E X$ be a neighborhood of x such that $\pi_E^*(\mu)(g) = 0$ for every $g \in C(\nu_E X, E)$ with $g(\nu_E X - U) = 0$. Since $x \in \operatorname{supp}(\mu)$, there exist a neighborhood $V \subset X$ of x and $f \in C(X,E)$ such that $\mathrm{cl}_{\nu_E X}(V) \subset U$, $f(X-V)=0$ and $\mu(f) \neq 0$. Let $h \in C(\nu_E X, E)$ be the continuous extension of f. Obviously $h(\nu_E X - U) = 0$, so $\pi_X^*(\mu)(h) = 0$. On the other hand $\pi_X^*(\mu)(h) = \mu(f) \neq 0$, which gives a contradiction.

Next, let us show that $X \cap \operatorname{supp}(\pi_X^*(\mu)) \subset \operatorname{supp}(\mu)$. Suppose not. Then there is

$$x \in \bigl(\operatorname{supp}(\pi_X^*(\mu)) \cap X\bigr) - \operatorname{supp}(\mu)$$

and an open neighborhood $V \subset X$ of x such that $\mu(f) = 0$ for each $f \in C(X,E)$ with $f(X-V) = 0$. Let $W \subset \nu_E X$ be an open extension of V. Since $x \in \operatorname{supp}(\pi_X^*(\mu))$, there exists $g \in C(\nu_E X, E)$ such that $g(\nu_E X - W) = 0$ and $\pi_X^*(\mu)(g) \neq 0$. But $g(X-V)=0$, so $\mu(g|X) = 0$. This contradicts $\pi_X^*(\mu)(g) = \mu(g|X)$. □

Proposition 2.2. *Let* μ *be a nonzero element of* $M_k(X,E,F)$. *Then*:

(i) supp(μ) *is a nonempty compact element of* $\mathcal{S}(\mu)$;

(ii) *there is a positive constant* $N(\mu)$ *such that for every* $f \in C(X,E)$ *we have*

$$\|\mu(f)\| \leqslant N(\mu) \cdot \sup\bigl\{\|f(x)\|\colon\ x \in \operatorname{supp}(\mu)\bigr\};$$

(iii) *if, in addition* $\mu \in M_p(X,E,F)$, *then* supp(μ) *is finite.*

Proof. (i) Since μ is continuous at 0, there exist a nonempty compact set $K \subset X$ and $\varepsilon > 0$ such that $\|\mu(f)\| < 1$ for every $f \in A(0,K,\varepsilon)$. We claim that $K \in \mathcal{S}(\mu)$. Let $g \in C(X,E)$ and $g(K) = 0$. Then $\lambda \cdot g \in A(0,K,\varepsilon)$ for each $\lambda \in \mathbb{R}$, which implies $\|\mu(\lambda \cdot g)\| < 1$. Therefore $\mu(g) = 0$, i.e., $K \in \mathcal{S}(\mu)$. Consequently, supp(μ) is compact as a closed subset of K.

Claim 1. $W \in \mathcal{S}(\mu)$ *for any neighborhood* W *of* supp(μ) *in* X.

Proof. This is the case if $K \subset \operatorname{Int}(W)$. Let $P = K - \operatorname{Int}(W) \neq \emptyset$. Since $P \cap \operatorname{supp}(\mu) = \emptyset$, for every $x \in P$ there is a neighborhood $U_x \subset X$ of x such that $\mu(f) = 0$ for each

$f \in C(X, E)$ with $f(X - U_x) = 0$. Since P is compact, we can find $x(i) \in P$ and $h_i \in C^*(X)$, $i = 1, 2, \dots, n$, satisfying the following conditions:

$$P \subset \bigcup \{U_{x(i)}\colon\ i = 1, 2, \dots, n\}, \qquad h_i(X - U_{x(i)}) = 0 \quad \text{and}$$
$$h(x) = \sum \{h_i(x)\colon\ i = 1, 2, \dots, n\} = 1$$

for any $x \in P$. Now, let $g \in C(X, E)$ and $g(W) = 0$. Then $g|K = (g \cdot h)|K$, so $\mu(g) = \mu(g \cdot h)$ because $K \in \mathcal{S}(\mu)$. But $\mu(g \cdot h) = \sum\{\mu(g \cdot h_i)\colon\ i = 1, 2, \dots, n\}$ and each $g \cdot h_i$ is 0 outside $U_{x(i)}$. Therefore $\mu(g) = 0$. The claim is proved. □

It follows from Claim 1 that supp$(\mu) \neq \emptyset$ (otherwise $\emptyset \in \mathcal{S}(\mu)$, which implies $\mu = 0$). Now, let show that supp$(\mu) \in \mathcal{S}(\mu)$. We need the following result [19, Theorem 1.1]: if ζ is a continuous function on $C_k(X)$, then $\bigcap\{B\colon\ B \in \mathcal{H}(\zeta)\} \in \mathcal{H}(\zeta)$, where $\mathcal{H}(\zeta)$ is the family of all closed sets $B \subset X$ such that $\zeta(f) = \zeta(g)$ provided $f|B = g|B$ and $f, g \in C(X)$. The same proof works in case ζ is a continuous map from $C_k(X, E)$ into F, E and F normed spaces. So, in our case, $\bigcap\{\mathrm{cl}_X(B)\colon\ B \in \mathcal{S}(\mu)\} \in \mathcal{S}(\mu)$ and, by Claim 1, we have $\bigcap\{\mathrm{cl}_X(B)\colon\ B \in \mathcal{S}(\mu)\} =$ supp(μ). Consequently, supp$(\mu) \in \mathcal{S}(\mu)$.

(ii) Let $\pi : C_k(X, E) \to C_k(\mathrm{supp}(\mu), E)$ be the restriction map.

Claim 2. *π is a continuous open surjection.*

Proof. It is clear that π is continuous. To show that π is open, one can follow the proof of [19, Lemma 3]. Since every normed space is an absolute extensor for compact spaces [13], π is surjective. The claim is proved. □

Now, let us go back to the proof of (ii). Since π is an open, continuous surjection, and $\pi(f) = \pi(g)$ implies $\mu(f) = \mu(g)$, there is a continuous linear map

$$q : C_k(\mathrm{supp}(\mu), E) \to F$$

such that $q \circ \pi = \mu$. Because both $C_k(\mathrm{supp}(\mu), E)$ and F are normed, q is a bounded map. Then the inequality

$$\|\mu(f)\| \leqslant N(\mu) \cdot \sup\{\|f(x)\|\colon\ x \in \mathrm{supp}(\mu)\}$$

holds with $N(\mu) = \|q\|$.

(iii) As in the proof of (i), there exist finite $K \subset X$ and $\varepsilon > 0$ such that $\|\mu(f)\| < 1$ for each $f \in A(0, A, \varepsilon)$. Then $K \in \mathcal{S}(\mu)$, which implies supp$(\mu) \subset K$. □

Proposition 2.3. *A linear map $\mu \in M(X, E, F)$ belongs to $M_b(X, E, F)$ if and only if $\pi_X^*(\mu) \in M_b(\nu_E X, E, F)$ and there is bounded, closed $K \subset X$ such that* supp$(\pi_X^*(\mu)) \subset \mathrm{cl}_{\nu_E X}(K)$.

Proof. Suppose $\mu \in M_b(X, E, F)$. Clearly $\pi_X^*(\mu) \in M_b(\nu_E X, E, F)$. Since μ is b-continuous, there are a bounded, closed $K \subset X$ and $\varepsilon > 0$ such that $\|\mu(f)\| < 1$ for any $f \in A(0, K, \varepsilon)$. Next, $K \in \mathcal{S}(\mu)$ because μ is linear. This yields $\mathrm{cl}_{\nu_E X}(K) \in$

$\mathcal{S}(\pi_X^*(\mu))$, so $\operatorname{supp}(\pi_X^*(\mu)) \subset \operatorname{cl}_{\nu_E X}(K)$. Suppose now that $\pi_X^*(\mu) \in M_b(\nu_E X, E, F)$ and $\operatorname{supp}(\pi_X^*(\mu)) \subset \operatorname{cl}_{\nu_E X}(K)$ for some closed, bounded $K \subset X$. Since $\pi_X^*(\mu)$ is k-continuous on $C(\nu_E X, E)$, by Proposition 2.2(ii), there is $\varepsilon > 0$ such that $\|\pi_X^*(\mu)(g)\| < 1$ for every $g \in A(0, \operatorname{cl}_{\nu_E X}(K), \varepsilon)$. Hence $\|\mu(f)\| < 1$ for all $f \in A(0, K, \varepsilon/2)$, so μ is continuous on $C_b(X, E)$. □

Let $M_{\mathrm{ef}}(X, E, F)$ be the class of all effective linear maps from $C(X, E)$ into F. Following [1], $\mu \in M(X, E, F)$ is said to be effective if $\operatorname{supp}(\pi)$ is bounded in X and $U \in \mathcal{S}(\mu)$ for every neighborhood U of $\operatorname{supp}(\mu)$ in X. When $M_{\mathrm{ef}}(X, E, F)$ is considered as a topological space it is endowed with the pointwise topology. It is easy to see that for each $\mu \in M_{\mathrm{ef}}(X, E, F)$ the following assertions are true:

Fact 2.4. $\operatorname{supp}(\mu) = \emptyset$ *if and only if* $\mu = 0$.

