

## SPACES OF BOUNDED FUNCTIONS

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ABSTRACT. For a completely regular space  $X$  and a Banach space  $E$  let  $C_k^*(X, E)$  be the set of all  $E$ -valued bounded continuous maps on  $X$  endowed with the compact-open topology. We prove that some topological properties of  $X$  are determined by the linear topological structure of  $C_k^*(X, E)$ .

### 1. INTRODUCTION

Throughout this paper  $E$  and  $F$  are normed linear spaces unless stated otherwise; space always means a completely regular space;  $C(X, E)$  (resp.  $C^*(X, E)$ ) denotes the set of all continuous (resp., continuous and bounded)  $E$ -valued maps on  $X$ . If  $E$  is the real line  $\mathbb{R}$  we write  $C(X)$  or  $C^*(X)$ . When the function spaces are endowed with the compact open or pointwise convergence topology the subscripts  $k$  or  $p$ , respectively, are used.

There are many results treating topological properties of  $X$  which are determined by the linear topological structure of  $C_p(X)$  and  $C_k(X)$ , or equivalently, topological properties which are preserved by linear homeomorphisms between corresponding function spaces (see [1], [3], [6], [13]). The same question concerning bounded functions is no less interesting and it appears to be more complicated. For example, one of the most beautiful and nontrivial results in  $C_p^*$ -theory is that Čech completeness is preserved by continuous linear surjections  $\phi : C_p^*(X) \rightarrow C_p^*(Y)$  with  $X$  and  $Y$  metric spaces [5, Theorem 4.2]. The corresponding result for unbounded function spaces also holds [5, Theorem 3.3], but its proof is quite different from the proof of the first one.

The author [14] has shown that some topological properties are preserved by continuous linear surjections (or linear homeomorphisms) between  $C_k^*(X)$  and

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$C_k^*(Y)$ . The main purpose of this paper is to give further generalizations to some results from [14] and to present new ones in the more general situation of normed-valued function spaces.

The paper is organized as follows. Section 2 is devoted to the main technical tools. In Section 3 we prove that properties like compactness, realcompactness, local compactness,  $\mu$ -completeness and Čech completeness are preserved by continuous quotient linear surjections  $\phi : C_k^*(X, E) \rightarrow C_k^*(Y, F)$  under some restrictions on  $X$  and  $Y$ . A few words about these restrictions, most often we require  $X$  to be  $\mu$ -complete and  $Y$  to be a *generalized sq-space*. The class of *sq-spaces* was introduced in [14] and it is supposed to be of the same importance for bounded function spaces as the class of *wq-spaces* [13] is for unbounded function spaces. Generalized *sq-spaces* is the minimal class containing all *sq-spaces* and which is invariant under perfect images and perfect preimages. It is shown in Section 5 that these restrictions are essential. The restriction  $\phi$  to be quotient also can not be omitted because there is a natural continuous linear surjection from  $C_k^*(\beta X)$  onto  $C_k^*(X)$ .

Section 3 is devoted to properties which are preserved by linear injections and linear homeomorphisms. When  $\phi : C_k^*(X, E) \rightarrow C_k^*(Y, F)$  is a continuous linear injection, assertions of the following type are proved: if  $Y$  has a dense subset with a property  $\mathcal{P}$ , then  $X$  has a dense subset with the same property  $\mathcal{P}$ . When  $\phi$  is a linear homeomorphism and  $X, Y$  are  $\mu$ -complete generalized *sq-spaces*, then any perfect property (i.e. property which is preserved by perfect maps and by preimages of perfect maps) is a linear topological invariant (Corollary 4.6). Metrizable (which is not a perfect property) is another such an invariant provided  $E$  and  $F$  are separable (Theorem 4.7). Analogues of the last two results were proved by M. Choban [6] for unbounded function spaces.

As we already noted, Section 5 contains an example which shows that generalized *sq-space* property is essential in our results. In the final Section 6 we discuss the validity of our results for bounded function spaces with topologies weaker than the compact open topology. For  $\mu$ -complete spaces it is shown that every result of that type which holds for the compact open topology remains valid and for function spaces  $C_\alpha^*(X, E)$  such that  $\alpha$  is a family of compact sets in  $X$  containing all separable compacta (see Proposition 6.1). This section also contains some results about pointwise convergence topology and some questions.

## 2. PRELIMINARIES

Suppose  $\phi$  is a linear map from  $C^*(X, E)$  into  $C(Y, F)$ . For every  $y \in \beta Y$  we can define the maps  $\mu_y : C^*(X, E) \rightarrow \beta F$  and  $\varphi_y : C^*(X, E) \rightarrow [0, \infty]$  by  $\mu_y(f) = \beta\phi(f)(y)$  and  $\varphi_y(f) = \|\mu_y(f)\|$ , where  $\beta Z$  denotes the Čech-Stone compactification of a given space  $Z$  and  $\|\cdot\| : \beta F \rightarrow [0, \infty]$  is the continuous extension of the norm  $\|\cdot\|$  of  $F$  considered as a function from  $F$  into  $[0, \infty)$ . One can show that if  $\mu_y(f) \in F$  for every  $f \in C^*(X, E)$  (this is, for example, the case when  $y \in Y$ ), then  $\mu_y$  is a linear map from  $C^*(X, E)$  into  $F$ ; also  $\sup\{\|\phi(f)(z)\| : z \in Y\} \leq M$  for some  $M > 0$  yields  $\varphi_y(f) \leq M$  for every  $y \in \beta Y$ . Now we extend the operations  $a + b$  and  $|a - b|$  on  $[0, \infty]$  by defining  $\infty + a = \infty$  for every  $a \in [0, \infty]$ ,  $|\infty - a| = |a - \infty| = \infty$  for  $a \in [0, \infty)$  and  $|\infty - \infty| = 0$ .

**Lemma 2.1.** *Let  $y \in \beta Y$  and  $f, g \in C^*(X, E)$ . Then the following are true:  $\varphi_y(f + g) \leq \varphi_y(f) + \varphi_y(g)$ ,  $|\varphi_y(f) - \varphi_y(g)| \leq \varphi_y(f - g)$  and  $\varphi_y(\lambda \cdot f) = \lambda \cdot \varphi_y(f)$  for every  $\lambda \geq 0$ .*

PROOF. Take a net  $\{y_\alpha\}$  in  $Y$  converging to  $y$ . Then for each  $\alpha$  we have  $|\|\phi(f)(y_\alpha)\| - \|\phi(g)(y_\alpha)\|| \leq \|\phi(f - g)(y_\alpha)\|$ . This implies  $|\varphi_y(f) - \varphi_y(g)| \leq \varphi_y(f - g)$  because the net  $\{\phi(h)(y_\alpha)\}$  converges to  $\beta\phi(h)(y)$  for any  $h \in C^*(X, E)$  and the map  $\|\cdot\|$  is continuous. The same arguments can be used for the other inequality. To prove that  $\varphi_y(\lambda \cdot f) = \lambda \cdot \varphi_y(f)$  it is enough to observe that  $\varphi_{y_\alpha}(\lambda \cdot f) = \lambda \cdot \varphi_{y_\alpha}(f)$  for every  $\alpha$  and that the nets  $\{\varphi_{y_\alpha}(\lambda \cdot f)\}$ ,  $\{\lambda \cdot \varphi_{y_\alpha}(f)\}$  converge, respectively, to  $\varphi_y(\lambda \cdot f)$  and  $\lambda \cdot \varphi_y(f)$ .  $\square$

The space  $C^*(X, E)$  with the uniform convergence topology is denoted by  $C_u^*(X, E)$ .

**Lemma 2.2.** *Let  $\phi$  be a continuous linear map from  $C_k^*(X, E)$  into  $C_p^*(Y, F)$ , where  $E$  and  $F$  are Banach spaces. Then all  $\varphi_y$ ,  $y \in \beta Y$ , are continuous on  $C_u^*(X, E)$ . If, in addition,  $\phi$  is a continuous linear map from  $C_p^*(X, E)$  into  $C_p^*(Y, F)$ , then each  $\varphi_y$ ,  $y \in \nu Y$ , considered as a map from  $C_p(\beta X, E)$  into  $[0, \infty)$ , is also continuous.*

PROOF. Let  $\phi : C_k^*(X, E) \rightarrow C_p^*(Y, F)$  be continuous and linear. Fix  $y \in \beta Y$  and a convergent sequence  $\{f_n\}$  in  $C_u^*(X, E)$  with  $\lim f_n = f$ . By the Closed Graph Theorem,  $\phi$  considered as a map from  $C_u^*(X, E)$  onto  $C_u^*(Y, F)$  is also continuous. Thus,  $\{\phi(f_n)\}$  converges uniformly to  $\phi(f)$ . In particular, for any  $\varepsilon > 0$  there is  $m \in \mathbf{N}$  with  $|\|\phi(f)(z)\| - \|\phi(f_n)(z)\|| \leq \varepsilon$  for all  $z \in Y$  and  $n \geq m$ . This implies that  $\{\|\beta\phi(f_n)(y)\|\}$  converges to  $\|\beta\phi(f)(y)\|$ . Hence  $\varphi_y$  is continuous on  $C_u^*(X, E)$ .

