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FUNCTION SPACES AND DIEUDONNÉ COMPLETENESS

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ABSTRACT. For a completely regular space X and a normed space E let $C_k(X,E)$ (resp., $C_p(X,E)$) be the set of all E-valued continuous maps on X endowed with the compact-open (resp., pointwise convergence) topology. It is shown that the set of all F-valued linear continuous maps on $C_k(X,E)$ when equipped with the topology of uniform convergence on the members of some families of bounded subsets of $C_k(X,E)$ is a complete uniform space if F is a Banach space and X is Dieudonné complete. This result is applied to prove that Dieudonné completeness is preserved by linear quotient surjections from $C_k(X,E)$ onto $C_k(Y,F)$ (resp., from $C_p(X,E)$ onto $C_p(Y,F)$) provided E, F are Banach spaces and Y is a k-space.

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1. Introduction. Throughout this paper we consider only completely regular topological spaces and Hausdorff locally convex linear spaces; E and F are always normed linear spaces; $C(X,E)$ denotes the set of all E-valued continuous maps on a topological space X; when E is the real line $\mathbb{R}$ we simply write $C(X)$; a subset K of X is t-bounded in X if $f(K) \subset \mathbb{R}$ is bounded for every $f \in C(X)$. When every closed and t-bounded subset of X is compact we say that X is a μ-space. The notion of a t-bounded set has to be distinguished from the usual boundedness in a topological vector space.

Let $M(X,E,F)$ be the set of all continuous linear maps from $C_k(X,E)$ into F, where $C_k(X,E)$ is the set $C(X,E)$ with the compact-open topology. We also consider the pointwise convergence topology on $C(X,E)$ and denote this space by $C_p(X,E)$. If $\mathcal{A}$ is a cover of $C(X,E)$ invariant with respect to finite unions and such that every $A \in \mathcal{A}$ is a bounded set in $C_k(X,E)$, then $M_{\mathcal{A}}(X,E,F)$ stands for $M(X,E,F)$ with the topology of uniform convergence on the members of $\mathcal{A}$ and $M_{\mathcal{A},p}(X,E,F)$ is the subspace of all $\mu \in M_{\mathcal{A}}(X,E,F)$ which are continuous on $C_p(X,E)$. We consider the following specializations for $\mathcal{A}$:

the family $\mathcal{C}$ of all equicontinuous and pointwise bounded subsets of $C(X,E)$ and

the family $\mathcal{F} = \{A \in \mathcal{C} : \pi_H(A) \subset C_k(H,E)$ is compact for every compact $H \subset X\}$, where $\pi_H : C_k(X,E) \to C_k(H,E)$ is the restriction map.

* *Keywords:* Function spaces, Dieudonné completeness, linear maps.

It was proved by Uspenskij [U1, Theorem 5] that $M_{\mathcal{C}}(X, \mathbb{R}, \mathbb{R})$ is complete (as uniform space) and $M_{\mathcal{C},p}(X, \mathbb{R}, \mathbb{R})$ is Dieudonné complete [U2, Theorem 8] provided X is Dieudonné complete.

The purpose of this paper is to extend the result of Uspenskij to arbitrary Banach spaces. More precisely, in Section 2 we show that $M_{\mathcal{A}}(X, E, F)$ is a complete uniform space and $M_{\mathcal{A},p}(X, E, F)$ is Dieudonné complete if X is Dieudonné complete, E and F are Banach spaces, and $\mathcal{A}$ is a cover of $C(X, E)$ invariant with respect to finite unions and such that $\mathcal{F} \subset \mathcal{A} \subset \mathcal{C}$. In Section 3 the completeness of $M_{\mathcal{F}}(X, E, F)$ is applied to prove that Dieudonné completeness is preserved by quotient continuous linear surjections from $C_p(X, E)$ onto $C_p(Y, F)$, as well as from $C_k(X, E)$ onto $C_k(Y, F)$, if Y is a k-space.