Fact 2.5. $\operatorname{cl}_{\nu_E X}(\operatorname{supp}(\mu)) = \operatorname{supp}(\pi_X^*(\mu))$ *and it is compact.*

Fact 2.6. $\pi_X^*(\mu) \in M_{\mathrm{ef}}(\nu_E X, E, F)$.

Fact 2.7. $\mu \in M(X, E, F)$ *is effective if* $\mathcal{S}(\mu)$ *contains a compact element.*

Fact 2.7 follows from the proof of Claim 1, Proposition 2.2(i).

Summarizing the above facts and Proposition 2.3 we obtain the following

Proposition 2.8. *For any space* X *we have*:

$$M_k(X, E, F) \subset M_{\mathrm{ef}}(X, E, F) \cap M_b(X, E, F) \quad \textit{and}$$
$$\pi_X^*\bigl(M_b(X, E, F) \cup M_{\mathrm{ef}}(X, E, F)\bigr) \subset M_{\mathrm{ef}}(\nu_E X, E, F).$$

Recall that a set-valued map $\Phi : X \to P(Y)$ is lower semicontinuous (abbreviated LSC) whenever for every open $U \subset Y$ the set $\{x \in X\colon \Phi(x) \cap U \neq \emptyset\}$ is open in X. Note that it is possible $\Phi(x) = \emptyset$ for some $x \in X$.

Proposition 2.9. *For any space* X $\operatorname{supp} : M_{\mathrm{ef}}(X, E, F) \to P(X)$ *is LSC.*

Proof. The proof is similar to that of [6, Lemma 1.2.7]. □

Corollary 2.10. *For any space* X $\operatorname{supp} : M_k(X, E, F) \to P(X)$ *is LSC.*

Proof. Follows from Proposition 2.9 because $M_k(X, E, F) \subset M_{\mathrm{ef}}(X, E, F)$. □

Remark 2.11. We can show that $\operatorname{supp} : M_b(X, E, F) \to P(X)$ is LSC if and only if X is open in $\mu_1(X) = \bigcup\{\operatorname{supp}(\pi_X^*(\mu))\colon \mu \in M_b(X, E, F)\}$. By Proposition 2.3, $\mu_1(X)$ consists of all points $x \in \nu_E X$ such that $x \in \operatorname{cl}_{\nu_E X}(B)$ for some closed, bounded $B \subset X$. We can also show that the set $\mu(X) = \bigcup\{X_i\colon i \in \mathbb{N}\}$, where $X_1 = \mu_1(X)$ and $X_{i+1} = \mu_1(X_i)$, is the μ-completion of X (the smallest μ-space in νX containing X), and $\mu(X)$ has the following property: any continuous map from X into a μ-space Y can be continuously extended to a map from $\mu(X)$ into Y.

A set $K \subset M(X, E, F)$ is called w-bounded if $\{\mu(f)\colon \mu \in K\}$ is norm bounded in F for every $f \in C(X, E)$. For any $K \subset M(X, E, F)$ $\mathrm{supp}(K)$ stands for the set $\mathrm{cl}_X(\bigcup\{\mathrm{supp}(\mu)\colon \mu \in K\})$.

Suppose $u\colon C(X, E) \to F^Y$ is a linear map, where Y is a topological space. For every $y \in Y$ consider the projection π_y from F^Y onto F determined by y, and let $\mu_y = \pi_y \circ u$. We obtain a map $u^*\colon Y \to M(X, E, F)$, defined by $u^*(y) = \mu_y$. For any $K \subset Y$ we denote $\mathrm{supp}(u^*(K))$ again by $\mathrm{supp}(K)$. If $u^*(Y) \subset M_{\mathrm{ef}}(X, E, F)$, then u is said to be effective. Observe that the following simple assertions are true:

Fact 2.12. *For any $K \subset Y$ the set $u^*(K)$ is w-bounded if and only if $u(f)(K)$ is norm bounded in F for every $f \in C(X, E)$.*

Fact 2.13. *u^* is continuous if and only if $u(C(X, E)) \subset C(Y, F)$; u^* is a homeomorphism if and only if $u(C(X, E)) \subset C(Y, F)$ and $u(C(X, E))$ determines the topology of Y.*

Fact 2.14. *If u is b-continuous (respectively k-continuous), where F^Y is endowed with the product topology, then u^* is a map from Y into $M_b(X, E, F)$ (respectively into $M_k(X, E, F)$).*

3. Main constructions and basic results

The following proposition is an analog of [1, Proposition 2] (see also [6, Proposition 1.2.8]).

Proposition 3.1. *Let $K \subset M_{\mathrm{ef}}(X, E, F)$ be w-bounded. Then* $\mathrm{supp}(K)$ *is bounded in X.*

Proof. We follow very closely the proof of [6, Proposition 1.2.8]. Suppose $\mathrm{supp}(K) \subset X$ is not bounded. Then there exist $f \in C(X)$ and $x_n \in \bigcup\{\mathrm{supp}(\mu)\colon \mu \in K\}$, $n \in \mathbb{N}$, such that $\{f(x_n)\colon n \in \mathbb{N}\}$ is discrete and unbounded in $\mathbb{R}$. Embedding $\mathbb{R}$ in E we can assume that $f \in C(X, E)$. So, we can find an open family $\{V_n\colon n \in \mathbb{N}\}$ in X such that $x_n \in V_n$, $\{f(V_n)\}$ is a discrete family in E and $\sup\{\|f(y_n)\|\colon n \in \mathbb{N}\} = \infty$ for any sequence $\{y_n\}$ with $y_n \in V_n$. By induction we shall construct a sequence $\{\mu_k\colon k \in \mathbb{N}\} \subset K$, a subfamily $\{U_k\colon k \in \mathbb{N}\}$ of $\{V_n\colon n \in \mathbb{N}\}$ and a set $\{h_k\colon k \in \mathbb{N}\} \subset C(X, E)$ such that:

(1) $h_k(X - U) = 0$ for every k;
(2) $U_i \neq U_j$ for $i \neq j$;
(3) $\mathrm{supp}\{\mu_1, \ldots, \mu_{k-1}\} \cap \mathrm{cl}_X(U_k) = \emptyset$ for every $k \geqslant 2$;
(4) $\|\mu_k(h_k)\| = k + \|p_k\|$ for every k, where $p_k = \sum\{\mu_k(h_i)\colon i < k\}$ for $k \geqslant 2$ and $p_1 = 0$.

Let $\mu_1 \in K$ be such that $x_1 \in \operatorname{supp}(\mu)$ and $U_1 = V_1$. Pick a $h \in C(X,E)$ with $h(X-U_1) = 0$ and $\mu_1(h) \neq 0$. Let $\lambda_1 = 1/\|\mu_1(h)\|$ and $h_1 = \lambda_1 \cdot h$. Then $h_1(X-U_1) = 0$ and $\|\mu_1(h_1)\| = 1$.

Let $k \geqslant 2$ and suppose we have found $\mu_1, \ldots, \mu_{k-1}$, $U_1, \ldots, U_{k-1}$, and $h_1, \ldots, h_{k-1}$ satisfying (1)–(4). Set $H_k = \operatorname{supp}\{\mu_1, \ldots, \mu_{k-1}\}$. There exists $n \in \mathbb{N}$ such that $\operatorname{cl}_X(V_n) \cap H_k = \emptyset$ because $H_k \subset X$ is bounded. Take $\mu_k \in K$ with $x_n \in \operatorname{supp}(\mu_k)$, and let $U_k = V_n$. Since U_k is a neighborhood of x_n, there is $h \in C(X,E)$ such that $h(X - U_k) = 0$ and $\mu_k(h) \neq 0$. Put $\lambda_k = k + \|p_k\|/\|\mu_k(h)\|$ and $h_k = \lambda_k \cdot h$. Then $h_k(X - U_k) = 0$ and $\|\mu_k(h_k)\| = k + \|p_k\|$. Observe that (3) and $x_n \in \operatorname{supp}(\mu_k) \cap U_k$ imply $U_i \neq U_j$ for $i \neq j$, $i, j \in \{1, 2, \ldots, k\}$. That completes the inductive construction.

Let $g = \sum\{h_k\colon k \in \mathbb{N}\}$. Following the proof of [6, Proposition 1.2.8], we can check that $g \in C(X,E)$, and $\|\mu_k(g)\| \geqslant k$ for every k. This contradicts the fact that K is w-bounded. □

Corollary 3.2. *Let $K \subset M_b(X,E,F)$ be w-bounded. Then* $\operatorname{supp}(K) \subset X$ *is bounded.*

Proof. Clearly $\pi_X^*(K) \subset M_{\mathrm{ef}}(\nu_E X, E, F)$ is also w-bounded. Hence, by Proposition 3.1, $\operatorname{supp}(\pi_X^*(K)) \subset \nu_E X$ is bounded. Now, by Proposition 2.1,

$$\operatorname{supp}(K) \subset X \cap \operatorname{supp}\bigl(\pi_X^*(K)\bigr).$$

Therefore $\operatorname{supp}(K) \subset X$ is bounded. □

Corollary 3.3. *Let $u\colon C(X,E) \to F^Y$ be either effective or b-continuous (with respect to the product topology on F^Y), and $K \subset Y$ such that $u(f)$ is (norm) bounded on K for all $f \in C(X,E)$. Then* $\operatorname{supp}(K) \subset X$ *is bounded.*

Proof. Follows from Facts 2.12 and 2.14, Proposition 3.1 and Corollary 3.2. □

Now we can give an alternative proof, and at the same time a generalization, of [1, Theorem 1] (the case $E = F = \mathbb{R}$ is considered in [1]).