Suppose  $\phi : C_p^*(X, E) \rightarrow C_p^*(Y, F)$  is continuous and  $y \in \nu Y$ . Consider the map  $f \rightarrow \varphi_y(f|X)$ ,  $f \in C(\beta X, E)$ , which is denoted by  $\hat{\varphi}_y$ . To show that  $\hat{\varphi}_y$  is continuous on  $C_p(\beta X, E)$  choose  $B \subset C(\beta X, E)$  and  $f_0$  from the closure of  $B$  in  $C_p(\beta X, E)$ . Because the tightness of  $C_p(\beta X, E)$  is countable (see [2] for the case  $E = \mathbb{R}$  and  $X^n$  is Lindelöf for every  $n \in \mathbb{N}$ ; the same proof remains true if we consider  $C_p(X, E)$ , where  $E$  is a normed space), we can assume that  $B = \{f_n\}$  is countable. Then  $\phi(f_0|X)$  belongs to the closure (in  $C_p^*(Y, F)$ ) of the set  $\{\phi(f_n|X)\}$ . Because  $G = \bigcap \{z \in \beta Y : \hat{\varphi}_z(f_i) = \hat{\varphi}_y(f_i), i = 0, 1, 2, \dots\}$  is a  $G_\delta$ -subset of  $\beta Y$  containing  $y$ , there is  $z_0 \in Y \cap G$ . Since  $\phi(f_0|X)(z_0) \in cl_F \{\phi(f_n|X)(z_0)\}$  implies  $\hat{\varphi}_{z_0}(f_0) \in cl \{\hat{\varphi}_{z_0}(f_n)\}$  we have  $\hat{\varphi}_y(f_0) \in cl(\hat{\varphi}_y(B))$ . Thus  $\hat{\varphi}_y$  is continuous.  $\square$

**Remark.** In the proof of the second part of Lemma 2.2 we didn't use that  $\phi$  is linear, so it is true for any continuous map  $\phi$ .

Let  $\phi : C^*(X, E) \rightarrow C(Y, F)$  be a linear map. Then the support of  $\mu_y$  (resp.,  $\varphi_y$ ),  $y \in \beta Y$ , is the set  $supp(\mu_y)$  (resp.,  $supp(\varphi_y)$ ) of all  $x \in \beta X$  satisfying the condition that for every neighborhood  $U$  of  $x$  in  $\beta X$  there is  $f \in C^*(X, E)$  such that  $\beta f|(\beta X - U) = 0$  and  $\mu_y(f) \neq 0$  (resp.,  $\varphi_y(f) \neq 0$ ). Obviously,  $supp(\mu_y)$  and  $supp(\varphi_y)$  are closed in  $\beta X$ , and  $supp(\varphi_y) \subset supp(\mu_y)$  for every  $y \in \beta Y$ . Let us note the following very important observation: if  $y \in \beta Y$  is such that  $\mu_y(f) \in F$  for every  $f \in C^*(X, E)$ , then  $\mu_y(f) = 0$  if and only if  $\varphi_y(f) = 0$ . In such a case  $supp(\varphi_y) = supp(\mu_y)$ .

The following lemma is a generalization of [15, Lemma 2.1] (see also [12, Lemma 4] and we follow the same scheme of proof.

**Lemma 2.3.** *Let  $y \in \beta Y$  and  $U$  be a neighborhood of  $supp(\varphi_y)$  in  $\beta X$ . Then we have:*

- (i)  $\varphi_y(f) = 0$  for every  $f \in C^*(X, E)$  with  $\beta f|U = 0$ ;
- (ii)  $\varphi_y(f) = 0$  for every  $f \in C^*(X, E)$  with  $\beta f|supp(\varphi_y) = 0$  provided  $\varphi_y$  is continuous on  $C_u^*(X, E)$ .

**PROOF.** (i) For every  $x \notin supp(\varphi_y)$  take a neighborhood  $U(x)$  of  $x$  in  $\beta X$  such that  $\varphi_y(g) = 0$  provided  $g \in C^*(X, E)$  and  $\beta g|(\beta X - U(x)) = 0$ . We can suppose that all  $U(x)$  coincide with the interior of their closures in  $\beta X$  and are disjoint from  $supp(\varphi_y)$ . There is a finite cover  $\gamma = \{U, U(x_i) : i = 1, 2, \dots, k\}$  of  $\beta X$  and a partition of unity  $\{h, h_i : i = 1, \dots, k\}$  subordinated to  $\gamma$ . Now, suppose  $\beta f|U = 0$  for some  $f \in C^*(X, E)$ . Set  $g_0 = h \cdot f$  and  $g_i = h_i \cdot f$ . Obviously,  $g_0 \equiv 0$ , so  $\varphi_y(g_0) = 0$ . Observe that each  $g_i|(\beta X - U(x_i)) = 0$ ,  $i = 1, \dots, k$ , and because

$X - U(x_i)$  is dense in  $\beta X - U(x_i)$  we have  $\beta g_i | (\beta X - U(x_i)) = 0$ . Hence  $\varphi_y(g_i) = 0$  for all  $i = 0, 1, \dots, k$ . Finally, since  $f = \sum \{g_i : i = 0, 1, \dots, k\}$ , by Lemma 2.1, we obtain  $\varphi_y(f) \leq \sum \{\varphi_y(g_i) : i = 0, 1, \dots, k\}$ . Therefore  $\varphi_y(f) = 0$ .

(ii) First we prove that  $f \in C^*(X, E)$  and  $\beta f | \text{supp}(\varphi_y) = 0$  imply  $\varphi_y(f) \leq 1$ . Indeed, fix such  $f$  and choose  $\varepsilon > 0$  satisfying the condition  $\varphi_y(g) \leq 1$  for all  $g \in C^*(X, E)$  with  $\|g\|_u \leq \varepsilon$ , where  $\|g\|_u = \sup \{\|g(x)\| : x \in X\}$ . Let  $\eta : \beta E \rightarrow [0, \infty]$  be the continuous extension of  $\|\cdot\|$  considered as a function from  $E$  into  $[0, \infty)$ , and  $V = \{x \in \beta X : \eta((\beta f)(x)) < \varepsilon\}$ . Define  $f_\varepsilon \in C^*(X, E)$ ,  $f_\varepsilon(x) = f(x)$  if  $\|f(x)\| < \varepsilon$  and  $f_\varepsilon(x) = (\varepsilon \cdot f(x)) \cdot (\|f(x)\|)^{-1}$  otherwise. Since  $(f - f_\varepsilon)(V \cap X) = 0$  and  $V \cap X$  is dense in  $V$ ,  $(\beta(f - f_\varepsilon))(V) = 0$ . Observe that  $V$  is a neighborhood of  $\text{supp}(\varphi_y)$ . So, by (i), we have  $\varphi_y(f - f_\varepsilon) = 0$ . Because  $|\varphi_y(f) - \varphi_y(f_\varepsilon)| \leq \varphi_y(f - f_\varepsilon)$  (see Lemma 2.1), we obtain  $\varphi_y(f) = \varphi_y(f_\varepsilon)$ . But  $\|f_\varepsilon\|_u \leq \varepsilon$  implies  $\varphi_y(f) \leq 1$ , hence  $\varphi_y(f) \leq 1$ .

We are now in a position to finish the proof of (ii). Let  $f \in C^*(X, E)$  and  $\beta f | \text{supp}(\varphi_y) = 0$ . If  $\varphi_y(f) \neq 0$  we can choose  $n \in \mathbb{N}$  if necessary, such that  $\varphi_y(n \cdot f) > 1$  (see Lemma 2.1). It is easy to verify that  $\beta(n \cdot f) | (\text{supp}(\varphi_y)) = 0$ . This, according to the first part of our proof, implies  $\varphi_y(n \cdot f) \leq 1$ , a contradiction.  $\square$

**Corollary 2.4.** *Let  $\phi : C^*(X, E) \rightarrow C(Y, F)$  be a linear map such that either  $\phi(C^*(X, E))$  contains at least one constant non-zero map or it separates the points of  $Y$ . Then  $\text{supp}(\varphi_y) \neq \emptyset$  for every  $y \in \beta Y$ .*

PROOF. We consider only the case when  $\phi(C^*(X, E))$  contains a non-zero constant, the other one is similar. Assuming that  $\text{supp}(\varphi_y) = \emptyset$  for some  $y \in \beta Y$ , Lemma 2.3(i) implies  $\varphi_y(f) = 0$  for each  $f \in C^*(X, E)$ . On the other hand, let  $g = \phi(h) \in C^*(Y, F)$  be a constant non-zero map with  $h \in C^*(X, E)$ . Then  $\varphi_y(h) \neq 0$ , a contradiction.  $\square$

Recall that a set-valued map  $\Phi : Z \rightarrow T$ , possibly  $\Phi(z) = \emptyset$  for some  $z \in Z$ , is lower semi continuous (br., lsc) if  $\Phi^{-1}(U) = \{z \in Z : \Phi(z) \cap U \neq \emptyset\}$  is open in  $Z$  whenever  $U \subset T$  is open.