2. Completeness of some measure spaces. Let μ be a linear map from $C(X, E)$ into a linear space. The support of μ is the set $\text{supp}(\mu)$ of all $x \in \beta X$ satisfying the condition that for every neighborhood U of x in βX there is an $f \in C(X, E)$ such that $(\beta f)(\beta X - U) = 0$ and $\mu(f) \neq 0$, where βX is the Čech-Stone compactification of X and $\beta f : \beta X \to \beta E$ is the continuous extension of f. Obviously, $\text{supp}(\mu)$ is closed in βX, so it is always compact.

We denote by $C_u^*(X, E)$ the space of all bounded continuous E-valued maps on X with the uniform convergence topology. The normed spaces E and F are equipped with fixed norms. The same notation $\|\cdot\|$ is used for both norms and when E or F is required to be a Banach space we suppose that it is complete with respect to its fixed norm.

LEMMA 2.1. *Let μ be a linear map form $C(X, E)$ into a linear space and U be a neighborhood of* $\text{supp}(\mu)$ *in βX. Then:*

(i) $\mu(f) = 0$ for every $f \in C(X, E)$ with $(\beta f)(U) = 0$;

(ii) $\mu(f) = 0$ for every $f \in C(X, E)$ with $(\beta f)(\text{supp}(\mu)) = 0$ provided μ is continuous on $C_u^*(X, E)$.

Proof. (i) For every $x \notin \text{supp}(\mu)$ take a neighborhood $U(x)$ of x in βX such that $\mu(g) = 0$ provided $g \in C(X, E)$ and $(\beta g)(\beta X - U(x)) = 0$. We can suppose that all $U(x)$ coincide with the interior of their closures in βX and are disjoint from $\text{supp}(\mu)$. There is a finite cover $\gamma = \{U, U(x_i) : i = 1, 2, .., k\}$ of βX and a partition of unity $\{h, h_i : i = 1, .., k\}$ subordinated to γ. Now, suppose $(\beta f)(U) = 0$ for some $f \in C(X, E)$. Set $g_0 = h \cdot f$ and $g_i = h_i \cdot f$, $i = 1, .., k$. Since $f = \Sigma\{g_i : i = 0, 1, .., k\}$, $\mu(f) = \Sigma\{\mu(g_i) : i = 0, 1.., k\}$. Obviously, $g_0 \equiv 0$, so $\mu(g_0) = 0$. Observe that each g_i is 0 on $X - U(x_i)$, and because $X - U(x_i)$ is dense in $\beta X - U(x_i)$ we have $(\beta g_i)(\beta X - U(x_i)) = 0$. Hence $\mu(g_i) = 0$ for all $i = 0, 1, .., k$, which yields $\mu(f) = 0$.

(ii) First we prove that $f \in C(X, E)$ and $(\beta f)|\text{supp}(\mu) = 0$ imply $\|\mu(f)\| \leq 1$. Indeed, fix such f and choose $\varepsilon > 0$ satisfying the condition: $\|\mu(g)\| \leq 1$ for all $g \in C^*(X, E)$ whose uniform norm is $\leq \varepsilon$. Let $\eta : \beta E \to [0, \infty]$ be the continuous extension of $\|\cdot\| : E \to (0, \infty)$, and $V = \{x \in \beta X : \eta((\beta f)(x)) \leq \varepsilon\}$. Define $f_\varepsilon \in C^*(X, E)$, $f_\varepsilon(x) = f(x)$ if $\|f(x)\| \leq \varepsilon$ and $f_\varepsilon(x) = (\varepsilon \cdot f(x))/\|f(x)\|$ otherwise.

Since $(f-f_\varepsilon)|(V\cap X)=0$ and $V\cap X\subset V$ is dense, $(\beta(f-f_\varepsilon))(V)=0$. Observe that V is a neighborhood of $\mathrm{supp}(\mu)$. So, by (i), we have $\mu(f-f_\varepsilon)=0$, i.e. $\mu(f)=\mu(f_\varepsilon)$. Finally, because the uniform norm of f_ε is $\le\varepsilon$, we obtain $\|\mu(f)\|=\|\mu(f_\varepsilon)\|\le 1$.