Proposition 3.4. *Let $u\colon C_b(X,E) \to C_p(Y,F)$ be a continuous, effective linear map. If E and F are Banach spaces, then u can be lifted to a continuous map from $C_b(X,E)$ into $C_b(Y,F)$.*

Proof. Let $A(0,K,\varepsilon)$ be a neighborhood of 0 in $C_b(Y,F)$. By Facts 2.12 and 2.14, $u^*(K)$ is a w-bounded subset of $M_{\mathrm{ef}}(X,E,F) \cap M_b(X,E,F)$. Then $\pi_X^*(u^*(K)) \subset M_k(\nu_E X, E, F)$ is also w-bounded. So, by Proposition 3.1, $H = \operatorname{supp}(\pi_X^*(u^*(K))) \subset \nu_E X$ is closed and bounded; hence H is compact. Let $\pi_1 : C_k(\nu_E X, E) \to C_k(H,E)$, $\pi_2\colon C_p(Y,F) \to C_p(K,F)$ be the restriction maps, and let $\phi = \pi_2 \circ u \circ \pi_X$. Observe that, by Proposition 2.2, $\operatorname{supp}(\pi_X^*(u^*(y))) \in \mathcal{S}(\pi_X^*(u^*(y)))$ for every $y \in Y$. Consequently, for every $f, g \in C(\nu_E X, E)$, $\pi_1(f) = \pi_1(g)$ implies $\phi(f) = \phi(g)$. But π_1 is a continuous, open surjection (Claim 2, Proposition 2.2), so there is a continuous, linear map $q\colon C_k(H,E) \to C_p(K,F)$ with $q \circ \pi_1 = \phi$. By the Closed Graph Theorem, q can be lifted to a continuous map from $C_k(H,E)$ into $C_b(K,F)$; hence there exists $\eta > 0$ such that, for any $f \in C(\nu_E X, E)$ with $f \in A(0,H,\eta)$, $u(\pi_X(f)) \in A(0,K,\varepsilon)$. Now let $B = H \cap X$. Obviously, $B \subset X$ is closed and bounded. Because u is effective, by Fact 2.5, $\mathrm{cl}_{\nu_E X}(B) = H$. So, $u(g) \in A(0,K,\varepsilon)$ for every $g \in A(0,B,\eta/2)$. Therefore u, considered as a map from $C_b(X,E)$ into $C_b(Y,F)$, is also continuous. □

Since every continuous linear map from $C_k(X,E)$ into $C_p(Y,F)$ is effective, by Proposition 3.4, we have the following:

Corollary 3.5. *Every topological property which is preserved by continuous linear surjections from $C_b(X,E)$ onto $C_b(Y,F)$, where E and F are Banach spaces, is preserved also by continuous linear surjections from $C_k(X,E)$ (respectively $C_p(X,E)$) onto $C_p(Y,F)$.*

Proposition 3.6. *Let $u: C_k(X,E) \to C_p(Y,F)$ be a continuous linear map, where E and F are Banach spaces. Then u can be lifted to a continuous map from $C_k(X,E)$ into $C_k(Y,F)$ if and only if* supp(K) *is compact for any compact set $K \subset Y$.*

Proof. Suppose $K \subset Y$ is compact. One can show, using the proof of [6, Lemma 1.5.6], that supp$(K) \subset X$ is compact.

Now, suppose supp$(K) \subset X$ is compact for any compact $K \subset Y$. Let $A(0,K,\varepsilon)$ be a neighborhood of 0 in $C_k(Y,F)$. Then, by our assumption, $H = \mathrm{supp}(K) \subset X$ is compact. Repeating the proof of Proposition 3.4, with $C_b(X,E)$ replaced by $C_k(X,E)$, we obtain in this concrete situation that $A(0,H,\eta) \subset u^{-1}(A(0,K,\varepsilon))$ for some $\eta > 0$. Hence u is a continuous map from $C_k(X,E)$ into $C_k(Y,F)$. □

For any X there are natural embeddings of X in $M_{\mathrm{ef}}(X,E,E)$, $M_k(X,E,E)$ and $M_b(X,E,E)$, all defined by $x \to \delta_x$ with $\delta_x(f) = f(x)$ for $x \in X$ and $f \in C(X,E)$. It is easily seen that $\{\delta_x\colon x \in X\}$ is closed in $M_{\mathrm{ef}}(X,E,E)$. Indeed, if $\{\delta_{x(\alpha)}\colon \alpha \in \Lambda\}$ is a net in $M_{\mathrm{ef}}(X,E,E)$ converging to $\mu \in M_{\mathrm{ef}}(X,E,F)$, then $\{f(x(\alpha))\colon \alpha \in \Lambda\}$ converges to $\mu(f)$ for every $f \in C(X,E)$. This implies that $\mu \neq 0$, so by Fact 2.4, supp$(\mu) \neq \emptyset$. Because supp$(\delta_{x(\alpha)}) = x(\alpha)$, by Proposition 2.9, supp(μ) consists of only one point $x \in X$ and $\{x(\alpha)\colon \alpha \in \Lambda\}$ converges to x. Hence, $\mu = \delta_x$. The same proof shows that $\{\delta_x\colon x \in X\}$ is also closed in $M_k(X,E,E)$ and in $M_p(X,E,E)$. However, in general, $\{\delta_x\colon x \in X\}$ is not closed in $M_b(X,E,E)$. If, for example, X is pseudocompact and not compact, then $\{\delta_x\colon x \in \beta X\} \subset M_b(X,\mathbb{R},\mathbb{R})$ is homeomorphic to βX.

Proposition 3.7. *The space $M_k(X,E,E)$ is μ-complete if and only if X is a μ-space.*

Proof. Since X can be embedded as a closed subset of $M_k(X,E,E)$, we need only to show that $M_k(X,E,E)$ is a μ-space provided X is. Suppose X is μ-complete and $K \subset M_k(X,E,E)$ is closed and bounded. Then K is w-bounded, and by Corollary 3.2, $H = \mathrm{supp}(K) \subset X$ is closed and bounded. Hence H is compact. Since the restriction map π from $C_k(X,E)$ into $C_k(H,E)$ is an open surjection, its dual $\pi^*: M(H,E,E) \to M(X,E,E)$ is a closed linear embedding such that $K \subset \pi^*(M_k(H,E,E))$ and $\pi^*(M_k(H,E,E)) \subset M_k(X,E,E)$ is closed. Observe also that $\pi^*(M(H,E,E)$ is a closed linear subspace of $M(X,E,E)$ and it consists of all $\mu \in M(X,E,E)$ satisfying the following property: if $f_1, f_2 \in C(X,E)$ and $\pi(f_1) = \pi(f_2)$, then $\mu(f_1) = \mu(f_2)$.

Consider the G_δ-closure G of $M_k(X,E,E)$ in $E^{C(X,E)}$, i.e., the set of all $\mu \in E^{C(X,E)}$ such that any G_δ-subset of $E^{C(X,E)}$ containing μ meets $M_k(X,E,E)$. Then G is Dieudonné complete (homeomorphic to a closed subset of a product of metrizable spaces) ([9], see also [11]). Obviously, every closed and bounded subset of a Dieudonné complete space is compact. Since $K \subset G$ is bounded, to prove that K is compact, it suffices to show that $K \subset G$ is closed. To this end, let $\{\mu_\alpha\} \subset K$ be a net converging to $\mu \in G$. Then $\mu \in \pi^*(M(H,E,E))$, so $\mu = \pi^*(\nu)$ for some $\nu \in M(H,E,E)$. Observe that $\mu \in M_k(X,E,E)$ is equivalent to $\nu \in M_k(H,E,E)$ and implies $\mu \in K$. So, the proof is reduced to show that $\nu \in M_k(H,E,E)$. Towards this end, take $B \subset C_k(H,E)$ and g from the closure of B in $C_k(H,E)$. Because $C_k(H,E)$ is metrizable, there exists countable $A \subset B$ such that $g \in \mathrm{cl}_{C_k(X,E)}(A)$. □