**Lemma 2.5.** *Let  $\phi$  be a continuous linear map from  $C_k^*(X, F)$  into  $C_p(Y, F)$ . Then both  $\text{supp}(\varphi_y)$  and  $\text{supp}(\mu_y)$ , considered as set-valued maps from  $\beta Y$  to  $\beta X$ , are lsc and  $\text{supp}(\mu_y) \subset X$  for every  $y \in Y$ .*

PROOF. Let  $x \in \text{supp}(\varphi_y) \cap U$  for some open  $U \subset \beta X$  and  $y \in \beta Y$ . Choose  $f \in C^*(X, E)$  such that  $\beta f | (\beta X - V) = 0$  and  $\varphi_y(f) \neq 0$ , where  $V \subset \beta X$  is open with  $x \in V$  and  $cl_{\beta X}(V) \subset U$ . Let  $W = \{z \in \beta Y : \|\beta \phi(f)(z)\| > 0\}$ . Obviously,  $W$  is a neighborhood of  $y$  in  $\beta Y$ . Suppose  $\text{supp}(\varphi_{z_0}) \cap U = \emptyset$  for some

$z_0 \in W$ . Then, since  $\beta X - V$  is a neighborhood of  $\text{supp}(\varphi_{z_0})$ , Lemma 2.3(i) yields  $\varphi_{z_0}(f) = 0$ . The last one contradicts  $z \in W$ . Hence  $\text{supp}(\varphi_z) \cap U \neq \emptyset$  for any  $z \in W$ , i.e.  $\text{supp}(\varphi_y)$  is *lsc*. By similar arguments,  $\text{supp}(\mu_y)$  is also *lsc*.

It remains only to show that  $\text{supp}(\mu_y) \subset X$  for every  $y \in Y$ . And this is really true since  $\mu_y$  is a continuous linear map from  $C_k^*(X, E)$  into  $F$  when  $y \in Y$  (see [13, Proposition 2.2] for the case when  $\mu_y$  is continuous on  $C_k(X, E)$ ; the same proof is valid if  $\mu_y$  is continuous on  $C_k^*(X, E)$ ).  $\square$

For a space  $X$  and a cardinal  $\tau$  let  $\nu_\tau X$  be the set of all  $x \in \beta X$  such that every  $G_\tau$ -set in  $\beta X$  containing  $x$  meets  $X$ . If  $X = \nu_\tau X$  we say that  $X$  is  $\tau$ -realcompact. Obviously,  $\nu_\tau X \subset \nu_\omega X = \nu X$  for every infinite  $\tau$ .

**Corollary 2.6.** *In the hypotheses of Lemma 2.5,  $\text{supp}(\varphi_y) \subset \nu_\tau X$  for every  $y \in \nu_\tau Y$ .*

PROOF. This corollary in the particular case when  $E$  and  $F$  are the real line was proved in [14, Corollary 2.2]. The same scheme works and in the general case, we use only that  $\text{supp}(\varphi_y)$  is *lsc* and  $\text{supp}(\varphi_y) \subset X$  for every  $y \in Y$  (see Lemma 2.5).  $\square$

**Lemma 2.7.** *Let  $\phi$  be a quotient linear continuous surjection from  $C_k^*(X, E)$  onto a subspace  $\mathcal{A}$  of  $C_k^*(Y, F)$  (resp., of  $C_p^*(Y, F)$ ) with  $E$  and  $F$  Banach spaces and  $\mathcal{A}$  generates the topology of  $Y$ . Then for every compact set  $H \subset X$  the following holds:*

$$H^\# = \{y \in \beta Y : \text{supp}(\varphi_y) \subset H\} \text{ is a compact subset of } Y.$$

PROOF. Because  $\text{supp}(\varphi_y) : \beta Y \rightarrow \beta X$  is *lsc* (Lemma 2.5),  $H^\# \subset \beta Y$  is closed. It remains to show that  $H^\# \subset Y$ . To this end fix  $y \in H^\#$ .

**Claim.**  $\varphi_y$  is continuous on  $C_k^*(X, E)$ .

Since the restriction map  $\pi : C_k^*(X, E) \rightarrow C_k(H, E)$  is a continuous surjection (see [13, Proposition 2.2, Claim 2]) and  $\pi(f) = \pi(g)$  implies  $\varphi_y(f) = \varphi_y(g)$  (Lemmas 2.2 and 2.3(ii)), it is enough to show that the map  $\psi_y : C_k(H, E) \rightarrow [0, \infty)$ ,  $\psi_y(\pi(f)) = \varphi_y(f)$ , is continuous. And that is certainly true because  $\varphi_y$  is continuous on  $C_u^*(X, E)$  (Lemma 2.2) and  $\pi$ , considered as a map from  $C_u^*(X, E)$  onto  $C_k(H, E)$ , is continuous and open (by the Open Mapping Theorem). The claim is proved.

Let us finish the proof of Lemma 2.7. Because  $\phi$  is quotient and for any two  $f, g \in C^*(X, E)$  we have  $\varphi_y(f) = \varphi_y(g)$  provided  $\phi(f) = \phi(g)$ , the map  $\xi : \mathcal{A} \rightarrow [0, \infty)$ , defined by  $\xi(\phi(h)) = \varphi_y(h)$ , is continuous. Since the value of

$\xi$  at the constant function 0 is 0, there is a compact (resp., finite) set  $P \subset Y$  and  $\eta > 0$  such that  $\xi(g) < 1$  for every  $g \in \mathcal{A}$  with  $\|g(z)\| < \eta$  for all  $z \in P$ . This is equivalent to the following one:  $\|\beta g(y)\| < 1$  for every  $g \in \mathcal{A}$  with  $\sup\{\|g(z)\| : z \in P\} < \eta$ . Since  $\mathcal{A}$  generates the topology of  $Y$ , the last yields  $y \in P$ . Thus,  $H^\# \subset Y$ .  $\square$

**Corollary 2.8.** *In the hypotheses of Lemma 2.7 the set  $Y$  coincides with the set  $X^\# = \{y \in \beta Y : \text{supp}(\varphi_y) \subset K, K \subset X \text{ is compact}\}$ .*

PROOF. By Lemma 2.7,  $X^\# \subset Y$ . The converse inclusion follows from the fact that  $\text{supp}(\mu_y) = \text{supp}(\varphi_y) \subset X$  for each  $y \in Y$ .  $\square$

**Lemma 2.9.** *Let  $\phi$  be a quotient linear continuous surjection from  $C_p^*(X, E)$  onto a subspace  $\mathcal{A}$  of  $C_p^*(Y, F)$ , where  $E$  and  $F$  are Banach spaces and  $\mathcal{A}$  generates the topology of  $Y$ . Then, for every compact  $H \subset X$  we have  $H^* = \{y \in \nu Y : \text{supp}(\varphi_y) \subset H\} \subset Y$ .*

PROOF. Fix  $y \in H^*$ . Following the proof of Lemma 2.7, it is enough to show that  $\varphi_y$  is continuous on  $C_p^*(X, E)$ . This can be done in the same way as in Lemma 2.7 using the following facts: the restriction maps  $\pi : C_p^*(X, E) \rightarrow C_p(H, E)$  and  $\pi_1 : C_p(\beta X, E) \rightarrow C_p(H, E)$  are open and continuous surjections, and  $\varphi_y$ , considered as a map on  $C_p(\beta X, E)$ , is continuous (Lemma 2.2).  $\square$

Lemma 2.7 and Lemma 2.9 are not true if the requirement for  $\phi$  to be quotient is omitted. For example, the restriction map from  $C_k(\beta X)$  onto  $C_k^*(X)$  is always a linear continuous surjection, but  $X$  is not always compact.

### 3. LINEAR SURJECTIONS

In this section we consider topological properties which are preserved by continuous (quotient) linear surjections from  $C_k^*(X, E)$  onto  $C_k^*(Y, F)$  (resp., from  $C_p^*(X, E)$  onto  $C_p^*(Y, F)$ ) where  $E$  and  $F$  are normed (or Banach if explicitly stated) spaces. First corollary follows directly from Lemma 2.7.

**Corollary 3.1.** *Let  $\phi$  be a quotient continuous linear surjection from  $C_k^*(X, E)$  onto  $C_k^*(Y, F)$  (resp., onto  $C_p^*(Y, F)$ ) with  $E$  and  $F$  Banach spaces. Then  $Y$  is compact provided  $X$  is.*

**Corollary 3.2.** *Let  $\phi$  be a quotient continuous linear surjection from  $C_k^*(X, E)$  onto  $C_k^*(Y, F)$  (resp., from  $C_p^*(X, E)$  onto  $C_p^*(Y, F)$ ) with  $E$  and  $F$  Banach spaces. Then  $Y$  is  $\tau$ -realcompact whenever  $X$  is.*

PROOF. Suppose  $X$  is  $\tau$ -realcompact and there is  $y \in \nu_\tau Y - Y$ . Then  $\text{supp}(\varphi_y) \subset X$  (Corollary 2.6), and by Corollary 2.8 (resp., Lemma 2.9),  $y \in Y$ , a contradiction. Hence,  $Y = \nu_\tau Y$ .  $\square$

Note that the above corollaries were proved in [14, Corollary 2.4] in the case of real-valued functions.