We are now in a position to finish the proof of (ii). Let $f\in C(X,E)$ and $(\beta f)|\mathrm{supp}(\mu)=0$. If $\mu(f)\neq 0$ we can choose n if necessary, such that $\|\mu(nf)\|>1$. It is easy to verify that $(\beta(nf))(\mathrm{supp}(\mu))=0$. This, according to the first part of our proof, implies $\|\mu(nf)\|\le 1$, a contradiction. □

LEMMA 2.2. *Let $\mu: C(X,E)\to F$ be a linear map such that $\mu(A)$ is bounded in F for every $A\in\mathcal{F}$. Then $\mathrm{supp}(\mu)$ is contained in the Hewitt realcompactification νX of X.*

Proof. Suppose there is $x\in\mathrm{supp}(\mu)-\nu X$. Take a decreasing sequence $\{U_n\}$ of compact neighborhoods of x in βX such that $\bigcap_{n=1}^{\infty} U_n\subset\beta X-\nu X$ and for every n choose $f_n\in C(X,E)$ with $(\beta f_n)(\beta X-U_n)=0$ and $\mu(f_n)\neq 0$. Then for any $\lambda\in\mathbb{R}$ we have $(\beta(\lambda\cdot f_n))(\beta X-U_n)=0$. Therefore, multiplying f_n by a scalar, if necessary, we can assume that $\|\mu(f_n)\|\ge n$ for all n. Since any compact subset of X can meet only finitely many U_n, we have that $\{f_n: n\in\mathbb{N}\}\in\mathcal{F}$. But this contradicts the fact that $\{\mu(f_n): n\in\mathbb{N}\}$ is unbounded in F. □

Recall that X is said to be Diedonné complete if there is a complete uniformity generating the topology of X, or equivalently, X is homeomorphic to a closed subspace of a product of metrizable spaces ([D], [E]). It is well known [D] that Dieudonné completeness is hereditary with respect to closed subsets.

LEMMA 2.3. *Let μ be as in Lemma 2.2. Suppose X is Dieudonné complete and $\mathrm{supp}(\mu)\cap\beta X-X\neq\emptyset$. Then there exists $A\in\mathcal{F}$ satisfying the following condition:*

(*) for every compact set $H\subset X$ there is $g\in A$ such that $g(H)=0$ and $\|\mu(g)\|\ge 1$.

Proof. Part of our argument follows the reasoning of Uspenskij from the proof of Lemma 7 in [U1]. Since X is Dieudonné complete, it is an intersection of paracompact subsets of βX. So, there are paracompact $Z\subset\beta X$ containing X and $x^*\in K\cap(\beta X-Z)$, $K=\mathrm{supp}(\mu)$. We have $K\subset\nu X$ (by Lemma 2.2), which yields $K\cap Z$ is closed and t-bounded in Z. Thus, $K\cap Z$ is compact because Z is μ-complete (as a paracompact). Take open neighborhoods U, W of x^* in βX such that $\mathrm{cl}_{\beta X}(U)\subset W$ and $\mathrm{cl}_{\beta X}(W)\cap(K\cap Z)=\emptyset$. Next, using the paracompactness of Z we can construct a locally finite open cover $\lambda=\{V_\alpha:\alpha\in\Lambda\}$ of Z and a partition of unity $\{h_\alpha:\alpha\in\Lambda\}$ subordinated to λ such that:

(1) if $V_\alpha\cap U\neq\emptyset$ for some $\alpha\in\Lambda$, then $\mathrm{cl}_{\beta X}(V_\alpha)\subset W$;

(2) $cl_{\beta X}(V_\alpha)\cap\mathrm{cl}_{\beta X}(W)\cap K=\emptyset$ for all $\alpha\in\Lambda$.