Claim. *There are countable $D \subset C(X,E)$ and $f \in C(X,E)$ such that*

$$f \in \mathrm{cl}_{C_k(X,E)}(D), \qquad \pi(f) = g, \qquad \pi(D) = A.$$

Proof. Let $A_1 = A \cup \{g\}$ and $\beta X \subset I^T$ for some T, where $I = [0,1]$. For any $\Gamma \subset T$ let $p(\Gamma)$ be the natural projection from I^T onto I^Γ, $q(\Gamma) = p(\Gamma)|H$ and $H(\Gamma) = p(\Gamma)(H)$. Consider also the dual maps $p^*(\Gamma): C_k(I^\Gamma, E) \to C_k(I^T, E)$ and $q^*(\Gamma): C_k(H(\Gamma), E) \to C_k(H,E)$. Observe that both $p^*(\Gamma)$ and $q^*(\Gamma)$ are closed embeddings. Next step is to find countable sets $\Omega \subset T$ and $A_2 \subset C(H(\Omega), E)$ such that $q^*(\Omega)(A_2) = A_1$. That can be done as follows: extend each $h \in A_1$ to $h' \in C(I^T, E)$ (see Claim 2 from Proposition 2.2) and then use the fact that every continuous real-valued function on I^T can be factorized through I^M for some countable $M \subset T$ [15]. By Dugundji extension theorem [10], there is a continuous linear map $u: C_k(H(\Omega), E) \to C_k(I^\Omega, E)$ with $u(h)|H(\Omega) = h$ for each $h \in C(H(\Omega), E)$. Consider the continuous map $\xi = r \circ p^*(\Omega) \circ u \circ (q^*(\Omega))^{-1}$ from $q^*(\Omega)(C_k(H(\Omega), E))$ into $C_k(X,E)$, where r is the restriction map from $C_k(I^T, E)$ into $C_k(X,E)$. Then $f = \xi(g)$ and $D = \xi(A)$ satisfy the requirements of Claim. □

Let us, finally, finish the proof of Proposition 3.7. It remains only to show that $\nu(g) \in \mathrm{cl}_E(\nu(B))$. Since D is countable and μ belongs to the G_δ-closure of $M_k(X,E,E)$ in $E^{C(X,E)}$, there is $\mu_1 \in M_k(X,E,E)$ such that $\mu|D = \mu_1|D$ and $\mu(f) = \mu_1(f)$. Because μ_1 is continuous on $C_k(X,E)$ and $f \in \mathrm{cl}_{C_k(X,E)}(D)$, $\mu_1(f) \in \mathrm{cl}_E(\mu_1(D))$. Then $\mu(f) \in \mathrm{cl}_E(\mu(D))$. But $\mu(f) = \nu(g)$ and $\mu(h) = \nu(h|H)$ for each $h \in D$. Thus $\nu(g) \in \mathrm{cl}_E(\nu(A))$. Consequently, $\nu(g) \in \mathrm{cl}_E(\nu(B))$.

Next proposition is a generalization of the following result proved by Gul'ko and Okunev [12]: X is a μ-space if and only if $M_p(X, \mathbb{R}, \mathbb{R})$ is a μ-space. Our proof is different from theirs.

Proposition 3.8. *The space $M_p(X, E, E)$ is μ-complete if and only if X is a μ-space.*

Proof. Again, as in previous proposition, we need only to prove sufficiency. Suppose X is μ-complete and $K \subset M_p(X, E, E)$ is closed and bounded. Because $M_p(X, E, E) \subset M_k(X, E, E)$, K is also bounded in $M_k(X, E, E)$. According to Corollary 3.2 and our assumption on X, $H = \operatorname{supp}(K) \subset X$ is compact. We follow very closely the proof of Proposition 3.7. Let π be the restriction map from $C_p(X, E)$ onto $C_p(H, E)$. It is easily seen that π is open. Then $\pi^* : M(H, E, E) \to M(X, E, E)$ is a closed linear embedding such that $K \subset \pi_p^*(M_p(H, E, E))$ and $\pi_p^*(M_p(H, E, E))$ is closed in $M_p(X, E, E)$.

Consider the G_δ-closure G of $M_p(X, E, E)$ in $E^{C(X,E)}$. Then G is Dieudonné complete and it is enough to prove that $K \subset G$ is closed. Let $\{\mu_\alpha\} \subset K$ be a net converging to $\mu \in G$. Then there is $\nu \in M(H, E, E)$ with $\mu = \pi^*(\nu)$. To show that ν is continuous on $C_p(H, E)$ we choose $B \subset C_p(H, E)$ and g from the closure of B in $C_p(H, E)$. Now we need the following fact [3]: if, for every $n \in \mathbb{N}$, the Lindelöf degree of X^n is not greater than a cardinal number τ, then the tightness of $C_p(X)$ is not greater than τ. The same proof remains correct if $\mathbb{R}$ is replaced by any normed space. Thus, in our case, the tightness of $C_p(H, E)$ is countable; so there exists countable $A \subset B$ such that $g \in \operatorname{cl}_{C_p(H,E)}(A)$. □

Claim. *There are countable $D \subset C(X, E)$ and $f \in C(X, E)$ such that*

$$f \in \operatorname{cl}_{C_p(X,E)}(D), \qquad \pi(f) = g, \qquad \pi(D) = A.$$

Proof. The proof of Claim and continuity of ν is the same as in Proposition 3.7. We need only the following formal changes: $p^*(\Gamma)$ and $q^*(\Gamma)$ are, respectively the maps $p^*(\Gamma): C_p(I^\Gamma, E) \to C_p(I^T, E)$ and $q^*(\Gamma): C_p(H(\Gamma), E) \to C_p(H, E)$ for $\Gamma \subset T$, and $u: C_p(H(\Omega), E) \to C_p(I^\Omega, E)$ is a continuous linear extension operator (such u exists by Dugundji extension theorem [10]). □

Corollary 3.9. *Suppose X is a μ-space. Then the following conditions hold:*

(i) *both $M_k(X, E, F)$ and $M_p(X, E, F)$ are μ-spaces;*

(ii) *Y is a μ-space provided $C_k(Y, F)$ (respectively $C_p(Y, F)$) is a quotient space of $C_k(X, E)$ or $C_p(X, E)$.*

Proof. (i) By Proposition 3.7 and Proposition 3.8, it is enough to show that $M_k(X, E, F)$ (respectively $M_p(X, E, F)$) can be embedded as a closed subset of $M_k(X, E \times F, E \times F)$ (respectively $M_p(X, E \times F, E \times F)$). We shall consider only the case of k-topology, the other one is similar. Identifying E with $E \times \{0_F\}$ we can assume that $C_k(X, E)$ is a closed linear subspace of $C_k(X, E \times F)$. Let π be the projection from $E \times F$ onto $E \times \{0_F\}$ and define $r: C_k(X, E \times F) \to C_k(X, E)$ by $r(f) = \pi \circ f$. Then r is a continuous retraction, so its dual embeds $M_k(X, E, F)$ into $M_k(X, E \times F, F)$ as a closed linear subspace. Finally, identifying F and $\{0_E\} \times F$, we get a closed linear embedding of $M_k(X, E \times F, F)$ into $M_k(X, E \times F, E \times F)$.

(ii) Suppose there is a quotient continuous linear map u from $C_k(X,E)$ onto $C_k(Y,F)$. Then the map $u^*: M_k(Y,F,F) \to M_k(X,E,F)$ is a closed linear embedding. Hence, by (i), $M_k(Y,F,F)$ is a μ-space. According to Proposition 3.7, Y is also a μ-space. The other cases are similar. $\square$

Corollary 3.9(ii) becomes false if u is only assumed to be a continuous linear surjection, even if real-valued functions are considered. For example, let X be pseudocompact and noncompact. Then the restriction map from $C_k(\beta X)$ onto $C_k(X)$ is a continuous surjection, but X is not μ-complete.

Proposition 3.10 below is an analog of [6, Lemma 2.1] and we follow the same scheme of proof.

Proposition 3.10. *Let $u: C_b(X,E) \to F^Y$ be continuous, linear and effective with $C(Y,F) \subset u(C(X,E))$, and $K \subset X$ bounded. Then $H = \{y \in Y\colon \operatorname{supp}(y) \subset K\}$ is bounded in Y.*

Proof. First, let us show that each $\mu_y = u^*(y)$ is nontrivial. Indeed, for every $y \in Y$ there is $h \in C(Y,F)$ with $h(y) \neq 0$. Since $C(Y,F) \subset u(C(X,E))$, there exists $f \in C(X,E)$ such that $u(f) = h$. Then $\mu_y(f) = h(y)$, so $\mu_y \neq 0$. Hence, by Fact 2.4, $\operatorname{supp}(y) \neq \emptyset$.

By Fact 2.14, $\pi_X^*(\mu_y) \in M(\nu_E X, E, F)$, $y \in Y$. Therefore, by Proposition 2.2, to every $y \in Y$ corresponds an $N(y) > 0$ such that

$$\left\|\pi_X^*(\mu_y)(g)\right\| \leqslant N(y) \cdot \sup\left\{\left\|g(x)\right\|\colon\ x \in \operatorname{supp}(\pi_X^*(\mu_y))\right\}$$

for each $g \in C(\nu_E X, E)$. Because $\operatorname{supp}(\mu_y)$ is dense in $\operatorname{supp}(\pi_X^*(\mu))$ (Fact 2.5), the last inequality is equivalent to $\|\mu_y(f)\| \leqslant N(y)\cdot \sup\{\|f(x)\|\colon x \in \operatorname{supp}(\mu_y)\}$, $f \in C(X,E)$.