If  $\phi$  is a linear map from  $C^*(X, E)$  to  $C(Y, F)$  and  $A \subset \beta Y$  the set  $\text{supp}_{\beta X}(A) = \text{cl}_{\beta X} \cup \{\text{supp}(\varphi_y) : y \in A\}$  is called the support of  $A$ . When  $A \subset Y$  the set  $\text{cl}_X \cup \{\text{supp}(\varphi_y) : y \in A\}$  is denoted by  $\text{supp}(A)$ .

The following lemma was proved in [3, Lemma 1.5.6] when  $E = F = \mathbb{R}$ . The same arguments work and in the general case.

**Lemma 3.3.** *Let  $\phi$  be a continuous linear map from  $C_k^*(X, E)$  to  $C_k(Y, F)$  with  $E$  and  $F$  normed spaces. Then  $\text{supp}(K) \subset X$  is compact for every compact  $K \subset Y$ .*

A point  $x \in X$  is a *strongly  $q$ -point* (br., *sq-point*) in  $X$  if there exists a countable family  $\{U_n\}$  of neighborhoods of  $x$  in  $X$  such that any sequence  $\{x_n\}$  with  $x_n \in U_n$ ,  $n \in \mathbb{N}$ , has a compact closure in  $X$ . The set  $X^{(sq)}$  of all *sq-points* of  $X$  is called the *sq-derivative* of  $X$ . When  $X^{(sq)} = X$  we say that  $X$  is an *sq-space*. The class of *sq-spaces* was introduced in [14] and it contains all spaces of compact countable type (every compact  $K \subset X$  is contained in a compact  $H \subset X$  which is  $G_\delta$  in  $\beta X$ ), in particular all first countable spaces. The class of all *sq-spaces* can be extended by the same way as M.Choban [6] did for the class of *q-spaces*. We consider the minimal class of spaces containing all *sq-spaces* and which is stable with respect to perfect images and perfect preimages. The elements of that extended class are called *generalized sq-spaces*. Using the obvious fact that the class of *sq-spaces* is stable under taking perfect preimages, one can show that  $X$  is a generalized *sq-space* if and only if  $X$  is a perfect image of an *sq-space*.

A set  $H \subset X$  is called bounded in  $X$  if  $f(H)$  is bounded in  $\mathbb{R}$  for any  $f \in C(X)$ . When every closed and bounded set in  $X$  is compact we say that  $X$  is  $\mu$ -complete (or a  $\mu$ -space). The following criteria for boundedness is a modification of the well known one for pseudocompactness (see [8, Theorem 3.10.22]).

**Lemma 3.4.** *A set  $K \subset X$  is bounded if and only if every locally finite open family in  $X$  has a finite restriction on  $K$ .*

Lemma 3.5 below is an analogue to Lemma 3.1 from [14] (compare with [13, Proposition 3.14]). Recall that a set-valued map  $\psi : Y \rightarrow X$  is upper semi

continuous (br., *usc*) if the set  $\{y \in Y : \psi(y) \subset U\}$  is open in  $Y$  whenever  $U$  is open in  $X$ . Upper semi continuous compact-valued maps are called *usco* maps.

**Lemma 3.5.** *Let  $X$  be a  $\mu$ -space and  $\varphi : Y \rightarrow X$  be a set-valued map such that  $\varphi(K) \subset X$  is bounded whenever  $K$  is countable and has a compact closure in  $Y$ . Then for every generalized sq-subspace  $A$  of  $Y$  there is an usco map  $\theta : B \rightarrow \beta X$ , where  $B = cl_{\beta Y}(A)$ , such that  $\varphi(y) \subset \theta(y) \subset X$  for every  $y \in A$ .*

PROOF. Let  $H$  be an sq-space and  $g : H \rightarrow A$  a perfect surjection. Let  $h : P \rightarrow B$ ,  $P = \beta H$ , be the continuous extension of  $g$ . We define  $\psi : P \rightarrow \beta X$  by  $\psi(z) = \bigcap \{cl_{\beta X}(\varphi(h(U \cap H))) : U \in \mathcal{U}(z)\}$ , where  $\mathcal{U}(z)$  is the family of all neighborhoods of  $z$  in  $P$ . It is trivially to verify that  $\psi$  is *usco* and  $\varphi(h(z)) \subset \psi(z)$  for every  $z \in H$ .

Next step is to show that  $\psi(z) \subset X$  for every  $z \in H$ . To prepare for that proof, we need some preliminary considerations. For each  $z \in H$  choose a countable family  $\{U_n(z)\}$  of neighborhoods of  $z$  in  $H$  such that  $U_{n+1}(z) \subset U_n(z)$ ,  $n \in \mathbb{N}$ , and any sequence  $\{z_n\}$  with  $z_n \in U_n(z)$  has a compact closure in  $H$ . Let

$$\Phi(z) = \bigcap \{cl_X(\varphi(h(U_n(z)))) : n \in \mathbb{N}\}, z \in H.$$

**Claim 1.**  $\Phi(z)$  is compact for each  $z \in H$

Because  $X$  is a  $\mu$ -space, it suffices to show that  $\Phi(z)$  is bounded in  $X$ . Assuming the contrary, by Lemma 3.4, there exists a locally finite and open in  $X$  family  $\{V_n\}$  such that  $V_n \cap \Phi(z) \neq \emptyset$  for each  $n$ . Then for every  $n$  there are  $z_n \in U_n(z)$  and  $x_n \in \varphi(y_n) \cap V_n$ , where  $y_n = h(z_n)$ . Since  $\{z_n\}$  has a compact closure in  $H$ ,  $\{y_n\}$  has a compact closure in  $Y$ . Hence the closure of  $\varphi(\{y_n\})$  in  $X$  is also compact, and obviously, it meets each  $V_n$ . This contradicts the local finiteness of  $\{V_n\}$ .

**Claim 2.** If  $\Phi(z) \subset W$ ,  $W \subset X$  open, then  $\varphi(h(U_m(z))) \subset W$  for some  $m$ .

Since  $\Phi(z)$  is compact, there is open  $V$  in  $X$  with  $\Phi(z) \subset cl_X(V) \subset W$ . It is enough to prove that there is  $n$  such that  $\varphi(h(t)) \subset cl_X(V)$  for every  $t \in U_n(z)$ . Assuming this is not the case we can find  $z_n \in U_n(z)$  such that  $\varphi(h(z_n)) \cap (X - cl_X(V)) \neq \emptyset$ ,  $n \in \mathbb{N}$ . Let  $P_n = \{z_k : k \geq n\}$ . Since all  $P_n$  have compact closure in  $H$ , the sets  $K_n = cl_X(\varphi(h(P_n)))$  are compact. Then  $\gamma = \{K_n \cap (X - V)\}$  is a decreasing family of non-empty compact sets, so  $\bigcap \gamma \neq \emptyset$ . But  $\bigcap \gamma \subset \bigcap K_n \subset \Phi(z) \subset V$ , a contradiction.

**Claim 3.**  $\psi(z) \subset \Phi(z)$  for every  $z \in H$ .

Suppose there is  $x \in \psi(z) - \Phi(z)$  for some  $z \in H$ . Take an open set  $W \subset \beta X$  containing  $\Phi(z)$  with  $x \notin cl_{\beta X}(W)$ . By Claim 2, there is  $m$  such that  $cl_{\beta X}(\varphi(h(U_m(z)))) \subset W$ . Because  $\psi(z) \subset cl_{\beta X}(\varphi(h(U_m(z))))$  we have a contradiction.

It follows from the above claims that  $\psi$  is an *usco* map from  $P$  into  $\beta X$  such that  $\varphi(h(z)) \subset \psi(z) \subset X$  for every  $z \in H$ . Finally, we define  $\theta : B \rightarrow \beta X$  by  $\theta(y) = \psi(h^{-1}(y))$ . It is easy to check that  $\theta$  possesses all required properties.  $\square$

**Remark.** The proof of Lemma 3.5 shows that its modified version (with  $\varphi(K) \subset X$  bounded for every convergent sequence  $K$  in  $Y$ , and  $A$  a perfect image of a first countable space) is also true.

**Theorem 3.6.** *Let  $\phi$  be a quotient continuous linear map from  $C_k^*(X, E)$  onto  $C_k^*(Y, F)$  with  $E, F$  Banach spaces and  $X$   $\mu$ -complete. Then every closed generalized *sq*-subset of  $Y$  is Čech complete (resp., of compact countable type) provided  $X$  is Čech complete (resp., of compact countable type).*

PROOF. Let  $A$  be a closed generalized *sq*-subset of  $Y$  and  $B = cl_{\beta Y}(A)$ . Since, by Lemma 3.3, the map  $supp(\varphi_y) : A \rightarrow X$  satisfies the hypotheses of Lemma 3.5, there is an *usco* map  $\psi : B \rightarrow \beta X$  with  $supp(\varphi_y) \subset \psi(y) \subset X$  for every  $y \in A$ . Because  $\psi$  is *usc*,  $supp(\varphi_y)$  (considered as a set-valued map from  $B$  into  $\beta X$ ) is *lsc* and  $supp(\varphi_y) \subset \psi(y)$  for every  $y \in A$ , it follows that  $supp(\varphi_y) \subset \psi(y)$  for any  $y \in B$ .