Choose $f \in C(X,E)$ such that $(\beta f)(\beta X - U) = 0$ and $\mu(f) \neq \emptyset$. Multiplying f by a constant, if necessary, we can suppose that $\|\mu(f)\| \geq 1$. Let Γ be the set of all finite subsets of Λ and for each $\gamma \in \Gamma$ define $f_\gamma \in C(X,E)$ by $f_\gamma = f \cdot (1 - \varphi_\gamma)$, where $\varphi_\gamma = \Sigma\{h_\alpha : \alpha \in \gamma\}$. Observe that $A = \{f_\gamma : \gamma \in \Gamma\} \in \mathcal{F}$. Let $H \subset X$ be compact. Then there is $\gamma(H) \in \Gamma$ such that $H \cap V_\alpha \neq \emptyset$ implies $\alpha \in \gamma(H)$. Obviously $\varphi_{\gamma(H)}(H) = 1$, so $f_{\gamma(H)}(H) = 0$. It remains only to show that $\|\mu(f_{\gamma(H)})\| \geq 1$.

We claim that $\mu(f \cdot \varphi_{\gamma(H)}) = 0$. Let $\gamma_U(H)$ be the set $\{\alpha \in \gamma(H) : V_\alpha \cap U \neq \emptyset\}$. Define $P = \cup\{\mathrm{cl}_{\beta X}(V_\alpha) : \alpha \in \gamma_U(H)\}$ and $G = \beta X - P$. By (1), $P \subset W$ and, by (2), $K \cap W \subset G$. Hence $K \subset G$. Suppose $x \in G \cap X$. If $x \notin U$, then $f(x) = 0$; if $x \in U$, then $h_\alpha(x) = 0$ for all $\alpha \in \gamma(H) - \gamma_U(H)$ and, since $x \in G$, we have $h_\alpha(x) = 0$ for all $\alpha \in \gamma_U(H)$. Therefore $x \in U$ implies $\varphi_{\gamma(H)}(x) = 0$. Consequently, $(f \cdot \varphi_{\gamma(H)})(G \cap X) = 0$. Then $\beta(f \cdot \varphi_{\gamma(H)})(G) = 0$ because $G \cap X$ is dense in G. Now, Lemma 2.1(i) yields $\mu(f \cdot \varphi_{\gamma(H)}) = 0$. The claim is proved.

We proceed to the proof of Lemma 2.3. Since $\mu(f \cdot \varphi_{\gamma(H)}) = 0$ and $f_{\gamma(H)} = f - f \cdot \varphi_{\gamma(H)}$, we have $\mu(f_{\gamma(H)}) = \mu(f)$. Therefore, $\|\mu(f_{\gamma(H)})\| \geq 1$. □

THEOREM 2.4. *Let X be Dieudonné complete and F be a Banach space. Then $M_{\mathcal{A}}(X,E,F)$ is a complete uniform space for every $\mathcal{A} \subset \mathcal{C}$ such that $\mathcal{A}$ is invariant with respect to finite unions and contains $\mathcal{F}$.*

Proof. It is easily seen that the completion of $M_{\mathcal{A}}(X,E,F)$ is the set of all linear maps $\nu : C(X,E) \to F$ such that for every $\varepsilon > 0$ and $A \in \mathcal{A}$ there is $\mu \in M_{\mathcal{A}}(X,E,F)$ with $\|\mu(f) - \nu(f)\| \leq \varepsilon$ for all $f \in A$. Fix such ν.

Assume $\mathrm{supp}(\nu)$ meets $\beta X - X$. Since $\nu(A)$ is bounded in F for every $A \in \mathcal{A}$, we can apply Lemma 2.3. So, there is $B \in \mathcal{F}$ satisfying the condition (*) from Lemma 2.3. Take $\mu \in M_{\mathcal{A}}(X,E,F)$ with $\|\mu(f) - \nu(f)\| \leq 1/2$ for all $f \in B$. Because μ is continuous on $C_k(X,E)$, $K = \mathrm{supp}(\mu)$ is a compact set in X (see [V], Proposition 2.2). According to (*), $\|\nu(g)\| \geq 1$ and $g|K = 0$ for some $g \in B$. Since μ is continuous on $C_u^*(X,E)$, $\mu(g) = 0$ (by Lemma 2.1(ii)). Thus, $\|\mu(g) - \nu(g)\| \geq 1$, which contradicts $g \in B$. Therefore, $\mathrm{supp}(\nu) \subset X$.