Suppose now that $H \subset Y$ is not bounded, so there is $\varphi \in C(Y)$ and a sequence $\{y_n\} \subset H$ such that $\{\varphi(y_n)\colon n \in \mathbb{N}\} \subset \mathbb{R}$ is discrete and unbounded. Choose $z \in F$ with $\|z\| = 1$ and $\psi \in C(\mathbb{R}, F)$ with $\psi(\varphi(y_n)) = n \cdot N(y_n)z$ for each n. Then $h = \psi \circ \varphi \in C(Y,F)$, hence there is $f \in C(X,E)$ with $u(f) = h$. Since $K \subset X$ is bounded, $p = \sup\{\|f(x)\|\colon x \in K\} < \infty$; thus we can take $m \geqslant p+1$. Clearly $\sup\{\|f(x)\|\colon x \in \operatorname{supp}(\mu_y)\} \leqslant p$ for every $y \in H$, so we have

$$\left\|u(f)(y_m)\right\| = \left\|\mu_{y_m}(f)\right\| \leqslant N(y_m)\cdot p \leqslant (m-1)\cdot N(y_m).$$

On the other hand

$$\left\|u(f)(y_m)\right\| = \left\|h(y_m)\right\| = \|z\| \cdot mN(y_m) = m \cdot N(y_m),$$

a contradiction. $\square$

Remark 3.11. Let u and K satisfy the hypotheses of Proposition 3.10. If, in addition, $u(C(X,E)) = C(Y,F)$ and $K \subset X$ is closed, then $H \subset Y$ is also closed. That follows from Fact 2.13 and Proposition 2.9 because in this case $\operatorname{supp} \circ u^* : Y \to P(X)$ is LSC.

Corollary 3.12. *Let $u: C_b(X,E) \to F^Y$ be a continuous, linear and effective map with $C(Y,F) \subset u(C(X,E))$. Then Y is pseudocompact whenever X is.*

Corollary 3.13. *Let* $u: C_p(X,E) \to F^Y$ *be a continuous linear map such that*

$$C(Y,F) \subset u\big(C(X,E)\big).$$

If X *is* σ*-compact and* Y *is a* μ*-space, then* Y *is* σ*-compact.*

The above corollaries extend [6, Corollaries 2.2 and 2.3].

A point $x \in X$ is said to be a wq-point in X if there is a countable family $\{U_n\colon n \in \mathbb{N}\}$ of neighborhoods of x in X such that whenever $x_n \in U_n$ for each n, then $\{x_n\colon n \in \mathbb{N}\}$ is bounded in X. The set of all wq-points in X is called a wq-derivative of X and is denoted by $X^{(wq)}$. When $X = X^{(wq)}$ we say that X is a wq-space. The class of wq-spaces contains all locally bounded (each point has a bounded neighborhood) and all spaces of pointwise countable type (each point is contained in a compact set of a countable character), in particular all first countable spaces.

Recall that a set-valued map $\psi: Y \to P(X)$ is upper semicontinuous (br., u.s.c.) whenever for every open $U \subset X$ the set $\psi^{\#} = \{y \in Y\colon \psi(y) \subset U\}$ is open in Y. Upper semicontinuous compact valued maps are called u.s.c.o. maps.

Proposition 3.14. *Let* $u: C_k(X,E) \to C_p(Y,F)$ *be continuous, linear and* X μ*-complete. Suppose there is a set-valued map* ϕ *from a space* Z *into* Y *such that*:

($*$) *If* $\{z_n\} \subset Z$ *is bounded and* $y_n \in \phi(z_n)$ *for each* n, *then* $\{y_n\} \subset Y$ *is bounded.* *Then there is an u.s.c.o. map* $\psi: Z \to P(X)$ *such that* $\operatorname{supp}(\phi(z)) \subset \psi(z)$ *for all* $z \in Z$ *and* $\psi(z) \neq \emptyset$ *for all* $z \in Z_0$, *where* $Z_0 = Z^{(wq)} \cap \operatorname{cl}_Z(\{z \in Z\colon \operatorname{supp}(\phi(z)) \neq \emptyset\})$.

Proof. Let, for every $z \in Z$, fix a local base $\mathcal{U}(z)$ at z and define $\psi: Z^{(wq)} \to P(\beta X)$ by

$$\psi(z) = \bigcap \big\{\operatorname{cl}_{\beta X}\big(\operatorname{supp}(\phi(U))\big)\colon\ U \in \mathcal{U}(z)\big\}.$$

Obviously, $\operatorname{supp}(\phi(z)) \subset \psi(z)$ for $z \in Z^{(wq)}$ and $\psi(z) = \emptyset$ for $z \in Z^{(wq)} - Z_0$. Let show that $\psi(z) \neq \emptyset$ for $z \in Z_0$. Take arbitrary $z^* \in Z_0$. Then every $U \in \mathcal{U}(z^*)$ meets $B = \{z \in Z\colon \operatorname{supp}(\phi(z)) \neq \emptyset\}$, hence the family $\{\operatorname{cl}_{\beta X}(\operatorname{supp}(\phi(U)))\colon U \in \mathcal{U}(z^*)\}$ has the finite intersection property. Consequently, $\psi(z^*) \neq \emptyset$. □

That ψ is u.s.c. (as a set-valued map from $Z^{(wq)}$ into βX) is trivially seen. So, it remains only to show that $\psi(z) \subset X$ for $z \in Z_0$. Towards this end, for each $z \in Z^{(wq)}$ fix a countable family $\gamma(z) = \{U_n(z)\colon n \in \mathbb{N}\}$ of neighborhoods of z in Z such that whenever $z_n \in U_n(z)$ for every n, $\{z_n\} \subset Z$ is bounded. Now define

$$\varphi(z) = \bigcap \big\{\operatorname{supp}\big(\phi(U_n(z))\big)\colon\ n \in \mathbb{N}\big\}, \quad z \in Z_0.$$

Claim 3. $\varphi(z)$ *is compact.*

Proof. Since X is a μ-space, it suffices to show that $\varphi(z) \subset X$ is bounded. Suppose not. Then there is a discrete open in X family $\{V_n\colon n \in \mathbb{N}\}$ and $h \in C(X)$ such that $V_n \cap \varphi(z) \neq \emptyset$, $n \in \mathbb{N}$, and $\{h(V_n)\colon n \in \mathbb{N}\}$ is discrete in $\mathbb{R}$. For every n take $z_n \in U_n(z)$, $y_n \in \phi(z_n)$ and $x_n \in \text{supp}(y_n) \cap V_n$. By virtue of $(*)$ and Corollary 3.3, $\text{supp}(\{y_n\colon n \in \mathbb{N}\}) \subset X$ is bounded. Then $\{x_n\} \subset X$ is also bounded. This is a contradiction because $\{h(x_n)\} \subset \mathbb{R}$ is not bounded. □

Claim 4. $\psi(z) \subset \varphi(z)$.

Proof. Suppose there is $x \in \psi(z) - \varphi(z)$. Take an open $W \subset \beta X$ containing x such that $\text{cl}_{\beta X}(W) \cap \varphi(z) = \emptyset$. Then W meets each $\text{supp}(\phi(U))$, $U \in \mathcal{U}(z)$. Hence, for every n, we can find $z_n \in \bigcap\{U_k(z)\colon k \leqslant n\}$, $y_n \in \phi(z_n)$ and $x_n \in \text{supp}(y_n) \cap W$. Let $P_n = \{y_k\colon k \geqslant n\}$. By $(*)$, all $P_n \subset Y$ are bounded. Since X is a μ-space, by Corollary 3.3, $\text{supp}(P_n) \subset X$ are compact. Consequently, $\bigcap\gamma \neq \emptyset$, where $\gamma = \{\text{supp}(P_n) \cap \text{cl}_X(W)\colon n \in \mathbb{N}\}$. On the other hand

$$\bigcap\gamma \subset \bigcap\{\text{supp}(P_n)\colon n \in \mathbb{N}\} \subset \varphi(z).$$

Hence $\varphi(z) \cap W \neq \emptyset$, which is a contradiction. Claim 2 is proved.

It is clear now that the above two claims complete the proof of Proposition 3.14. □

Corollary 3.15. *If u, in the hypotheses of Proposition 3.14, is a surjection, then $\psi(z) \neq \emptyset$ for all $z \in Z^{(wq)}$.*

Proof. In this case $\text{supp}(y) \neq \emptyset$, $y \in Y$ (see the proof of Proposition 3.10), so $Z_0 = Z^{(wq)}$. □

Let us note that if $u\colon C_k(X,E) \to C_p(Y,F)$ is a continuous linear surjection, sometimes the existence of an u.s.c.o. map $\psi\colon Y \to P(X)$ with $\text{supp}(y) \subset \psi(y)$ for any $y \in Y$ is sufficient for Y to be a wq-space (see Corollary 4.7 below).

4. Some applications of the main results

4.1. Properties preserved by continuous linear surjections

In this section we apply the main results from Section 3 to show that some topological properties are preserved under continuous linear maps from $C_k(X,E)$ (respectively $C_b(X,E)$) onto $C_p(Y,F)$.

As previously proved, pseudocompactness is preserved by continuous effective linear surjections from $C_b(X,E)$ onto $C_p(Y,F)$ (see Corollary 3.12), and μ-completeness is preserved by continuous quotient linear surjections from $C_k(X,E)$ onto $C_k(Y,F)$, as well as from $C_p(X,E)$ onto $C_p(Y,F)$ (Corollary 3.9). Hence, we have

Corollary 4.1. *Let E and F be normed spaces. Then compactness is preserved by continuous quotient linear surjections from $C_k(X,E)$ onto $C_k(Y,F)$, as well as from $C_p(X,E)$ onto $C_p(Y,F)$.*

The example following Corollary 3.9 shows that the requirement to consider quotient maps in Corollary 4.1 is essential. It was known that compactness is preserved by linear homeomorphisms between $C_k(X)$ and $C_k(Y)$ [6, Theorem 1.5.7] and, in the case of p-topology, by linear homeomorphisms between $C_p(X)$ and $C_p(Y)$ [1], as well as, by continuous linear surjections from $C_p(X)$ onto $C_p(Y)$ provided Y is a μ-space [5].

