LINDELÖF DEGREE AND FUNCTION SPACES

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Abstract. We prove that if X and Y are completely regular perfect spaces and Φ is a continuous linear homeomorphism between $C_p(X)$ and $C_p(Y)$ (resp., $C_p^*(X)$ and $C_p^*(Y)$) then the Lindelöf degrees of X and Y are the same. When Φ is positive the above result remains true for any completely regular X and Y. It is also shown that pseudocompactness and compactness are preserved by continuous linear homeomorphisms between $C_p^*(X)$ and $C_p^*(Y)$.

Key Words. Function spaces, Lindelöf degree, linear homeomorphism.

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1. Introduction

For a completely regular space X, $C(X)$ (resp., $C^*(X)$) denotes the set of all real-valued continuous (resp., continuous and bounded) functions on X. When $C(X)$ and $C^*(X)$ are endowed with the pointwise convergence topology we write $C_p(X)$ and $C_p^*(X)$, respectively. Two spaces X and Y (we shall consider only completely regular spaces) are called l_p-equivalent if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic.

In [1], Arkhangel'skiî announced the following result proved by Veličko: if X and Y are l_p-equivalent then X is Lindelöf if and only if Y is. According to Arkhangel'skiî [1], Veličko's proof does not work for higher Lindelöf degrees (the Lindelöf degree $l(Z)$ of a space Z is the least cardinal $\tau \geq \omega$ such that each open cover of Z has a subcover of cardinality at most τ). Baars [4] proved that Veličko's theorem is true for higher Lindelöf degrees provided X and Y are first countable paracompact spaces. Recently first author [12] generalized Baars' result in different directions: in the class of μ-complete wq-spaces the Lindelöf degree is preserved by linear homeomorphisms between $C_p(X,E)$ and $C_p(Y,F)$, as well as between $C_k(X,E)$ and $C_k(Y,F)$, where E and F are normed spaces. Here $C_p(X,E)$ (resp., $C_k(X,E)$) denotes the space of all continuous E-valued maps on X with the pointwise convergence (resp., compact open) topology. The class of wq-spaces [12] is closely related to that of q-spaces [7], and it contains all locally compact and all first countable spaces. Let us also note that μ-complete spaces are substantially more general than paracompact ones. In the present paper we prove that if Φ is a linear homeomorphism between $C_p(X)$ and $C_p(Y)$ (resp., $C_p^*(X)$ and $C_p^*(Y)$), where X and Y are perfect spaces, then $l(X) = l(Y)$. It is shown that the requirement for X and Y to be perfect can be omitted if Φ is positive.

We also prove that compactness and pseudocompactness are preserved by continuous linear homeomorphisms between $C_p^*(X)$ and $C_p^*(Y)$.

2. Preliminaries

Let $L_p(X)$ (resp., $L_p^*(X)$) be the continuous dual of $C_p(X)$ (resp., $C_p(X)$) with the pointwise convergence topology. The common notation $D_p(X)$ will be adopted for either of the spaces $L_p(X)$ and $L_p^*(X)$. It is well known that, for every $\mu \in D_p(X) - \{0\}$ there are finitely many points x_1,..., x_n from X and λ_{x_1},..,$\lambda_{x_n} \in \mathbf{R} - \{0\}$ (see [2] or [5]) such that $\mu = \Sigma\{\lambda_{x_i} \cdot \delta_{x_i} : i = 1, .., n\}$, where δ_{x_i} denotes the evaluation map at x_i defined by $\delta_{x_i}(f) = f(x_i)$. We define $supp(\mu)$ to be the set $\{x_i : i = 1, .., n\}$ if $\mu = 0$, and $supp(\mu) = \emptyset$ if $\mu = 0$. In the sequel the following notations will be considered :
$supp(\mu^+) = \{x \in supp(\mu) : \lambda_x > 0\}$, $\|\mu^+\| = \Sigma\{\lambda_x : x \in supp(\mu^+)\}$, $supp(\mu^-) =$
$\{x \in supp(\mu) : \lambda_x < 0\}$, $\|\mu^-\| = \Sigma\{|\lambda_x| : x \in supp(\mu^-)\}$.
For every $x \in \beta X$ let $\mathcal{A}(x) = \{f \in C(\beta X, I) : x \in Int_{\beta X}(f^{-1}(1))\}$, where βX is the Čech-Stone compactification of X and $I = [0, 1]$. When $x \in \beta X$ and $\mu \in D_p(X)$ we define $s_\mu(x) = \inf\{\mu(f) : f \in \mathcal{A}(x)\}$ and $\mathcal{S}_\mu(x) = \sup\{\mu(f) : f \in \mathcal{A}(x)\}$, where $\mu(f)$ stands for $\mu(f|X)$ for all $f \in A(x)$.