Because every convergent in $C_u^*(X,E)$ sequence together with its limit belongs to $\mathcal{F} \subset \mathcal{A}$, one can verify that $\nu|C_u^*(X,E)$ is continuous. Thus, by Lemma 2.1(ii), $\nu(f) = 0$ for each $f \in C(X,E)$ such that $f(H) = 0$, where $H = \mathrm{supp}(\nu)$. Since the restriction map $\pi_H : C_k(X,E) \to C_k(H,E)$ is surjective (see [V], Proposition 2.2, Claim 2), every $g \in C(H,E)$ has an extension $\bar{g} \in C(X,E)$. Then the equality $\nu_1(g) = \nu(\bar{g})$, defines a linear map $\nu_1 : C(H,E) \to F$. So, $\nu = \nu_1 \circ \pi_H$. Therefore, to show that ν is continuous on $C_k(X,E)$ it suffices to prove that ν_1 is continuous on $C_k(H,E)$ (recall that π_H is continuous). Take a convergent sequence $\{g_n\}$ in $C_k(H,E)$ and let g be its limit. Applying the Claim from Proposition 3.7 in [V], we can find a convergent sequence $\{f_n\} \subset C_u^*(X,E)$ such that $f_n|H = g_n$ for all n and $f|H = g$, where $f = \lim f_n$. Since $\{\nu(f_n)\}$ converges to $\nu(f)$, $\{\nu_1(g_n)\}$ converges to $\nu_1(g)$. Thus ν_1 is continuous. □

COROLLARY 2.5. *Let E be a Banach space and $\mathcal{A}$ be as in Theorem 2.4. Then $M_{\mathcal{A}}(X,E,E)$ is complete if and only if X is Dieudonné complete.*

Proof. Since the map $x \to \delta_x$, $\delta_x(f) = f(x)$ for every $f \in C(X,E)$, is a closed embedding of X into $M_{\mathcal{A}}(X,E,E)$, we have that X is Dieudonné complete provided $M_{\mathcal{A}}(X,E,E)$ is complete. The other implication follows from Theorem 2.4. □

Let Z be a subset of a space X. The G_δ-closure of Z in X is the set $G_\delta(Z)$ of all $x \in X$ satisfying the condition that every G_δ-subset of X containing x meets Z. When $G_\delta(Z) = Z$ we say that Z is G_δ-closed in X.

PROPOSITION 2.6. *Let $\mathcal{A}$ be a family of bounded sets in $C_k(X,E)$ which is invariant with respect to finite unions and covers $C(X,E)$. Then $M_{\mathcal{A},p}(X,E,F)$ is G_δ-closed in $M_{\mathcal{A}}(X,E,F)$.*

Proof. Let μ^* belong to the G_δ-closure of $M_{\mathcal{A},p}(X,E,F)$ in $M_{\mathcal{A}}(X,E,F)$. We claim that $\mu^* \in M_{\mathcal{A},p}(X,E,F)$. Denote by π the restriction map from $C_p(X,E)$ onto $C_p(H,E)$, $H = \operatorname{supp}(\mu^*)$. Then $\mu^* = \nu \circ \pi$ for some linear map $\nu : C(H,E) \to F$. Because π is continuous, it suffices to show that ν is continuous. To this end take $B \subset C_p(H,E)$ and g from the closure of B in $C_p(H,E)$. Since the tightness of $C_p(H,E)$ is countable (see [Ar1] for the case $E = \mathbb{R}$; the same proof remains true if $\mathbb{R}$ is replaced by any normed space) we can assume B is countable. Then, by the Claim in Proposition 3.8 from [V], there is countable $A \subset C(X,E)$ and $f \in C(X,E)$ such that $f \in \operatorname{cl}_{C_p(X,E)}(A)$, $f|H = g$ and $h|H \in B$ for every $h \in A$.