Suppose  $X$  is Čech complete. Then  $\psi^*(X) = \{y \in B : \psi(y) \subset X\}$  is  $G_\delta$  in  $B$  and contains  $A$  (recall that  $supp(\varphi_y) \subset X$  for every  $y \in Y$ ). To prove that  $A$  is Čech complete it suffices to show that  $\psi^*(X) = A$ . And that is certainly true because  $y \in \psi^*(X)$  implies  $supp(\varphi_y) \subset \psi(y) \subset X$  and, by Corollary 2.8,  $y \in A$ .

Let  $X$  be of compact countable type and  $K \subset A$  be compact. Then  $\psi(K) \subset X$  is also compact (because  $\psi$  is *usco*), so there is a compact subset  $H \subset X$  which is  $G_\delta$  in  $\beta X$  and contains  $\psi(K)$ . As in the previous case, we have  $K \subset \psi^*(H) \subset A$  and  $\psi^*(H)$  is  $G_\delta$  in  $B$ . Consequently, we can find a compact  $G_\delta$ -subset  $D \subset B$  with  $K \subset D \subset A$ . Hence  $A$  is of compact countable type.  $\square$

**Corollary 3.7.** *Let  $X$  and  $Y$  be metric spaces and  $C_k^*(Y, E)$  be a quotient linear continuous image of  $C_k^*(Y, F)$ , where  $E$  and  $F$  are Banach. Then  $Y$  is complete provided  $X$  is.*

The real-valued case of the following proposition was proved in [14, Theorem 3.4]. The same proof works in our present situation, the only difference is that

we have to use Lemma 2.7 and Lemma 3.5 instead of, respectively, Lemma 2.3 and Lemma 3.1 from [14].

**Theorem 3.8.** *Let  $X$  be a locally compact  $\mu$ -space and there is a quotient continuous linear surjection from  $C_k^*(X, E)$  onto  $C_k^*(Y, F)$  with  $E$  and  $F$  Banach spaces. Then:*

- (i) *every closed generalized sq-subset of  $Y$  is locally compact;*
- (ii)  *$Y^{(sq)}$  is locally compact and open in  $Y$ .*

**Theorem 3.9.** *Let  $\phi$  be a continuous linear map from  $C_k^*(X, E)$  into  $C_k^*(Y, F)$  and let  $K \subset Y$ .*

- (i) *If  $Y$  is generalized sq and  $K \subset Y$  is bounded, then  $\text{supp}(K) \subset X$  is bounded;*
- (ii) *If  $K$  is a pseudocompact generalized sq-space, then  $\text{supp}(K)$  is pseudocompact.*

PROOF. (i) Let  $g : Z \rightarrow Y$  be a perfect surjection with  $Z$  an sq-space. Suppose  $K \subset Y$  is bounded but  $H = \text{supp}(K)$  is not bounded in  $X$ . According to Lemma 3.4 there is a locally finite open family  $\gamma$  in  $X$  such that its restriction on  $H$  is infinite. We can assume that  $\gamma = \{W_n\}$  is countable and  $\{W_n \cap H\}$  is infinite and consists of nonempty sets. Let  $V_n = \{y \in Y : \text{supp}(\phi_y) \cap W_n \neq \emptyset\}$ .

**Claim.**  $\eta = \{V_n\}$  is open and locally finite in  $Y$ .

Since  $\text{supp}(\phi_y) : Y \rightarrow X$  is lsc,  $\eta$  is open. Since  $g$  is perfect, to prove that  $\eta$  is locally finite in  $Y$  it is enough to show that the family  $\{G_n\}$ ,  $G_n = g^{-1}(V_n)$ , is locally finite in  $Z$ . For every  $z \in Z$  fix a family  $\{U_n(z)\}$  of neighborhoods of  $z$  in  $Z$  such that any sequence  $\{z_n\}$  with  $z_n \in U_n(z)$  has a compact closure in  $Z$ . It suffices to show that for any  $z \in Z$  there is  $n$  such that  $\text{supp}(g(U_n(z)))$  meets only finitely many elements of  $\gamma$ . And that is really true, otherwise for some  $z \in Z$  and every  $n \in \mathbb{N}$  there would be  $z_n \in U_n(z)$  with  $\text{supp}(\{g(z_n)\})$  intersecting infinitely many  $W_i$ . Since  $\text{supp}(\{g(z_n)\}) \subset X$  is compact (see Lemma 3.3), it can meet only finitely many elements of  $\gamma$ , a contradiction.

By Claim and Lemma 3.4, the restriction of  $\eta$  on  $K$  is finite. Because  $\eta$  is locally finite in  $Y$ , this means that only finitely many elements of  $\eta$  can meet  $K$ . Let show that the last one is a contradiction. Indeed, for every  $n$  we can choose  $x_n \in W_n \cap H$  and  $y_n \in K$  such that  $x_n \in \text{supp}(\phi_{y_n})$ , so  $y_n \in V_n \cap K$ .

(ii) In this case we follow the notations and the arguments from (i). Let  $g : Z \rightarrow K$  be a perfect surjection with  $Z$  an sq-space. Take  $\gamma$  to be open and locally finite in  $H$ . Then  $\eta = \{V_n\}$  is open and locally finite in  $K$ , where

$V_n = \{y \in K : \text{supp}(\varphi_y) \cap W_n \neq \emptyset\}$ . Consequently,  $\eta$  is finite which implies that  $\gamma$  is also finite. Hence  $H$  is pseudocompact.  $\square$

**Remark.** The requirement in Theorem 3.9 that  $g$  is perfect can be omitted in both cases if we additionally assume that  $K = g(P)$  for some bounded  $P \subset Z$  (in the first case), and  $Z$  is pseudocompact (in the second case). The proof is the same.

**Corollary 3.10.** *Let  $\phi$  be a quotient continuous linear map from  $C_k^*(X, E)$  onto  $C_k^*(Y, F)$  with  $Y$  a generalized sq-space and  $E, F$  Banach spaces. If  $X$  is  $\mu$ -complete, then  $Y$  is also  $\mu$ -complete.*

**PROOF.** Suppose  $K \subset Y$  is closed and bounded. Then, by Theorem 3.9,  $H = \text{supp}(K)$  is bounded in  $X$ . Since  $H \subset X$  is closed and  $X$  is a  $\mu$ -space,  $H$  is compact. Now, by Lemma 2.7,  $H^\# = \{y \in \beta Y : \text{supp}(\varphi_y) \subset H\}$  is a compact subset of  $Y$  and obviously  $K \subset H^\#$ . Hence  $K$  is compact.  $\square$

For a space  $X$  let  $\mathcal{K}(X)$  be the family of all compact subsets of  $X$ . A set  $\mathcal{B} \subset \mathcal{K}$  is cofinal in  $\mathcal{K}(X)$  if for any  $K \in \mathcal{K}(X)$  there is  $B \in \mathcal{B}$  such that  $K \subset B$ . The cofinality of  $\mathcal{K}(X)$  is defined by

$$\text{cof } \mathcal{K}(X) = \min \{\text{card } \mathcal{B} : \mathcal{B} \text{ is cofinal in } \mathcal{K}(X)\}.$$

**Theorem 3.11.** *Let  $C_k^*(Y, F)$  be a quotient image of  $C_k^*(X, E)$  under a continuous linear map. If  $E$  and  $F$  are Banach spaces, then  $\text{cof } \mathcal{K}(X) \leq \text{cof } \mathcal{K}(Y)$ .*

**PROOF.** For real-valued functions it was proved in [14, Corollary 2.5]. The same arguments can be applied, the only difference is that instead of Lemma 2.3 from [14] we make use of Lemma 2.7.  $\square$

#### 4. LINEAR INJECTIONS AND LINEAR HOMEOMORPHISMS

Suppose  $\phi : C_k^*(X, E) \rightarrow C_k^*(Y, F)$  is a continuous linear injection (resp., homeomorphism), where  $E$  and  $F$  are normed or Banach spaces. We are going to prove that for some topological properties  $\mathcal{P}$  the following is true: if  $Y$  is  $\mathcal{P}$ -dense (resp.,  $Y \in \mathcal{P}$ ), then  $X$  is  $\mathcal{P}$ -dense (resp.,  $X \in \mathcal{P}$ ), where a space is called  $\mathcal{P}$ -dense if it has a dense subset with the property  $\mathcal{P}$ . We also say that  $X$  is dense  $\sigma$ - $\mathcal{P}$  if  $X$  contains a dense subset which is a countable union of closed in  $X$  sets with the property  $\mathcal{P}$ .

We begin with the following lemma which can be proved by standard arguments, see for example [3, Lemma 1.2.5].

**Lemma 4.1.** *Let  $\phi : C_k^*(X, E) \rightarrow C_p(Y, F)$  be a continuous linear injection and  $D \subset Y$  be dense. Then  $\text{supp}(D) = X$ .*

**Corollary 4.2.** *Let  $\phi : C_k^*(X, E) \rightarrow C_k^*(Y, F)$  be a continuous linear injection. Then we have:*

- (i)  *$X$  is compact (resp., dense  $\sigma$ -compact set) if  $Y$  is compact (resp., dense  $\sigma$ -compact);*
- (ii)  *$X$  is pseudocompact (resp., dense  $\sigma$ -pseudocompact) if  $Y$  is a pseudocompact (resp., dense  $\sigma$ -pseudocompact) generalized sq-space.*

PROOF. The proof of (i) follows from Lemmas 4.1 and 3.3. Condition (ii) is an implication of Lemma 4.1 and Theorem 3.9.  $\square$

A space  $X$  is said to be  $Z$ -analytic ( $Z$  is a given space) if there is an *usco* map from  $Z$  onto  $X$ .