Lemma 2.1. *For every $\mu \in D_p(X)$ we have:*

(i) $s_\mu(x) = \lambda_x - \|\mu^-\|$ *and* $\mathcal{S}_\mu(x) = \|\mu^+\|$ *if* $x \in supp(\mu^+)$*;*
(ii) $= s_\mu(x) - \|\mu^-\|$ *and* $\mathcal{S}_\mu(x) = \lambda_x + \|\mu^+\|$ *if* $x \in supp(\mu^-)$*;*
(iii) $s_\mu(x) = -\|\mu^-\|$ *and* $\mathcal{S}_\mu(x) = \|\mu^+\|$ *if* $x \notin supp(\mu)$.

Proof. Follows from the representation $\mu = \Sigma\{\lambda_x \cdot \delta_x : x \in supp(\mu)\}$.□

For any $a \in \mathbf{R}$ and $\mu \in D_p(X)$ let $\varphi_a(\mu) = \{x \in \beta X : s_\mu(x) \geq a\}$ and $\psi_a(\mu) = \{x \in \beta X : \mathcal{S}_\mu(x) \leq a\}$. Observe that φ_a and ψ_a are set-valued maps from $D_p(X)$ into βX. Recall that a set-valued map $\theta : Y \to P(Z)$ is upper-semi continuous (written, *usc*) if the set

$\theta^{\#}(U) = \{y \in Y : \theta(y) \subset U\}$, is open in Y whenever U is open in Z. Note that $P(Z)$ stands for the set of all subsets of Z, so the possibility $\theta(y) = \emptyset$, for some $y \in Y$, is not excluded. We say that θ is *usco* if θ is usc and $\theta(y)$ is compact for every $y \in Y$.

Lemma 2.2. *The set-valued maps φ_a and ψ_a , $a \in \mathbf{R}$, are usc.*

Proof. We shall consider only φ_a, the other case is similar. Suppose $\varphi_a(\mu) \subset U$ for some open $U \subset \beta X$, $a \in \mathbf{R}$ and $\mu \in D_p(X)$. For every $x \in \beta X - U$ we have $s_\mu(x) < a$ so there is $f_x \in \mathcal{A}(x)$ with $\mu(f_x) < a$. Then $\{V_x \colon x \in \beta X - U\}$ is an open cover of $\beta X - U$, where $V_x = Int_{\beta X}(f_x^{-1}(1))$. Choose finitely many points $x(i) \in \beta X - U$, $i = 1, ..., n$, such that $\{V_{x(i)} : i = 1, .., n\}$ covers $\beta X - U$. Since $\mu(f_{x(i)}) < a$ for all i, there exists a neighborhood $W \subset D_p(X)$ of μ such that $\nu(f_{x(i)}) < a$ for every $\nu \in W$ and $i = 1, ..., n$. The proof will be completed if $\nu \in W$ implies $\varphi_a(\nu) \subset U$. To prove this implication it suffices to show that for a fixed $\nu \in W$ we have $\varphi_a(\nu) \cap V_{x(i)} = \emptyset$, $i = 1, ..., n$. Suppose not. Then there exist $j \in \{1, ..., n\}$ and $x_0 \in \varphi_a(\nu) \cap V_{x(j)}$. Since $s_\nu(x_0) \geq a$, and $x_0 \in V_{x(j)}$ yields $f_{x(j)} \in \mathcal{A}(x_0)$, we obtain $\nu(f_{x(j)}) \geq a$. The last inequality contradicts $\nu \in W$.$\square$