To finish the proof observe that the set

$$D = \{\mu \in M_{\mathcal{A}}(X,E,F) : \mu(h) = \mu^*(h), h \in A \cup \{f\}\}$$

is G_δ in $M_{\mathcal{A}}(X,E,F)$. So, there is $\mu_1 \in M_{\mathcal{A},p}(X,E,F) \cap D$. Because μ_1 is continuous on $C_p(X,E)$ and $f \in \operatorname{cl}_{C_p(X,E)}(A)$, we have $\mu_1(f) \in \operatorname{cl}_F(\mu_1(A))$. Since $\mu_1 \in D$, $\mu^*(f) \in \operatorname{cl}_F(\mu^*(A))$. But $\mu^*(f) = \nu(g)$ and $\mu^*(A) = \nu(B)$. Thus, $\nu(g) \in \operatorname{cl}_F(\nu(B))$. Finally, ν is continuous on $C_p(H,E)$. □

COROLLARY 2.7. *Let F be a Banach space and $\mathcal{A} \subset \mathcal{C}$ be invariant with respect to finite unions and $\mathcal{A}$ contain $\mathcal{F}$. Then $M_{\mathcal{A},p}(X,E,F)$ is Dieudonné complete if and only if X is Dieudonné complete.*

Proof. Sufficiency follows from Theorem 2.4, Proposition 2.6 and the fact that every G_δ-closed subset of a Dieudonné complete space is also Dieudonné complete (for the last fact see [D] and [E]). Necessity is trivial because X is homeomorphic to a closed set in $M_{\mathcal{A},p}(X,E,F)$. □

3. Dieudonné completeness. It is well known that every paracompact space is Dieudonné complete and that Dieudonné completeness implies μ-completeness. The first author proved that μ-completeness is preserved by continuous quotient linear surjections from $C_k(X,E)$ onto $C_k(Y,F)$, as well as from $C_p(X,E)$ onto $C_p(Y,F)$ [V] (see also [GO] for the case when $C_p(X)$ is linearly homeomorphic to $C_p(Y)$). On the other hand it was Rezničenko who showed [Ar2] that paracompactness is not preserved by linear homeomorphisms between $C_p(X)$ and $C_p(Y)$ (see also [AC]). It follows from [U2] that Dieudonné completeness is preserved by linear homeomorphisms between $C_k(X)$ and $C_k(Y)$ if X or Y is a k-space. In the frame of the above results the following question seems to be interesting:

Is it true that Dieudonné completeness is preserved by linear homeomorphisms between $C_k(X,E)$ and $C_k(Y,F)$ (resp., $C_p(X,E)$ and $C_p(Y,F)$)?

The next two theorems answer this question positively in the class of k-spaces.

THEOREM 3.1. *Let $u : C_k(X,E) \to C_k(Y,F)$ be a continuous quotient linear surjection, where F is a Banach space and Y is a k-space. Then Y is Dieudonné complete provided X is.*

Proof. Define $u^* : Y \to M(X,E,F)$, $u^*(y)(f) = u(f)(y)$ for every $f \in C(X,E)$. Since $M_{\mathcal{F}}(X,E,F)$ is complete (by Theorem 2.4), it is enough to show that u^* is a closed embedding into $M_{\mathcal{F}}(X,E,F)$. It is easily seen that u^* is injective (because u is a surjection) and that $(u^*)^{-1}$ is continuous. So, it remains to prove that u^* is continuous and $u^*(Y) \subset M_{\mathcal{F}}(X,E,F)$ is closed.