**Theorem 4.3.** *Let  $\phi : C_k^*(X, E) \rightarrow C_k^*(Y, F)$  be a continuous linear injection with  $X$   $\mu$ -complete. If  $Y$  is dense  $Z$ -analytic, where  $Z$  is an sq-space, then  $X$  is also dense  $Z$ -analytic.*

PROOF. Let  $\theta : Z \rightarrow Y$  be an *usco* map such that  $\theta(Z) = \cup\{\theta(z) : z \in Z\} \subset Y$  is dense. Define  $\varphi : Z \rightarrow X$  by  $\varphi(z) = \text{supp}(\theta(z))$ . Then  $\varphi$  satisfies the hypotheses of Lemma 3.5. Hence there is an *usco* map  $\psi$  from  $Z$  into  $X$  with  $\varphi(z) \subset \psi(z)$  for any  $z \in Z$ . It is easily seen that  $\varphi(Z) \subset X$  is dense, so  $X$  is dense  $Z$ -analytic.  $\square$

We say that a topological property is perfect if it is preserved by continuous perfect maps and by preimages of continuous perfect maps. It is well known that if  $\theta : X \rightarrow Y$  is an *usco* map with  $Y = \phi(X)$ , then  $X$  is a perfect image of the graph of  $\theta$ ; in such a case if  $X$  has the perfect property  $\mathcal{P}$ , then  $Y$  is a continuous image of a space with the property  $\mathcal{P}$ . Combining this fact and Theorem 4.3, we obtain the following

**Corollary 4.4.** *Let  $\phi : C_k^*(X, E) \rightarrow C_k^*(Y, F)$  be a continuous linear injection with  $X$   $\mu$ -complete. If  $Y$  is dense  $Z$ -analytic, where  $Z$  is an sq-space with the perfect property  $\mathcal{P}$ , then  $X$  contains a dense subset which is a continuous image of a space with the property  $\mathcal{P}$ .*

If  $\phi : C^*(X, E) \rightarrow C^*(Y, F)$  is a linear bijection, then we can define  $(\varphi)_x^{-1}$  and  $\text{supp}((\varphi)_x^{-1})$ ,  $x \in \beta X$ , analogously to  $\varphi_y$  and  $\text{supp}(\varphi_y)$  (considering  $\phi^{-1}$  instead of  $\phi$ ). For  $A \subset X$  we denote  $cl_Y \cup \{\text{supp}((\varphi)_x^{-1}) : x \in A\}$  by  $\text{supp}(A)$ . It is easy to check that for all  $x \in X$  and  $y \in Y$  we have

- (1)  $x \in \text{supp}(\text{supp}((\varphi)_x^{-1}))$ ,
- (2)  $y \in \text{supp}(\text{supp}(\varphi_y))$ .

**Proposition 4.5.** *Let  $\phi : C_k^*(X, E) \rightarrow C_k^*(Y, F)$  be a linear homeomorphism with  $X$   $\mu$ -complete and  $Y$  a generalized  $sq$ -space. Then there exists an usco map  $\psi$  from  $Y$  onto  $X$  such that  $y \in \text{supp}(\psi(y))$  and  $\text{supp}(y) \subset \psi(y)$  for every  $y \in Y$ .*

PROOF. By Lemma 3.5, there is an usco map  $\psi : Y \rightarrow X$  such that  $\text{supp}(\varphi_y) \subset \psi(y)$  for any  $y \in Y$ . Since  $\text{supp}((\varphi)_x^{-1})$  is compact (Lemma 3.3) and  $\psi$  is usc,  $\psi(\text{supp}((\varphi)_x^{-1}))$  is also compact, so it contains  $\text{supp}(\text{supp}((\varphi)_x^{-1}))$ . Then, by (1),  $x \in \psi(\text{supp}((\varphi)_x^{-1}))$ ,  $x \in X$ . Hence  $\psi(Y) = X$ . Finally, because of (2) and  $\text{supp}(\varphi_y) \subset \psi(y)$ ,  $y \in \text{supp}(\psi(y))$  for every  $y \in Y$ .  $\square$

M. Choban [6, Theorem 6.1] proved that any perfect property is invariant under linear homeomorphisms between  $C_k(X, E)$  and  $C_k(Y, F)$  with both  $X$  and  $Y$   $\mu$ -complete  $wq$ -spaces. Next corollary shows that a similar result is also valid for bounded function spaces.

**Corollary 4.6.** *If  $C_k^*(X, E)$  and  $C_k^*(Y, F)$  are linearly homeomorphic with both  $X$  and  $Y$   $\mu$ -complete generalized  $sq$ -spaces, then  $X \in \mathcal{P}$  if and only if  $Y \in \mathcal{P}$  for any perfect property  $\mathcal{P}$ .*

PROOF. By Proposition 4.5 there exist usco maps  $\psi$  from  $Y$  onto  $X$  and  $\theta$  from  $X$  onto  $Y$  such that  $x \in \text{supp}(\theta(x))$ ,  $\text{supp}(x) \subset \theta(x)$ ,  $x \in X$ , and  $y \in \text{supp}(\psi(y))$ ,  $\text{supp}(y) \subset \psi(y)$ ,  $y \in Y$ . The sets  $\psi(\theta(x))$ ,  $x \in X$ , are compact as images of compact set under the usco map  $\psi$ . Hence, by the above inclusions, each  $\psi(\theta(x))$  contains  $x$ ; similarly,  $y \in \theta(\psi(y))$ ,  $y \in Y$ . M. Choban [6] called such a pair of usco maps (satisfying the last two inclusions) a perfect-reflective pair and showed ([6, Theorem 2.5]) that there is a space  $Z$  and perfect surjections  $f : Z \rightarrow X$  and  $g : Z \rightarrow Y$ . This implies that for any perfect property  $\mathcal{P}$  we have  $X \in \mathcal{P}$  if and only if  $Y \in \mathcal{P}$ .  $\square$

Now we are going to show that metrizability is also preserved by linear homeomorphisms between  $C_k^*(X, E)$  and  $C_k^*(Y, F)$  in the class of  $\mu$ -complete generalized  $sq$ -spaces. Note that metrizability is not a perfect property, so we can not apply Corollary 4.6. For unbounded function spaces with the pointwise or compact open topology it was proved by Choban [6, Corollary 5.12]. We follow the same scheme of proof adapted to our present situation.

**Theorem 4.7.** *Let  $C_k^*(X, E)$  and  $C_k^*(Y, F)$  be linearly homeomorphic with  $Y$  a generalized  $sq$ -space and  $E$  separable. If  $X$  is metrizable, then  $Y$  is also metrizable.*

PROOF. Since paracompact  $p$ -space property is perfect, by Corollary 4.6,  $Y$  is a paracompact  $p$ -space, in particular  $Y$  is a  $k$ -space. Metrizability of  $X$  and separability of  $E$  imply that  $C_k^*(X, E)$  is dense  $\sigma$ -compact (we use the following result [6, Property 5.7]: if  $X$  is submetrizable and  $E$  a separable normed space, then  $C_k(X, E)$  is dense  $\sigma$ -compact; actually, in the proof of that result a  $\sigma$ -compact subset of  $C_k^*(X, E)$  was constructed which is dense in  $C_k(X, E)$ , so  $C_k^*(X, E)$  is dense  $\sigma$ -compact). Then, by [6, Property 5.6] (see also [10, Theorem 4.3.1] for real-valued functions),  $C_k(C_k^*(X, E), F)$  is submetrizable. Using that  $Y$  is a  $k$ -space and following the arguments of [10, Theorem 2.3.5(a), Theorem 2.3.6] we can show that the map  $y \rightarrow \mu_y$  embeds  $Y$  in  $C_k(C_k^*(X, E), F)$ . Therefore  $Y$  is submetrizable. So,  $Y$  is a submetrizable paracompact  $p$ -space, which yields that it is metrizable.  $\square$

5. AN EXAMPLE

In this section we establish an example which shows that the requirement in our previous results  $Y$  to be a generalized  $sq$ -space is essential.

Suppose  $\{X_n : n \in \mathbb{N}\}$  is a sequence of spaces and  $F_n \subset C_k^*(X_n, E)$ . The space  $\{(f_n) \in \prod F_n : \sup \|f_n\| < \infty\}$  with the topology of coordinatewise convergence is denoted by  $(F_n)_0$ . Here  $\|f_n\| = \sup \{\|f_n(x)\| : x \in X_n\}$ . When  $K \subset X$  and  $f \in C^*(X, E)$  we also use the notation  $\|f\|_K = \sup \{\|f(x)\| : x \in K\}$ . For closed  $A \subset X$ ,  $C_k^*(X|A, E)$  denotes the subspace of all functions from  $C_k^*(X, E)$  vanishing on  $A$ .