Let us note the following observations:

Lemma 2.3. *Let $a \in \mathbf{R}$ and $\mu \in D_p(X)$. Then $\varphi_a(\mu)$ is either βX or a finite subset of $supp(\mu^+)$ and:*

(i) $\varphi_a(\mu) \cap supp(\mu)$ consists of all $x \in supp(\mu^+)$ with $-\|\mu^-\| < a \leq \lambda_x - \|\mu^-\|$;

(ii) $\varphi_a(\mu) = \emptyset$ iff $\lambda_x - \|\mu^-\| < a$ for every $x \in supp(\mu^+)$;

(iii) $\varphi_a(\mu) = \beta X$ iff $-\|\mu^-\| \geq a$.$\square$

Lemma 2.4. *For $a \in \mathbf{R}$ and $\mu \in D_p(X)$ either $\psi_a(\mu)$ is a finite subset of $supp(\mu^-)$ or $\psi_a(\mu) = \beta X$ and:*

(i) $\psi_a(\mu) \cap supp(\mu)$ consists of all $x \in supp(\mu^-)$ with $\lambda_x + \|\mu^+\| \leq a < \|\mu^+\|$;

(ii) $\psi_a(\mu) = \emptyset$ iff $\lambda_x + \|\mu^+\| > a$ for every $x \in supp(\mu^-)$;

(iii) $\psi_a(\mu) = \beta X$ iff $\|\mu^+\| \leq a$.$\square$

Lemma 2.5. *For every $\mu \in D_p(X)$ and $a, b \in \mathbf{R}$ with $a < b$ we have $\varphi_b(\mu) \subset \varphi_a(\mu)$ and $\psi_a(\mu) \subset \psi_b(\mu)$.*$\square$

3. Lindelöf Degrees and Linear Homeomorphisms

Let $\Phi : C_p(X) \to C_p(Y)$ (resp., $\Phi : C_p^*(X) \to C_p^*(Y)$) be a linear continuous map. For every $y \in Y$ the function $\mu_y : C_p(X) \to \mathbf{R}$ (resp., $\mu_y : C_p^*(X) \to \mathbf{R}$), where $\mu_y(f) = \Phi(f)(y)$, is continuous and linear.

Therefore we have a map $\Phi^* : Y \to D_p(X)$ with $\Phi^*(y) = \mu_y$. It is easily seen that Φ^* is continuous, and Φ^* is an embedding when Φ is surjective. We shall consider the set-valued maps $\varphi^*_a, \psi^*_a : Y \to P(\beta X)$, $a \in \mathbf{R}$, defined by $\varphi^*_a(y) = \varphi_a(\mu_y)$ and $\psi^*_a(y) = \psi_a(\mu_y)$. According to Lemma 2.2, φ^*_a and ψ^*_a are usc. Let $U_a = \{y \in Y : \varphi^*_a(y) \subset X\}$ and $V_a = \{y \in Y : \psi^*_a(y) \subset X\}$.