Before proving continuity of u^* let us note that the topology on $M_{\mathcal{F}}(X,E,F)$ is generated by the family of all seminorms $p_A(\mu) = \sup\{\|\mu(f)\| : f \in A\}$, $A \in \mathcal{F}$. Fix a compact set $K \subset Y$ and a net $\{y_\alpha : \alpha \in \Lambda\} \subset K$ converging to $y_0 \in K$. We claim that $\{u^*(y_\alpha) : \alpha \in \Lambda\}$ converges to $u^*(y_0)$. Set $H = \mathrm{cl}_{\beta X}(\cup\{\mathrm{supp}(u^*(y)) : y \in K\})$ and let $A \in \mathcal{F}$. By [V], Corollary 3.3, $H \subset X$ is t-bounded. Hence H is compact because X is Dieudonné complete. Both $\pi_H : C_k(X,E) \to C_k(H,E)$ and $\pi_K : C_k(Y,F) \to C_k(K,F)$ are continuous open surjections and since $\pi_H(f) = \pi_H(g)$ implies $\pi_K(u(f)) = \pi_K(u(g))$ for every $f, g \in C(X,E)$, we have that $\varphi = \pi_K \circ u \circ (\pi_H)^{-1} : C_k(H,E) \to C_k(K,F)$ is continuous. If $A \in \mathcal{F}$, then $\varphi(\pi_H(A))$ is compact, so it is equicontinuous (by the Ascoli Theorem). The last yields that $\{p_A(u^*(y_\alpha) - u^*(y_0)) : \alpha \in \Lambda\}$ converges to 0 for all $A \in \mathcal{F}$. So, $\{u^*(y_\alpha) : \alpha \in \Lambda\}$ converges to $u^*(y_0)$. We proved that u^* is continuous on every compact $K \subset Y$. Consequently, u^* is continuous because Y is a k-space.

To show that $u^*(Y) \subset M_{\mathcal{F}}(X,E,F)$ is closed let $\{u^*(y(\gamma)) : \gamma \in \Gamma\}$ be a net in $u^*(Y)$ converging to $\mu \in M_{\mathcal{F}}(X,E,F)$. Then $\mu(f) = \lim u(f)(y(\gamma))$ for every $f \in C(X,E)$. So we can define $\nu \in M(Y,F,F)$ by $\nu(h) = \mu(f)$, where $h \in C(Y,F)$ and $f \in C(X,E)$ with $u(f) = h$. Since u is quotient, ν is continuous on $C_k(Y,F)$ and $\nu(h) = \lim h(y(\gamma))$ for each $h \in C(Y,F)$. Let $M_p(Y,F,F)$ denote the set $M(Y,F,F)$ with the pointwise convergence topology. Then $\delta(Y) = \{\delta_y : y \in Y\} \subset M_p(Y,F,F)$ is closed and homeomorphic to Y. Observe that $\{\delta_{y(\gamma)} : \gamma \in \Gamma\}$ converges to ν in $M_p(Y,F,F)$. Therefore, $\nu \in \delta(Y)$. Consequently, there is $z \in Y$ such that $\nu = \delta_z$ and $\mu = u^*(z)$. So, $u^*(Y)$ is closed in $M_{\mathcal{F}}(X,E,F)$. □

THEOREM 3.2. *Let $u : C_p(X, E) \to C_p(Y, F)$ be a continuous quotient linear surjection, where E and F are Banach spaces and Y is a k-space. Then Y is Dieudonné complete if X is.*

Proof. We follow very closely the proof of Theorem 3.1. By [V, Proposition 3.6], $u : C_k(X, E) \to C_k(Y, F)$ is also continuous. Observe that the proof of the continuity of u^* in Theorem 3.1 does not require u to be quotient, it suffices u to be a continuous surjection and X to be Dieudonné complete. So, in our case u^* is continuous. Since $u : C_p(X, E) \to C_p(Y, F)$ is continuous, u^* maps Y into $M_{\mathcal{F},p}(X, E, F)$. Using that u is quotient (as a map between $C_p(X, E)$ and $C_p(Y, F)$) and repeating the same arguments as in the previous theorem we can show that $u^*(Y) \subset M_{\mathcal{F},p}(X, E, F)$ is closed. So, Y is homeomorphic to a losed subset of $M_{\mathcal{F},p}(X, E, F)$. Because $M_{\mathcal{F},p}(X, E, F)$ is Dieudonné complete (by Corollary 3.7), Y is also Dieudonné complete. □

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