**Lemma 5.1.** *Let  $\phi : C_k^*(A, E) \rightarrow C_k^*(X, E)$  be a continuous linear extension map (i.e.,  $\phi(f)(x) = f(x)$  for all  $x \in A$ ) such that  $A \subset X$  is closed and  $E$  a Banach space. Then  $(C_k^*(X, E))_0$  is linearly homeomorphic to  $(C_k^*(A, E))_0 \times (C_k^*(X|A, E))_0$ .*

PROOF. By the Closed Graph Theorem,  $\phi$  considered as a map between  $C_u^*(A, E)$  and  $C_u^*(X, E)$  is continuous. Hence there is a constant  $M$  with  $\|\phi(f)\| \leq M \cdot \|f\|$  for all  $f \in C^*(A, E)$ . Thus  $(f_1, f_2, \dots, f_n, \dots) \in (C_k^*(A, E))_0$  if and only if  $(\phi(f_1), \phi(f_2), \dots, \phi(f_n), \dots) \in (C_k^*(X, E))_0$ . Consider the linear homeomorphism  $u$  from  $C_k^*(A, E) \times C_k^*(X|A, E)$  onto  $C_k^*(X, E)$  defined by  $u(f, g) = \phi(f) + g$ . If  $(f_1, f_2, \dots, f_n, \dots) \in (C_k^*(A, E))_0$  and  $(g_1, g_2, \dots, g_n, \dots) \in (C_k^*(X|A, E))_0$ , then  $(u(f_1, g_1), u(f_2, g_2), \dots, u(f_n, g_n), \dots) \in (C_k^*(X, E))_0$ . Therefore we have a linear

map from  $(C_k^*(A, E))_0 \times (C_k^*(X|A, E))_0$  onto  $(C_k^*(X, E))_0$  which is a homeomorphism.  $\square$

If  $X$  is metrizable and  $A \subset X$  closed, then for any Banach space  $E$  there is a continuous linear extension operator from  $C_k^*(A, E)$  into  $C_k^*(X, E)$  (see [7]). The same holds when  $X$  is arbitrary and  $A$  is an one-point set in  $X$ . Therefore we have the following

**Corollary 5.2.** *Lemma 5.1 is true for any couple  $(X, A)$  with  $X$  metrizable and  $A \subset X$  closed, or  $X$  arbitrary and  $A \subset X$  a point set.*

**Lemma 5.3.** *If  $X$  is the disjoint sum of the spaces  $X_n$ ,  $n \in \mathbb{N}$ , then  $C_k^*(X, E)$  is linearly homeomorphic to  $(C_k^*(X_n, E))_0$ .*

PROOF. The required linear homeomorphism  $u : C_k^*(X, E) \rightarrow (C_k^*(X_n, E))_0$  is defined by  $u(f) = (f_1, f_2, \dots, f_n, \dots)$ , where each  $f_n$  is the restriction  $f|X_n$ .  $\square$

Let  $X = [1, \omega] \times \mathbb{N}$  and  $Y$  be the space obtained from  $X$  by identifying all points of the set  $A = \{\omega\} \times \mathbb{N}$ . It is known (see [3, Example 2.4.10] and [5]) that  $C_p(X)$  is linearly homeomorphic to  $C_p(Y)$ . We are going to show that  $C_k^*(X, E)$  is linearly homeomorphic to  $C_k^*(Z, E)$  for any Banach space  $E$ , where  $Z$  is the disjoint sum of countably many copies of  $Y$ .

**Lemma 5.4.** *The quotient map  $f : X \rightarrow Y$  is compact-covering, i.e. any compact set in  $Y$  is an image (under  $f$ ) of a compact set in  $X$ .*

PROOF. Because  $f$  is the identity map between  $X - A$  and  $Y - \{\infty\}$ , where  $\infty$  is the point  $f(A)$ , it is enough to consider only the case when  $K \subset Y$  is compact and contains  $\infty$ . Using the arguments from the proof that  $Y$  is not first countable given in [3, Example 2.4.10] we can show that there is no sequence  $\alpha_{i_n} < \omega$  such that  $f(\{\alpha_{i_n}\} \times \{i_n\})$  converges to  $\infty$  in  $Y$ . This yields that  $K$  is contained in the union of finitely many of the sets  $f([1, \omega] \times \{n\})$ . Therefore there exists a compact set  $H \subset X$  with  $f(H) = K$ .  $\square$

**Lemma 5.5.** *Maintaining the notions of Lemma 5.4 we have that  $(C_k^*(X|A, E))_0$  is linearly homeomorphic to  $(C_k^*(Y|\{\infty\}, E))_0$  for every Banach space  $E$ .*

PROOF. Since  $f$  is compact-covering, the linear map  $v : C_k^*(Y, E) \rightarrow C_k^*(X, E)$ ,  $v(h) = h \circ f$ , is an embedding. Therefore  $v$  is a linear homeomorphism between  $C_k^*(Y|\{\infty\}, E)$  and  $C_k^*(X|A, E)$ . Because  $\|v(h)\| = \|h\|$  for any  $h \in C_k^*(Y|\{\infty\}, E)$ , the formula  $(h_1, h_2, \dots, h_n, \dots) \rightarrow (v(h_1), v(h_2), \dots, v(h_n), \dots)$  defines a linear homeomorphism from  $(C_k^*(Y|\{\infty\}, E))_0$  onto  $(C_k^*(X|A, E))_0$ .  $\square$

**Proposition 5.6.** *Let  $X = [1, \omega] \times \mathbb{N}$  and  $Y$  be the quotient space  $X/A$  with  $A = \{\omega\} \times \mathbb{N}$ . Then  $C_k^*(X, E)$  is linearly homeomorphic to  $C_k^*(Z, E)$  for every Banach space  $E$ , where  $Z$  is the disjoint sum of countably many copies of  $Y$ .*

PROOF. Everywhere in our proof the sign  $\sim$  stands for the phrase "linearly homeomorphic". Notice that  $X$  is a disjoint sum of countably many copies of itself, so, by Lemma 5.3,  $C_k^*(X, E) \sim (C_k^*(X, E))_0$ . Another observation, since  $A$  is countable and discrete,  $C_k^*(A, E) \sim (E)_0$  (by Lemma 5.3). Then we have

$$\begin{aligned}
 C_k^*(X, E) &\sim (C_k^*(X, E))_0 \\
 &\sim (C_k^*(X|A, E))_0 \times (C_k^*(A, E))_0 \quad \text{by Corollary 5.2} \\
 &\sim (C_k^*(Y|\{\infty\}, E))_0 \times (C_k^*(A, E))_0 \quad \text{by Lemma 5.5} \\
 &\sim (C_k^*(Y|\{\infty\}, E))_0 \times (E)_0 \\
 &\sim (C_k^*(Y|\{\infty\}, E) \times E)_0 \\
 &\sim (C_k^*(Y|\{\infty\}, E) \times C_k^*(\{\infty\}, E))_0 \\
 &\sim (C_k^*(Y, E))_0 \quad \text{by Corollary 5.2} \\
 &\sim C_k^*(Z, E) \quad \text{by Lemma 5.3.} \quad \square
 \end{aligned}$$

Since  $X$  is a metric space and  $Y$  is not metrizable (it is not even first countable, see [3]), it follows from Theorem 4.7 that the space  $Z$  in Example 5.6 is not a generalized  $sq$ -space. Therefore generalized  $sq$ -space property is essential in our results.

**Remark.** Using similar arguments we can show that  $C_p^*(X, E)$  is linearly homeomorphic to  $C_p^*(Z, E)$  for any Banach space  $E$ .

## 6. APPENDIX

First, let us discuss the validity of our results for more general spaces  $C_\alpha^*(X, E)$ , where  $\alpha$  is a compact set-ring on  $X$  (i.e. a finitely additive and hereditary closed family of compact subsets of  $X$  covering  $X$ ) and  $C_\alpha^*(X, E)$  is the set  $C^*(X, E)$  with the topology of uniform convergence on the elements of  $\alpha$ .

**Proposition 6.1.** *Let  $\phi : C_\alpha^*(X, E) \rightarrow C_\beta(Y, F)$  be a continuous linear map such that  $E$  and  $F$  are Banach spaces,  $\alpha$  and  $\beta$  compact set-rings on  $X$  and  $Y$ , respectively, and  $\beta$  contains all separable compact subsets of  $Y$ . If  $X$  is  $\mu$ -complete, then  $\phi$  can be lifted to a continuous map from  $C_k^*(X, E)$  into  $C_k(Y, F)$ .*

PROOF. In this situation we can show that  $\text{supp}(K) \in \alpha$  for every  $K \in \beta$ , in particular  $\text{supp}(K)$  is compact for every separable compact  $K \subset Y$ . Using this fact (instead of Lemma 3.3) in the proof of Theorem 3.9(ii), we obtain that  $\text{supp}(K)$  is compact for any compact set  $K \subset Y$ . Then, the arguments from the proof of [13, Proposition 3.6] can complete our proof.  $\square$

Proposition 6.1 is not true if the requirement for  $\beta$  to contain all separable compacta in  $Y$  is omitted. There is an example in [4] of a continuous linear map  $\phi : C_p^*(X) \rightarrow C_p^*(Y)$  with  $X$  and  $Y$  metric spaces, and a point  $y \in Y$  such that  $\text{supp}(y)$  is not compact. Therefore, according to Lemma 3.3,  $\phi$  can not be lifted to a continuous map between  $C_k^*(X)$  and  $C_k^*(Y)$ .