Lemma 3.1. *Let* $\Phi : C_p(X) \to C_p(Y)$ *(resp.,* $\Phi : C^*_p(X) \to C^*_p(Y)$*) be continuous and linear. Then* U_a *and* V_a *,* $a \in \mathbf{R}$*, are open in* Y*.*

Proof. Let $a \in \mathbf{R}$. By Lemma 2.3, $Y - U_a$ consists of all $y \in Y$ with $\varphi^*_a(y) = \beta X$. We claim that $Y - U_a \subset Y$ is closed. Let $\{y_\alpha\colon \alpha \in \Lambda\}$ be a net in $Y - U_a$ converging to a point $y \in Y$. If $\varphi^*_a(y) \neq \beta X$, we can choose an open set $W \subset \beta X$ such that $\varphi^*_a(y) \subset W$ and $W \neq \beta X$. Then, since φ^*_a is usc and $\{y_\alpha\colon \alpha \in \Lambda\}$ converges to y, there is $\gamma \in \Lambda$ with $\varphi^*_a(y_\gamma) \subset W$. This is a contradiction because $\varphi^*_a(y_\gamma) = \beta X$. Hence $Y - U_a \subset Y$ is closed. By using Lemma 2.4 instead of Lemma 2.3 in this proof we can show that V_a is open.□

Lemma 3.2. *Let* $\Phi : C_p(X) \to C_p(Y)$ *(resp.,* $\Phi : C^*_p(X) \to C^*_p(Y)$ *) be continuous, linear. Then, for every* $y \in Y$*, there exist rational numbers* $p, q \in Q$ *such that* $\varphi^*_p(y) = supp(\mu^+_y)$ *and* $\psi^*_q(y) = supp(\mu^-_y)$*.*

Proof. Suppose both $supp(\mu^+_y)$ and $supp(\mu^-_y)$ are non-empty. Then $\left\|\mu^-_y\right\| \neq 0$ and $\left\|\mu^+_y\right\| \neq 0$, so we can take $p, q \in Q$ such that $-\left\|\mu^-_y\right\| < p \leq \lambda_x - \left\|\mu^-_y\right\|$ for all $x \in supp(\mu^+_y)$ and $\lambda_x + \left\|\mu^+_y\right\| \leq q < \left\|\mu^+_y\right\|$ for all $x \in supp(\mu^-_y)$. By Lemma 2.3(i) and Lemma 2.4(i), we have $\varphi^*_p(y) = supp(\mu^+_y)$ and $\psi^*_q(y) = supp(\mu^-_y)$. If $supp(\mu^+_y) = \emptyset$ (resp., $supp(\mu^-_y) = \emptyset$) we choose $p \in Q$ (resp., $q \in Q$) with $-\left\|\mu^-_y\right\| < p$ (resp., $q < \left\|\mu^+_y\right\|$) and apply Lemma 2.3(ii) (resp., Lemma 2.4(ii)).□

Recall that a space X is perfect if every open set in X is a union of countably many closed subsets of X.

Theorem 3.3. *Let* $\mathcal{P}$ *be a class of spaces with the following properties:*

(a) if $Y \in \mathcal{P}$ *and* $F \subset Y$ *is closed, then* $F \in \mathcal{P}$*;*

(b) if $X = \cup\{X_n : n \in \mathbf{N}\}$ *with each* $X_n \in \mathcal{P}$*, then* $X \in \mathcal{P}$*;*

(c) if $F \in \mathcal{P}$ *and there is usco* $\theta : F \to P(X)$ *such that* $X = \cup\{\theta(z) : z \in F\}$*, then* $X \in \mathcal{P}$*.*

Let $Y \in \mathcal{P}$ *be a perfect space. If* $C_p(X)$ *and* $C_p(Y)$ *(resp.,* $C^*_p(X)$ *and* $C^*_p(Y)$*) are linearly homeomorphic, then* $X \in \mathcal{P}$*.*

Proof. Let $\Phi : C_p(X) \to C_p(Y)$ (resp., $\Phi : C^*_p(X) \to C^*_p(Y)$) be a linear homeomorphism. Let $H_a = \cup\{\varphi^*_a(y) : y \in U_a\}$ and $F_a =$