For a space X let $\mathcal{K}(X)$ be the family of all compact subsets of X considered as a poset under inclusion. We say that $\mathcal{B} \subset \mathcal{K}(X)$ is cofinal in $\mathcal{K}(X)$ if for any $K \in \mathcal{K}(X)$ there is $B \in \mathcal{B}$ with $K \subset B$. The cofinality of $\mathcal{K}(X)$ is defined by

$$\operatorname{cof}\mathcal{K}(X) = \min\{\operatorname{card}(\mathcal{B})\colon\ \mathcal{B} \text{ is cofinal in } \mathcal{K}(X)\}.$$

Proposition 4.2. *Let $u: C_k(X,E) \to C_p(Y,F)$ be a continuous, linear surjection. If X and Y are μ-spaces, then* $\operatorname{cof}\mathcal{K}(Y) \leqslant \operatorname{cof}\mathcal{K}(X)$.

Proof. Let $\{K_j\colon\ j \in J\} \subset \mathcal{K}(X)$ be cofinal and, for each $j \in J$, let

$$H_j = \{y \in Y\colon\ \operatorname{supp}(y) \subset K_j\}.$$

Then, by Proposition 3.10 and Remark 3.11, H_j are compact. It remains to show that $\{H_j\colon\ j \in J\} \subset \mathcal{K}(Y)$ is cofinal. To this end, let $B \subset Y$ be compact. By virtue of Corollary 3.3, $\operatorname{supp}(B)$ is compact in X. Hence, $\operatorname{supp}(B) \subset K_j$ for some $j \in J$, and then $B \subset H_j$. □

Proposition 4.2 was proved in [6, Theorem 1.5.3(d)] in case u is a surjection from $C(X)$ onto $C(Y)$.

One of our starting points to write the present paper is the following result [7, Theorem 3.3]: in the class of metrizable spaces completeness is preserved by continuous linear surjections from $C_p(X)$ onto $C_p(Y)$. This becomes false outside the class of metrizable spaces. For example, let $X = \omega(\omega+1)$ and Y be obtained from X by identifying all accumulation points of X to one point. Then $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic [6, Example 2.4.10], and X is countable metric and locally compact while Y is paracompact and σ-metrizable (a countable union of closed metrizable subspaces) but not Čech complete. We are going to show that the above result of Baars, de Groot and Pelant is true for a strictly larger class of spaces than metrizable ones.

Proposition 4.3. *Let $u: C_k(X,E) \to C_p(Y,F)$ be a continuous, linear surjection, X and Y μ-spaces, and $Y_0 \subset Y$ a wq-subspace. If X is Čech complete, then there is a Čech complete subspace of* $\operatorname{cl}_Y(Y_0)$ *containing* Y_0.

Proof. By Corollary 3.15 (with $Z = Y_0$ and $\phi = \mathrm{id}$), there is u.s.c.o. $\psi: Y_0 \to P(X)$ such that $\operatorname{supp}(y) \subset \psi(y)$ for every $y \in Y_0$. Denote $H = \operatorname{cl}_{\beta Y}(Y_0)$ and define an u.s.c.o. extension $\varphi: H \to P(\beta X)$ of ψ by $\varphi(y) = \bigcap\{\operatorname{cl}_{\beta X}(\psi(U \cap Y_0))\colon\ U \in \mathcal{U}(y)\}$, where $\mathcal{U}(y)$ is a local base at y in H. Since X is Čech complete and φ extends ψ, $\varphi^{\#}(X) = \{y \in H\colon\ \varphi(y) \in X\} \subset H$ is G_δ and contains Y_0. It suffices to show that $\varphi^{\#}(X) \subset \operatorname{cl}_Y(Y_0)$. To prove that we need the following lemma:

Lemma 4.4. *Let $u : C_b(X, E) \to C_p(Y, F)$ be linear, continuous and effective, Y μ-complete and $Y_0 \subset Y$. Suppose there is an u.s.c.o. map $\varphi : H \to P(\beta X)$ with $\operatorname{supp}(y) \subset \varphi(y)$ for every $y \in Y_0$, where $H = \mathrm{cl}_{\beta Y}(Y_0)$. Then $K \cap \mathrm{cl}_Y(Y_0)$ is compact for any compact $K \subset \varphi^{\#}(X)$.*

Proof. Let $D = \mathrm{cl}_Y(Y_0)$. First, let us show that $\operatorname{supp}(y) \subset \varphi(y)$ for every $y \in D$. Suppose there is $y^* \in D$ and $x \in X$ with $x \in \operatorname{supp}(y^*) - \varphi(y^*)$ and take disjoint neighborhoods U and V (in βX) of $\varphi(y^*)$ and x, respectively. Since supp is LSC and φ is u.s.c., there is a neighborhood $W \subset D$ of y^* such that $\varphi(y) \subset U$ and $\operatorname{supp}(y) \cap V \neq \emptyset$ for every $y \in W$. To obtain a contradiction, pick a point $y_1 \in W \cap Y_0$ and, then observe that $\operatorname{supp}(y_1) \subset \varphi(y_1) \subset U$ and $\operatorname{supp}(y_1) \cap V \neq \emptyset$.

Clearly $\varphi(K) \subset X$ is compact (as an image of a compact set under an u.s.c.o. map). By Proposition 3.10 and Remark 3.11, $B = \{y \in Y\colon \operatorname{supp}(y) \subset \varphi(K)\}$ is certainly compact. Since $\operatorname{supp}(y) \subset \varphi(y)$ for every $y \in D$, $K \cap D \subset B$ is closed, and therefore $K \cap D$ is compact. □

Now, let us go back to the proof of Proposition 4.3. Suppose there is a point $y^* \in \varphi^{\#}(X) - \mathrm{cl}_Y(Y_0)$. Since $\varphi^{\#}(X) \subset H$ is G_δ, there exists $f \in C(H)$ such that $f^{-1}(f(y^*)) \subset \varphi^{\#}(X)$. Without loss of generality we can assume that $f^{-1}(f(y^*)) \cap \mathrm{cl}_Y(Y_0) \neq \emptyset$. According to Lemma 4.4, $B = f^{-1}(f(y^*)) \cap \mathrm{cl}_Y(Y_0)$ is compact. Choose $h \in C(H)$ with $h(y^*) = 0$ and $h(B) = 1$, and let g be the diagonal product of f and h. Then g separates y^* and $\mathrm{cl}_Y(Y_0)$ and $g^{-1}(g(y^*)) \subset f^{-1}(f(y^*)) \subset \varphi^{\#}(X)$. Since $g(Y_0) \subset g(H)$ is dense, there exists a sequence $\{y_n\} \subset Y_0$ such that $\{g(y_n)\}$ converges to $g(y^*)$. So, $\{y_n\} \cup g^{-1}(g(y^*)) \subset \varphi^{\#}(X)$ is compact and its restriction on $\mathrm{cl}_Y(Y_0)$ is $\{y_n\}$. Thus, by Lemma 4.4, $\{y_n\}$ is compact. This implies $g(y^*) \in \{g(y_n)\}$, which is a contradiction. □

Corollary 4.5. *Let $u : C_k(X, E) \to C_p(Y, F)$ be a continuous, linear surjection, and X a Čech complete μ-space. If Y is μ-complete, then every closed wq-subspace of Y is Čech complete.*

Proof. Follows from Proposition 4.3. □

Corollary 4.6. *Let $u : C_k(X) \to C_p(Y)$ be a continuous, linear surjection, X completely metrizable, and Y paracompact. Then every closed wq-subset of Y is completely metrizable.*

Proof. Suppose $K \subset Y$ is a closed wq-subset. According to Corollary 4.5, K is Čech complete. On the other hand, by Proposition 3.6 and Corollary 3.3, $u : C_k(X) \to C_k(Y)$ is also continuous. Since $C_k(X)$ contains a dense σ-compact subset [17, Theorem 5.6.2], so is $C_k(Y)$. This implies [17, Theorem 5.6.3] that Y is submetrizable (i.e., there is one-to-one continuous surjection from Y onto a metric space). Hence K is a Čech complete and submetrizable paracompact. Consequently, K is completely metrizable. □

Corollary 4.6 partially answers the following question [4, Problem 20]:

Suppose $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, X metrizable and Y first countable. Is it true that Y is also metrizable?

In fact, we can prove that the answer is positive if X is Čech complete. Indeed, by a result of Uspenskiĭ [22], Y is paracompact. Then Corollary 4.6 completes the proof.

A space X is said to be of compact countable type if every compact $H \subset X$ is contained in a compact set $K \subset X$ which is G_δ in βX. Every space of compact countable type is a wq-space.

Corollary 4.7. *Let $u: C_k(X,E) \to C_p(Y,F)$ be a continuous, linear surjection, X and Y μ-spaces, and X a space of compact countable type. Then closed $Y_0 \subset Y$ is of compact countable type if and only if there is an u.s.c.o. map $\psi: Y_0 \to P(X)$ with* $\operatorname{supp}(y) \subset \psi(y)$ *for every $y \in Y_0$.*

Proof. Necessity follows from Corollary 3.15 with $Z = Y_0$ and $\varphi = \mathrm{id}$.