Proposition 6.1 yields that all results in this paper concerning linear injections or linear homeomorphisms between function spaces with the compact open topology remain true and for linear homeomorphisms (resp., continuous linear injections)  $\phi : C_\alpha^*(X, E) \rightarrow C_\beta^*(Y, F)$  with  $E$  and  $F$  Banach spaces provided that  $\alpha$  and  $\beta$  contain all separable compacta in  $X$  and  $Y$  (resp.,  $\beta$  contains all separable compacta in  $Y$ ), and  $X, Y$  (resp.,  $X$ )  $\mu$ -complete. When  $\phi : C_\alpha^*(X, E) \rightarrow C_\beta^*(Y, F)$  is quotient, we can not derive the same conclusion directly from Proposition 6.1 because the lifting of  $\phi$  may not be quotient. But we can use a modified version of Lemma 2.7 (with  $H \in \alpha$ ) to see that Theorem 3.6, Corollary 3.7 and Theorem 3.9 also have true analogues with  $C_k^*(X, E)$  and  $C_k^*(Y, F)$  replaced, respectively, by  $C_\alpha^*(X, E)$  and  $C_\beta^*(Y, F)$ , where  $\beta$  contains all separable compacta in  $Y$ . Finally, let us note that because of the remark after Lemma 3.5, if everywhere instead of generalized  $sq$ -spaces we consider perfect images of first countable spaces, then it suffices  $\beta$  to contain all convergent sequences.

In general, our paper follows the ideas from [13], where unbounded function spaces were considered. It appears that the bounded case is much more complicated. The first difference is that many properties which are preserved by continuous linear surjections  $\phi$  between  $C_k(X, E)$  and  $C_k(Y, F)$  (resp.,  $C_p(X, E)$  and  $C_p(Y, F)$ ) are no longer preserved by continuous linear surjections between bounded function spaces, unless we require some additional properties for  $\phi$  (see the remark after Lemma 2.9).

We couldn't prove pointwise versions of the results in Section 3 and Section 4 (except Corollary 3.2). The reason is that our technique is based on two major lemmas, Lemma 2.7 and Lemma 3.3. The pointwise analogue of Lemma 3.3 is not valid even for metrizable  $X$  and  $Y$  and real-valued functions [4]. A pointwise analogue of Lemma 2.7 is known only for metric spaces  $X$  and  $Y$  and real-valued functions, see [5, Lemma 5.4].

It is clear now that the pointwise case for bounded functions requires completely new ideas. One of the most beautiful and nontrivial results in this direction is that Čech completeness is preserved by continuous linear surjections between  $C_p^*(X)$  and  $C_p^*(Y)$  for metrizable  $X$  and  $Y$  (see [5, Theorem 4.2]). If we restrict our attention to linear homeomorphisms, we can prove the following

**Proposition 6.2.** *Let  $\phi : C_p^*(X) \rightarrow C_p^*(Y)$  be a linear homeomorphism with  $X$  and  $Y$  Lindelöf  $p$ -spaces. If  $X$  is Čech complete, then  $Y$  is Čech complete.*

PROOF. It suffices to show that  $Y$  admits a perfect map onto a completely metrizable space. Take a perfect continuous surjection  $f : Y \rightarrow M$  with  $M$  a separable metric space. Following the proof of Theorem 4.3 from [16] we can find a perfect surjection  $h : X \rightarrow X_0$  with  $X_0$  a separable metric space such that  $\phi(h_p^*(C_p^*(X_0))) = f_p^*(C_p^*(M))$ , where  $f_p^* : C_p^*(M) \rightarrow C_p^*(Y)$  and  $h_p^* : C_p^*(X_0) \rightarrow C_p^*(X)$  are the duals of  $f_p$  and  $h_p$  respectively. Since  $f_p^*$  and  $h_p^*$  are linear embeddings, we have a linear homeomorphism between  $C_p^*(X_0)$  and  $C_p^*(M)$ . Observe that  $X_0$  is completely metrizable (as a perfect image of  $X$ ). Hence, by above mentioned result of J. Baars, J. de Groot and J. Pelant [5, Theorem 4.2],  $M$  is also completely metrizable.  $\square$

The analogue of Proposition 6.2 for compact open topology is also true, and the same scheme works, but it is covered by Theorem 3.6. Let us mention another application of the arguments from Proposition 6.2.

**Proposition 6.3.** *Let  $\mathcal{P}$  be a class of separable metrizable spaces with the following properties:*

- (i)  $\mathcal{P}$  contains all compact metrizable spaces;
- (ii)  $\mathcal{P}$  is closed under finite products;
- (iii)  $\mathcal{P}$  is hereditary with respect to closed subsets;
- (iv) If  $X$  is a metrizable space which is a countable union of closed subsets  $X_n \in \mathcal{P}$  then  $X \in \mathcal{P}$ .

Let  $X$  and  $Y$  be Lindelöf  $p$ -spaces. If  $C_p(X)$  and  $C_p(Y)$  (resp.,  $C_p^*(X)$  and  $C_p^*(Y)$ ) are uniformly homeomorphic and  $X$  admits a perfect map onto a space from  $\mathcal{P}$ , then there is a perfect map from  $Y$  onto a space from  $\mathcal{P}$ .

PROOF. Maintaining the notations from the proof of Proposition 6.2, there are perfect surjections  $h : X \rightarrow X_0$  and  $f : Y \rightarrow M$ , where  $X_0$  and  $M$  are separable metric spaces, such that  $C_p(X_0)$  and  $C_p(M)$  (resp.,  $C_p^*(X_0)$  and  $C_p^*(M)$ ) are uniformly homeomorphic. Since  $X$  admits a perfect map onto a space from  $\mathcal{P}$  and  $\mathcal{P}$  is closed under taking metric perfect preimages [9, Corollary 3.2], we can assume that  $X_0 \in \mathcal{P}$ . Then, by [9, Corollary 3.2],  $M \in \mathcal{P}$ .  $\square$

W. Marciszewski and J. Pelant applied their result [9, Corollary 3.2] to prove that absolute Borel classes  $\mathcal{A}_\alpha$  and  $\mathcal{M}_\alpha$ ,  $\alpha > 1$ , are preserved by uniformly continuous surjections between  $C_p(X)$  and  $C_p(Y)$  (resp.,  $C_p^*(X)$  and  $C_p^*(Y)$ ) for metric  $X$  and  $Y$ , as well as that the same is true for projective classes  $\Sigma_n^1$ ,  $\Pi_n^1$  and separable metric spaces  $X$  and  $Y$  [9, Theorems 3.4 and 3.5]. Hence we have

**Corollary 6.4.** *Let  $\mathcal{P}$  be one of the classes  $\mathcal{A}_\alpha$ ,  $\mathcal{M}_\alpha$ ,  $\alpha > 1$ , or  $\Sigma_n^1$ ,  $\Pi_n^1$ . Suppose  $C_p(X)$  and  $C_p(Y)$  (resp.,  $C_p^*(X)$  and  $C_p^*(Y)$ ) are uniformly homeomorphic with  $X$  and  $Y$  Lindelöf  $p$ -spaces. If  $X$  admits a perfect map onto a space from  $\mathcal{P}$ , then  $Y$  does.*

We didn't discuss the preservation of the class  $\mathcal{A}_1$  (i.e.  $\sigma$ -compactness) under quotient linear surjections because of the following general result which implies that  $\sigma$ -compactness is preserved even by continuous (not necessary linear) surjection.

**Proposition 6.5.** *Suppose  $\phi$  is an open map from a subset  $A$  of  $C_p(X, E)$  (resp.,  $C_p^*(X, E)$ ) onto  $C_p(Y, F)$  (resp.,  $C_p^*(Y, F)$ ). Then there exists a compact space  $K$  such that  $Y$  is the continuous image of an  $F_\sigma$ -subset  $B$  of the product  $K \times \exp_\omega(X)$ . The same holds for the spaces  $C_k(X, E)$  and  $C_k(Y, F)$  (resp.,  $C_k^*(X, E)$  and  $C_k^*(Y, F)$ ) with  $\exp_\omega(X)$  replaced by  $\exp_c(X)$ .*

Here  $\exp_\omega(X)$  (resp.,  $\exp_c(X)$ ) denotes the set of all finite (resp., compact) subsets of  $X$  with the Vietoris topology. When bounded Banach-valued function spaces are considered (both with pointwise or compact open topology),  $A$  is a linear subspace and  $\phi$  is quotient and linear, then we can take  $B$  to be closed.

Proposition 6.5 can be proved by using Okunev's technique [11]. Its linear variant for the compact open topology gives a little bit stronger result than Lemma 2.7.

**Question.** Let  $C_k^*(X, E)$  and  $C_k^*(Y, F)$  (resp.,  $C_p^*(X, E)$  and  $C_p^*(Y, F)$ ) be linearly homeomorphic. Are  $X$  and  $Y$  simultaneously pseudocompact?

The answer of this question is affirmative if  $X$  and  $Y$  are generalized  $sq$ -spaces (see Corollary 4.2(ii)), or  $E$  and  $F$  are the reals (see [16, Theorem 3.5, Theorem 4.3]).

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