$\cup\{\psi^*_a(y) : y \in V_a\}$ for all $a \in \mathbf{R}$. By Lemma 3.1, U_a and V_a are open in Y, hence each of them is a union of countably many closed subsets of Y. Consequently, $U_a, V_a \in \mathcal{P}$ for any $a \in\mathbf{R}$. Since φ^*_a , considered as a set-valued map from U_a onto H_a , is usco, we have $H_a \in \mathcal{P}$. Similarly, $F_a \in \mathcal{P}$, $a \in\mathbf{R}$. To prove that $X \in \mathcal{P}$ it is enough to show that X is a union of the countable family $\gamma = \{H_q, F_q\colon q \in Q\}$. To this end, fix $x \in X$. Because Φ is a linear homeomorphism, there is $y \in Y$ with x$\in$ $supp(\mu_y)$ (see [5]). Then, by Lemma 3.2, we have $\varphi^*_p(y) = supp(\mu^+_y)$ and $\psi^*_q(y) = supp(\mu^-_y)$ for some $p, q \in Q$. Thus $x \in supp(\mu_y) = supp(\mu^+_y) \cup supp(\mu^-_y) \subset H_p \cup F_q$, so $X = \cup\gamma$.$\square$

Corollary 3.4. *Let $C_p(X)$ and $C_p(Y)$ (resp., $C^*_p(X)$ and $C^*_p(Y)$) be linearly homeomorphic with X and Y perfect. Then $l(X) = l(Y)$.*

Proof. By Theorem 3.3, it suffices to show that $l(X) \leq \tau$, where $\tau = l(Y)$. Let $\mathcal{P}_\tau$ be the class of all spaces having Lindelöf degree not greater that τ. It is easily seen that $\mathcal{P}_\tau$ satisfies the hypotheses of Theorem 3.3, hence $X \in \mathcal{P}_\tau$.$\square$

Proposition 3.5. *Let $\mathcal{P}$ be a class of spaces such that:*

(a) $\mathcal{P}$ is hereditary with respect to closed subspaces;
(b) if $X\in \mathcal{P}$ and Y is a continuous image of X, then $Y \in \mathcal{P}$;
(c) if $X, Y \in \mathcal{P}$, then $X \times Y \in \mathcal{P}$;
(d) $\mathcal{P}$ is closed with respect to taking countable unions;
(e) $\mathbf{R}\in \mathcal{P}$.

Suppose $\Phi : C^*_p(X) \to C^*_p(Y)$ is a quotient, continuous linear surjection. Then $Y \in \mathcal{P}$ provided $X \in \mathcal{P}$.

Proof. Arkhangel'skiî [2] proved that, for any class $\mathcal{B}$ satisfying (b)-(e), $X \in \mathcal{B}$ implies $L_p(X) \in \mathcal{B}$. So, $L_p(X) \in \mathcal{P}$. Since the inclusion map $i : C^*_p(X) \to C_p(X)$ induces a continuous surjection $i^* : L_p(X) \to L^*_p(X)$, we have $L^*_p(X) \in \mathcal{P}$.

We claim that $\Phi^* : Y \to L^*_p(X)$ is a closed embedding. Indeed, since Φ is a continuous surjection, Φ^* is an embedding. To show that $\Phi^*(Y) \subset L^*_p(X)$ is closed choose a net $\{\mu_{y(\alpha)}\colon\alpha \in \Lambda\} \subset \Phi^*(Y)$ converging to $\mu \in L^*_p(X)$. Then for any $f, g \in C^*_p(X)$ with $\Phi(f) = \Phi(g)$ we have $\mu(f) = \mu(g)$ because $\mu_{y(\alpha)}(f) = \mu_{y(\alpha)}(g)$, $\mu(f) = lim\ \mu_{y(\alpha)}(f)$ and $\mu(g) = lim\ \mu_{y(\alpha)}(g)$. Since Φ is quotient, there is $\nu \in L^*_p(Y)$ with $\mu = \nu \circ \Phi$. It is easily seen that the evaluation map $y \to \delta_y$ is a closed embedding from Y into $L^*_p(Y)$. Then $\{\mu_{y(\alpha)}\}$ converges to μ yields $\{\delta_{y(\alpha)}\}$ converges to ν. Therefore $\nu = \delta_y$ for some $y \in Y$, so $\mu = \mu_y \in \Phi^*(Y)$.