Now let $\psi: Y_0 \to P(X)$ be u.s.c.o. with $\operatorname{supp}(y) \subset \psi(y)$ for every $y \in Y_0$, and $K \subset Y_0$ compact. As in the proof of Proposition 4.3, define an u.s.c.o. extension $\varphi: H \to P(\beta X)$ of ψ, where $H = \mathrm{cl}_{\beta Y}(Y_0)$. Then $\varphi(K) \subset X$ is compact, so there is a sequence $\{V_i\}$ of open in βX sets each of them containing $\varphi(K)$ and such that $\bigcap\{V_i\colon i \in \mathbb{N}\} \subset X$. Because every $U_i = \varphi^{\#}(V_i) \subset H$ is open and contains K, we need only to show that $G = \bigcap\{U_i\colon i \in \mathbb{N}\} \subset Y_0$. Suppose there is $y^* \in G - Y_0$. Since $G \subset H$ is G_δ and $G \subset \varphi^{\#}(X)$, we can repeat the final part of the proof of Proposition 4.3 (with $\varphi^{\#}(X)$ replaced by G) to obtain a contradiction. □

Gul'ko and Okunev [12], and R. McCoy and I. Ntantu [18] proved that local compactness is preserved by l_p-equivalence in the class of paracompact spaces of pointwise countable type. Another proof of that result was also given by Baars and de Groot [6, Theorem 1.5.10], who asked if paracompactness is essential [6, Question 3, p. 37] and whether the same is true for l_k-equivalence [6, Question 4, p. 37]. Our next proposition answers affirmatively the second question and shows that the above result is true in a more general situation, in particular paracompactness can be weakened to μ-completeness and pointwise countable type to wq-space property.

Proposition 4.8. *Suppose $u: C_k(X,E) \to C_p(Y,F)$ is a continuous, linear surjection and X is a locally compact μ-space. If Y is μ-complete, then*:

(i) *every closed wq-subspace $K \subset Y$ is locally compact*;

(ii) *$Y^{(wq)} \subset Y$ is locally compact and open.*

Proof. (i) Take a closed wq-subspace $K \subset Y$. By Corollary 3.15 (with $Z = K$ and $\varphi = \mathrm{id}$), there is u.s.c.o. $\psi: K \to P(X)$ such that $\operatorname{supp}(y) \subset \psi(y)$ for all $y \in K$. Fix $y^* \in K$ and a compact neighborhood $U \subset X$ of $\psi(y^*)$. Then, since ψ is u.s.c., we can find a closed neighborhood $V \subset K$ of y^* with $\psi(y) \subset U$ for any $y \in V$. By Proposition 3.10, $H = \{y \in Y\colon \operatorname{supp}(y) \subset U\}$ is bounded in Y and, because $V \subset H$, $V \subset Y$ is also bounded. Hence V is compact.

(ii) Following the notations from the proof of Proposition 3.14 (with $Z = Y$ and $\phi = \mathrm{id}$), for a fixed $y \in Y^{(wq)}$ define $\varphi(y) = \bigcap\{\mathrm{supp}(U_n(y))\colon n \in \mathbb{N}\}$. Since $\varphi(y) \subset X$ is compact, there is an open neighborhood $W \subset X$ of $\varphi(y)$ with a compact closure. If every $\mathrm{supp}(U_n(y))$ meets $X - \mathrm{cl}_X(W)$, using the arguments from the proof of Claim 2, Proposition 3.14, we can get a contradiction. So, $\mathrm{supp}(U_i(y)) \subset \mathrm{cl}_X(W)$ for some $i \in \mathbb{N}$. Now, Proposition 3.10 implies that $\mathrm{cl}_Y(U_i(y))$ is compact. □

It follows from [6, Example 2.4.10, p. 67] that the wq-space property in Proposition 4.8 cannot be dropped.

4.2. Continuous linear injections

We are going to prove that some topological properties are invariant under continuous linear injections in the following sense: if there is a continuous linear injection from $C_k(X, E)$ into $C_p(Y, F)$ and if Y has a dense subset with a given property, then X has also a dense subset with the same property.

First we need the following analog of [6, Lemma 1.2.5] (see also [5, Lemma 2.6]), which can be derived by the same arguments.

Proposition 4.9. *Let $u : C_b(X, E) \to C_p(Y, F)$ be a continuous, effective linear injection. If $D \subset Y$ is dense, then* $\mathrm{supp}(D) = X$.

Corollary 4.10. *Let $u : C_b(X, E) \to C_p(Y, F)$ be a continuous, effective linear injection. Then*

(i) *X is pseudocompact provided Y is;*

(ii) *X contains a dense σ-bounded subset (a countable union of closed and bounded in X sets) provided Y contains such a dense set.*

Proof. (i) It follows from the combination of Corollary 3.3 and Proposition 4.9.

(ii) Suppose $B = \bigcup\{B_i\colon i \in \mathbb{N}\} \subset Y$ is σ-bounded and dense with each $B_i \subset Y$ closed and bounded. Then, by Corollary 3.3, $H_i = \mathrm{supp}(B_i) \subset X$ are closed and bounded. Because $H = \bigcup\{H_i\colon i \in \mathbb{N}\} \subset \mathrm{supp}(B)$ is dense and $\mathrm{supp}(B) = X$ (by Proposition 4.9), $H \subset X$ is also dense. □

Let Z be a given space. We say that a space X is Z-analytic if X is an u.s.c.o. image of Z.

Proposition 4.11. *Let $u : C_k(X, E) \to C_p(Y, F)$ be a continuous, linear injection, X a μ-space, and Z a μ-complete wq-space. If Y contains a dense Z-analytic set, then X also contains such a dense set.*

Proof. Let $\phi : Z \to P(Y)$ be an u.s.c.o. map such that $Y_0 = \bigcup\{\phi(z)\colon z \in Z\} \subset Y$ is dense. We claim that ϕ satisfies $(*)$ from Proposition 3.14. Indeed, let $\{z_n\} \subset Z$ be bounded and $y_n \in \phi(z_n)$ for each n. Because Z is a μ-space, $\mathrm{cl}_Z(\{z_n\})$ is compact.

Then $\phi(\text{cl}_Z(\{z_n\}))$ is also compact, so $\{y_n\} \subset Y$ is bounded. Now we can apply Proposition 3.14 to get u.s.c.o. $\psi: Z \to P(X)$ with $\text{supp}(\phi(z)) \subset \psi(z)$ for any $z \in Z$; in particular

$$\bigcup\{\text{supp}(y):\ y \in Y_0\} \subset \bigcup\{\psi(z):\ z \in Z\}.$$

By Proposition 4.9

$$\bigcup\{\text{supp}(y):\ y \in Y_0\} \subset X$$

is dense. Therefore $\psi(Z) = \bigcup\{(z):\ z \in Z\}$ is dense in X. □

We consider three specifications for Z: when Z is a complete separable metric space, a separable metric space or a disjoint union of τ many compact spaces. Each of these three classes is hereditarily with respect to closed subsets and consists of μ-complete wq-spaces. It is well known that if Z belongs to the first (respectively second) class, then Z-analytic spaces are called K-analytic [8] (respectively countably K-determined, or Lindelöf Σ-spaces [20]). It is also clear that X is τ-compact (a union of τ many compact sets) if and only if X is Z-analytic, where Z is a disjoint union of τ many compact spaces. Hence, by Proposition 4.11, we have the following

Corollary 4.12. *Let X be a μ-space and $u: C_k(X, E) \to C_p(Y, F)$ be a continuous linear injection. Then X contains a dense subset which is K-analytic (respectively Lindelöf Σ-space, or τ-compact) provided Y contains such a dense subset.*

We say that a space X is τ-Lindelöf if the Lindelöf degree of X is not greater than τ, i.e., every open cover of X contains a subcover of cardinality $\leqslant \tau$.