Because Y is homeomorphic to the closed set $\Phi^*(Y) \subset L_p^*(X)$ and $L_p^*(X) \in \mathcal{P}$, we finally obtain $Y \in \mathcal{P}$.$\square$

Note that the requirement in Proposition 3.5 for Φ to be quotient cannot be dropped. If, for example, X is not σ-compact, then the restriction map $\Phi : C_p(\beta X) \to C_p^*(X)$ is a continuous, linear surjection, βX is σ-compact and the class of σ-compact spaces has all above properties (a)-(e).

Corollary 3.6. *Let $C_p^*(Y)$ be an image of $C_p^*(X)$ under a continuous quotient linear surjection and let X have one of the following properties: X is σ-compact, X is a Lindelöf Σ-space, X^n is Lindelöf for every $n \in$**N**. Then Y has the same property.*$\square$

The analog of Corollary 3.6 for the spaces $C_p(X)$ and $C_p(Y)$ is also true, even in the more general case when $C_p(X)$ and $C_p(Y)$ are homeomorphic (see Okunev [8]).

Uspenskiî [11] proved that if $C_p(X)$ and $C_p(Y)$ are uniformly homeomorphic, then X is compact iff Y is. Let us note that every linear homeomorphism between function spaces is uniformly continuous. Concerning the bounded functions, the authors know only the following result of Baars, de Groot and Pelant [6]: if X and Y are metrizable and there is a continuous linear surjection from $C_p^*(X)$ onto $C_p^*(Y)$, then X compact yields Y compact. The next theorem shows that this result remains true for any X and Y in the case of linear homeomorphisms.

Theorem 3.7. *Let $C_p^*(X)$ and $C_p^*(Y)$ be linearly homeomorphic. Then X is compact if and only if Y is compact.*

Proof. Suppose X is compact. By Corollary 3.6, Y is σ-compact. Since pseudocompactness is preserved by linear homeomorphisms between $C_p^*(X)$ and $C_p^*(Y)$ (see Theorem 4.3 below), Y is pseudocompact. Hence Y is compact.$\square$

The remarks following Proposition 3.5 show that we cannot extend Theorem 3.7 to continuous linear surjections. The question is whether it is true for continuous quotient linear surjections.

4. Appendix

First, let us show that the requirement in Theorem 3.3 and Corollary 3.4 for X and Y to be perfect can be omitted if $C_p(X)$ and $C_p(Y)$ (resp., $C_p^*(X)$ and $C_p^*(Y)$) are homeomorphic under positive or negative linear homeomorphisms.

Theorem 4.1. *Let $\mathcal{P}$ be a class of spaces satisfying the hypotheses of Theorem 3.3. If there is either a positive or negative linear homeomorphism between $C_p(X)$ and $C_p(Y)$ (resp., $C_p^*(X)$ and $C_p^*(Y)$), then $X \in \mathcal{P}$ if and only if $Y \in \mathcal{P}$.*

Proof. Suppose $\Phi : C_p(X) \to C_p(Y)$ (resp., $\Phi : C_p^*(X) \to C_p^*(Y)$) is a positive linear homeomorphism. Since all μ_y are positive, we have $supp(\mu_y^-) = \emptyset$ and $\left\|\mu_y^-\right\| = 0$ for each $y \in Y$. Then $s_{\mu_y}(x) = \lambda_x$ if $x \in supp(\mu_y^+)$ and $s_{\mu_y}(x) = 0$ otherwise. Consequently, $\varphi_a^*(y)$ is a finite subset of $supp(\mu_y^+)$ if $a > 0$ and $\varphi_a^*(y) = \beta X$ if $a \leq 0$. Let $F_a = \{y \in Y : \varphi_a^*(y) \neq \emptyset\}$ for $a > 0$. Since $F_a = Y - \{y \in Y : \varphi_a^*(y) = \emptyset\}$ and φ_a^* is usc, $F_a \subset Y$ is closed. Consider the family $\gamma = \{F_q : q \in Q^+\}$, where Q^+ is the set of all positive rational numbers. The arguments from Lemma 3.2 show that , for every $y \in Y$, there is $q(y) \in Q^+$ with $\varphi_{q(y)}^*(y) = \sup p(\mu_y^+)$. This yields $y \in F_{q(y)}$. We claim that $X = \cup\{\varphi_q^*(F_q) : q \in Q^+\}$. Indeed, for every $x \in X$ there exists $y \in Y$ such that $x \in supp(\mu_y^+)$ (see the proof of Theorem 3.3). Choosing $q \in Q^+$ with $\varphi_q^*(y) = supp(\mu_y^+)$ we obtain $x \in \varphi_q^*(F_q)$.

To finish the proof it suffices to show that $Y \in \mathcal{P}$ implies $X \in \mathcal{P}$. Since each $F_q \subset Y$ is closed and $\varphi_q^* : F_q \to P(\varphi_q^*(F_q))$ is usco, we have $\varphi_q^*(F_q) \in \mathcal{P}$. Therefore $X \in \mathcal{P}$.

If Φ is a negative linear homeomorphism, then $-\Phi$ is positive and we can apply the proof above.□

Corollary 4.2. *If there exists a positive continuous linear homeomorphism between $C_p(X)$ and $C_p(Y)$ (resp., $C_p^*(X)$ and $C_p^*(Y)$), then $l(X) = l(Y)$.*□

Now, let us show that pseudocompactness is preserved by linear homeomorphisms between $C_p^*(X)$ and $C_p^*(Y)$. It is well known that if f is a map from a space X onto a set Y, then the quotient topology on Y (with respect to f) is not always completely regular. But the strongest among all completely regular topologies on Y with respect to which f is continuous exists and it is called the **R**-*quotient* [2]. The set Y with the **R**-quotient topology is denoted by Y^q and $q(f)$ is the corresponding continuous map from X onto Y^q. We also adopt the following notations. For a space X let $\mathcal{F}_X$ denote the family of all continuous maps from X onto spaces of countable weight, and direct this family by the relation: $g \succ f$ if there is a continuous surjection $h(g, f)$ from $g(X)$ onto $f(X)$ such that $h(g, f) \circ g = f$. For $f \in \mathcal{F}_X$ let $X_f = f(X)$, $C_p(f) = f_p^*(C_p^*(X))$ and $C_p^q(f) = q_p^*(f)(C_p^*(X_f^q))$, where

$f_p^* : C_p^*(X_f) \to C_p^*(X)$ and $q_p^*(f) : C_p^*(X_f^q) \to C_p^*(X)$ are the continuous duals of f and $q(f)$, respectively. When the uniform convergence topology is considered similar notation is used, the only difference is that the subscript p is replaced by u. Note that f_p^* , $q_p^*(f)$, f_u^* and $q_u^*(f)$ are embeddings. We need the following observations:

Fact 1. *$C_p^*(X) = \cup\{C_p(f) : f \in \mathcal{F}_X\}$ and $C_p(f) \subset C_p^q(f)$, $f \in \mathcal{F}_X$.*$\square$

Fact 2. *For every countable $A \subset C^*(X)$ there exists $f \in \mathcal{F}_X$ such that $A \subset C_p(f)$.*$\square$

Fact 3. *$C_p^q(f) \subset C_p^*(X)$ is closed and separable for every $f \in \mathcal{F}_X$.*

Proof. $C_p^q(f) \subset C_p^*(X)$ is closed because $q(f)$ is $\mathbf{R}$-quotient [2]. Since there is a one-to-one map from X_f^q onto X_f, $C_p(f) \subset C_p^q(f)$ is dense [2]. On the other hand $C_p(f)$ is homeomorphic to $C_p^*(X_f)$ and $C_p^*(X_f)$ is separable because X_f has a countable weight.$\square$

An analog of the following theorem was proved in [13] for function spaces with the compact-open topology. The same arguments work in the case of pointwise convergence topology.