Corollary 4.13. *Let X be a μ-space and $u: C_k(X, E) \to C_p(Y, F)$ be a continuous linear injection. Then X contains a dense τ-Lindelöf subset provided Y contains a dense τ-Lindelöf wq-subspace.*

Proof. In the proof of Proposition 4.11 we needed Z to be a μ -space only to show that ϕ satisfies $(*)$. Hence, if $Z \subset Y$ is a dense τ-Lindelöf, wq-subspace and $\phi = \text{id}$, we can apply Proposition 4.11 to obtain a dense Z-analytic $B \subset X$. Then B is τ-Lindelöf as an u.s.c.o. image of Z. □

4.3. Properties preserved by linear homeomorphisms

Proposition 4.14. *Let X be a μ-space and u be a linear homeomorphism between $C_k(X, E)$ and $C_k(Y, F)$. If Y is a wq-space, then X is Y-analytic.*

Proof. By Proposition 3.14 (with $\phi = \text{id}$), there is u.s.c.o. $\psi: Y \to P(X)$ such that $\text{supp}(y) \subset \psi(y)$ for all $y \in Y$. We have only to show that $\psi(Y) = X$. First, let prove that $x \in \text{supp}(\text{supp}(x))$ for every $x \in X$, where $\text{supp}(x) \subset Y$ is the support of x with respect to u^{-1}. If this is not the case for some $x \in X$, there is $f \in C(X, E)$ with $f(x) \neq 0$ and $f|\mathrm{supp}(\mathrm{supp}(x)) = 0$. Consequently, $u(f)|\mathrm{supp}(x) = 0$, which yields $f(x) = 0$, a contradiction. Next, fix a point $x^* \in X$. By Corollary 3.9, Y is a μ-space, so $\mathrm{supp}(x^*) \subset Y$ is compact (see Corollary 3.3). Then $\psi(\mathrm{supp}(x^*))$ is compact, and because it contains $\bigcup\{\mathrm{supp}(y)\colon y \in \mathrm{supp}(x^*)\}$, it contains also $\mathrm{supp}(\mathrm{supp}(x^*))$. Hence, $x^* \in \psi(Y)$. □

Remark 4.15. If u is a linear homeomorphism between $C_p(X, E)$ and $C_p(Y, F)$ and Z is a μ-complete wq-space, then X is Z-analytic provided Y is Z-analytic. Indeed, let $\phi: Z \to P(Y)$ be u.s.c.o. As in the proof of Proposition 4.11, there is u.s.c.o. $\psi: Z \to P(X)$ with $\mathrm{supp}(\phi(z)) \subset \psi(z)$ for all $z \in Z$. Next, take $x \in X$. By Proposition 2.2(iii), $\mathrm{supp}(x)$ is finite and, because $x \in \mathrm{supp}(\mathrm{supp}(x))$, there is $y \in \mathrm{supp}(x)$ with $x \in \mathrm{supp}(y)$. Finally, choose $z \in Z$ such that $y \in \phi(z)$. Then $x \in \psi(z)$.

In [2, Theorem 3.7] the following result of Velicko is announced: Lindelöfness is preserved by linear homeomorphisms between $C_p(X)$ and $C_p(Y)$. According to [2], Velicko's proof does not work for higher Lindelöf degrees. Our next result shows this is true in the class of μ-complete wq-spaces.

Corollary 4.16. *Lindelöf degree and τ-compactness are preserved by linear homeomorphisms between $C_k(X, E)$ and $C_k(Y, F)$ (respectively $C_p(X, E)$ and $C_p(Y, F)$) in the class of μ-complete wq-spaces.*

Proof. Suppose $C_k(X, E)$ and $C_k(Y, F)$ are linearly homeomorphic. Because both τ-lindelöfness and τ-compactness are preserved by u.s.c.o. maps, the proof follows from Proposition 4.14. If $C_p(X, E)$ and $C_p(Y, F)$ are linearly homeomorphic we can apply Remark 4.15 instead of Proposition 4.14. □

We should note that wq-space property is not preserved even by linear homeomorphisms between $C_k(X)$ and $C_k(Y)$. We already observed [6, Example 2.4.10], that there exist countable l_k-equivalent spaces X and Y such that X is metrizable and locally compact and Y is not locally compact. By Proposition 3.6, X and Y are l_k-equivalent. Since Y is not locally compact, it follows from Proposition 4.8 that Y is not a wq-space.

Corollary 4.17. *The following properties are preserved by linear homeomorphisms between $C_k(X, E)$ and $C_k(Y, F)$ in the class of wq-spaces: Lindelöfness, σ-compactness, K-analyticity and Lindelöf Σ-space property.*

Proof. Because each of the above properties is preserved by u.s.c.o. maps and implies μ-completeness, the proof follows from Proposition 3.9(ii) and Proposition 4.14. □

Concerning σ-compactness, Corollary 4.17 gives a positive partial answer to a question from [6, Question 1, p. 35] whether σ-compactness is preserved by linear homeomorphisms between $C_k(X)$ and $C_k(Y)$.

Corollary 4.18. *K-analyticity, σ-compactness and Lindelöf Σ-space property are preserved by linear homeomorphisms between $C_p(X, E)$ and $C_p(Y, F)$.*

Proof. Suppose Y has one of the above properties. Then Y is μ-complete and Z-analytic, where Z is a μ-complete wq-space. So, by Corollary 3.9, X is also μ-complete. Finally, Remark 4.15 completes the proof. □

The continuous images of separable complete metric spaces are called analytic. It is well known [14] that the class of analytic spaces coincides with the class of K-analytic spaces having a countable network. On the other hand, if E is a separable metric space, then $C_p(X, E)$ has a countable network if and only if X has a countable network [17]. Combining these two facts, we obtain from Corollary 4.18 the following

Corollary 4.19. *If E and F are separable normed spaces, then analyticity is preserved by linear homeomorphisms between $C_p(X, E)$ and $C_p(Y, F)$.*

Concerning Corollaries 4.18 and 4.19, Okunev [21] has a much stronger result in the case when both E and F are the real line: K-analyticity, analyticity, σ-compactness and Lindelöf Σ-property are preserved by homeomorphisms between $C_p(X)$ and $C_p(Y)$.

Our last proposition answers affirmatively a question of Baars and de Groot [6, Question 1, p. 35].

Proposition 4.20. *If $C_k(X, E)$ and $C_k(Y, F)$ are linearly homeomorphic, then*

$$\operatorname{cof} \mathcal{K}(X) = \operatorname{cof} \mathcal{K}(Y).$$

Proof. It is enough to show that $\operatorname{cof} \mathcal{K}(Y) \leqslant \operatorname{cof} \mathcal{K}(X)$. Let $\{K_j\colon j \in J\} \subset \mathcal{K}(X)$ be cofinal and $H_j = \operatorname{supp}(K_j)$, $j \in J$. Now we need the following fact which follows from the proof of [6, Lemma 1.5.6]: if $K \subset X$ and $H \subset Y$ are compact sets, then $\operatorname{supp}(K) \subset Y$ and $\operatorname{supp}(H) \subset X$ are also compact provided that $C_k(X, E)$ and $C_k(Y, F)$ are linearly homeomorphic. So, all $H_j \subset Y$ are compact. We claim that $\{H_j\colon j \in J\} \subset \mathcal{K}(Y)$ is cofinal. Take compact $B \subset Y$. Then, by the above fact, $\operatorname{supp}(B) \subset X$ is compact. Hence, $\operatorname{supp}(B) \subset K_j$ for some $j \in J$. Since $B \subset \operatorname{supp}(\operatorname{supp}(B))$, we finally have $B \subset H_j$. □

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References

[1] A. Arkhangel'skiĭ, On linear homeomorphisms of function spaces, Soviet Math. Dokl. 25 (1982) 852–855.

[2] A. Arkhangel'skiĭ, C_p-theory, in: M. Husek and J. van Mill, eds., Recent Progress in General Topology (Elsevier, Amsterdam, 1992) 1–56.

[3] A. Arkhangel'skiĭ, Topological Function Spaces (Kluwer, Dordrecht, 1992).
[4] A. Arkhangel'skiĭ, A survey of C_p-theory, Questions Answers Gen. Topology 5 (1987) 1–109.
[5] J. Baars, A note on linear mappings between function spaces, Comment. Math. Univ. Carolin. 34(4) (1993) 711–715.
[6] J. Baars and J. de Groot, On Topological and Linear Equivalence of Certain Function Spaces, CWI Tracts 86 (Centrum Wisk. Inform., Amsterdam, 1992).
[7] J. Baars, J. de Groot and J. Pelant, Function spaces of completely metrizable spaces, Trans. Amer. Math. Soc. 340(2) (1993) 871–879.
[8] G. Choquet, Lectures on Analysis, Vol. I (Benjamin, New York, 1969).
[9] J. Diedonné, Sur les espaces uniformes completes, Ann. Sci. École Norm. Sup. 56 (1939) 277–291.
[10] J. Dugundji, An extension of Tietze's theorem, Pacific J. Math. 1 (1951) 353–367.
[11] R. Engelking, General Topology (PWN, Warszawa, 1977).
[12] S. Gul'ko and O. Okunev, Local compactness and M-equivalence, in: A.V. Ivanov et al., eds., Questions of Geometry and Topology (Petrozavodsk State University, Petrozavodsk, 1986) 14–23 (in Russian).
[13] S.T. Hu, Theory of Retracts (Wayne State Univ. Press, Detroit, 1965).
[14] J. Jayne and C. Rogers, Analytic Sets (Academic Press, London, 1980).
[15] Y. Mibu, On Baire functions on infinite product spaces, Proc. Imp. Acad. Tokyo 20 (1944) 661–663.
[16] E. Michael, A quintuple quotient quest, Gen. Topology Appl. 2 (1972) 91–138.
[17] R. McCoy and I. Ntantu, Topological Properties of Spaces of Continuous Functions, Lecture Notes in Math. 1315 (Springer, Berlin, 1988).
[18] R. McCoy and I. Ntantu, Completeness properties of function spaces, Topology Appl. 22 (1986) 191–206.
[19] K. Morishita, The minimal support for a continuous functional on a function space II, Tsukuba J. Math. 16(2) (1992) 495–501.
[20] K. Nagami, Σ-spaces, Fund. Math. 65(2) (1969) 169–192.
[21] O. Okunev, Weak topology of an associated space and t-equivalence, Math. Notes 46(1–2) (1990) 534–538.
[22] V. Uspenskiĭ, On the topology of a free locally convex space, Soviet Math. Dokl. 27 (1983) 781–785.