Theorem 4.3. *Let $C_p^*(X)$ be linearly homeomorphic to $C_p^*(Y)$. Then X is pseudocompact if and only if Y is.*

Proof. Suppose X is pseudocompact and $\Phi : C_p^*(X) \to C_p^*(Y)$ is a linear homeomorphism. To show that Y is pseudocompact it is enough to prove that $g(Y)$ is compact for every $g \in \mathcal{F}_Y$. So, let $g_1 \in \mathcal{F}_Y$ be fixed. Since, by Fact 3, $C_p^q(g_1) \subset C_p^*(Y)$ is closed and separable, Facts 1 and 2 imply that $\Phi^{-1}(C_p^q(g_1)) \subset C_p^q(f_1)$ for some $f_1 \in \mathcal{F}_X$. Again using Facts 1, 2 and 3 we can find $g_2 \in \mathcal{F}_Y$ such that $\Phi(C_p^q(f_1)) \subset C_p^q(g_2)$. We can assume $g_2 \succ g_1$ (otherwise instead of g_2 we take the diagonal product of g_1 and g_2). Proceeding in this way two sequences $\{f_n\} \subset \mathcal{F}_X$, $\{g_n\} \subset \mathcal{F}_Y$ will be constructed such that:

(1) $f_{n+1} \succ f_n$ and $g_{n+1} \succ g_n$;
(2) $\Phi(C_p^q(f_n)) \subset C_p^q(g_{n+1})$ and $\Phi^{-1}(C_p^q(g_n)) \subset C_p^q(f_n)$.

Let f (resp., g) be the diagonal product of all f_n (resp., g_n). It is easily seen that $\cup_{n\in\mathbf{N}} h_p^*(f, f_n)(C_p^*(f_n(X)))$ is dense in $C_p^*(X_f)$, where $h_p^*(f, f_n) : C_p^*(f_n(X))) \to C_p^*(X_f)$ is the dual of the map $h(f, f_n) : X_f \to f_n(X)$. Because $C_p(f)$ is dense in $C_p^q(f)$, we have that $\cup_{n\in\mathbf{N}} C_p^q(f_n)$ is dense in $C_p^q(f)$ and, similarly, $\cup_{n\in\mathbf{N}} C_p^q(g_n)$ is dense in $C_p^q(g)$. Then, by (2), $\Phi(C_p^q(f)) = C_p^q(g)$. By the closed graph theorem, Φ considered as a map between $C_u^*(X)$ and $C_u^*(Y)$, is a homeomorphism, so $C_u^q(f)$ and $C_u^q(g)$ are homeomorphic. Consequently, $C_u^*(X_f^q)$ and $C_u^*(Y_g^q)$ are linearly homeomorphic (recall that $q_u^*(f)$ and $q_u^*(g)$ are also linear homeomorphisms). But X pseudocompact implies X_f^q is homeomorphic to X_f ([3], see also [10]). Hence $C_u^*(X_f)$ is homeomorphic to $C_u^*(Y_g^q)$. So, $C_u^*(Y_g^q)$ is separable (since X_f compact yields $C_u^*(X_f)$ separable). Therefore, since Y_g^q is compact [9], so is Y_g. Then $g_1(Y)$ is also compact because $g \succ g_1$.□

Added in proof. Recently first author proved that Theorem 3.3 remains true if $C_p(X,E)$ and $C_p(Y,F)$ (resp., $C_p^*(X,E)$ and $C_p^*(Y,F)$) are uniformly homeomorphic, where E and F are normed spaces.

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