Journal of Mathematical Analysis and Applications **242,** 1–17 (2000)
doi:10.1006/jmaa.1999.6625, available online at http://www.idealibrary.com on IDEAL®

Barrelled and Bornological Function Spaces

Julian Dontchev[1]

Department of Mathematics, University of Helsinki, 00014 Helsinki 10, Finland
E-mail: dontchev@cc.helsinki.fi

Sergio Salbany

Department of Mathematics, University of Botswana, Private Bag 022,
Gabarone, Botswana
E-mail: SalbanyS@NOKA.UB.BW

and

Vesko Valov

Department of Mathematics, University of Swaziland, Private Bag 4,
Kwaluseni, Swaziland

Submitted by Ying-ming Liu

Received April 27, 1998

For a completely regular space X and a normed space E let $C_\lambda(X, E)$ be the set of all E-valued continuous maps on X endowed with the topology of uniform convergence on the members of λ, where λ is a family of closed and t-bounded subsets of X such that λ is a cover of X and it is invariant with respect to finite unions. Necessary and sufficient conditions in the terms of the topology of X are given in order $C_\lambda(X, E)$ to be barrelled, quasi-barrelled, bornological, or ultra-bornological space. © 2000 Academic Press

Key Words: function spaces; supports; bounded sets; barrelled, quasi-barrelled, bornological, and ultrabornological spaces.

---

[1] Partially supported by the Magnus Ehrnrooth Foundation at the Finnish Society of Science and Letters.

0022-247X/00 $35.00

1. INTRODUCTION

Throughout this paper we consider only completely regular topological spaces and Hausdorff locally convex linear spaces; E and F are always normed linear spaces; $C(X, E)$ denotes the sets of all E-valued continuous maps on a topological space X; when E is the real line $\mathbf{R}$ we simply write $C(X)$; a subset K of X is *t-bounded* in X if $f(K)$ is a bounded subset of $\mathbf{R}$ for every $f \in C(X)$. When every closed and t-bounded subset of X is compact we say that X is a *μ-space*. Let λ be a cover of a space X with the following properties: every element of λ is closed and t-bounded in X and λ is invariant with respect to finite unions. Recall that an *ideal of closed sets* [8] is a collection of closed sets closed under heredity and finite additivity. Note that often closed ideals have the required properties of the cover λ (for example, the ideals of all finite or compact sets). The set $C(X,E)$ with the topology of uniform convergence on members of λ is denoted by $C_\lambda(X, E)$. We also denoted by $\nu_\lambda(X)$ the space $\bigcup\{\mathrm{cl}_{\nu X}(K): K \in \lambda\}$ with the topology inherited from the Hewitt realcompactification νX of X. Observe that each $\mathrm{cl}_{\nu X}(K)$, $K \in \lambda$, is compact because the members of λ are t-bounded in X. If Z is a subspace of νX containing $\nu_\lambda(X)$, then a set $B \subseteq Z$ is said to be *λ-bounded* in Z if every lower semi-continuous (abbreviated lsc) function φ: $Z \to [0, \infty)$ is bounded on B provided it is bounded on each $K \in \lambda$. Both notions, t-boundedness and λ-boundedness, have to be distinguished from the usual boundedness in a topological vector space.

The aim of this paper is to give necessary and sufficient conditions in the terms of the topological structure of X in order $C_\lambda(X, E)$ to be a barrelled, quasi-barrelled, bornological, or ultrabornological space. Such conditions were given in [5], when E is the space of the real or complex numbers and λ is the family of all compact subsets of X. A more recent paper [12], contains characterizations of the above linear topological properties of $C_\lambda(X)$ in the terms of filters on X. The following are our main results:

THEOREM 1. *Let E be a Banach space. Then $C_\lambda(X, E)$ is barrelled if and only if every t-bounded subset of $\nu_\lambda(X)$ is contained in $\mathrm{cl}_{\nu X}(K)$ for some $K \in \lambda$.*

THEOREM 2. *$C_\lambda(X, E)$ is quasi-barrelled if and only if every λ-bounded subset of $\nu_\lambda(X)$ is contained in $\mathrm{cl}_{\nu X}(K)$ for some $K \in \lambda$.*

THEOREM 3. *$C_\lambda(X, E)$ is bornological if and only if every λ-bounded subset of νX is contained in $\mathrm{cl}_{\nu X}(K)$ for some $K \in \lambda$.*

THEOREM 4. *Let E be a Banach space. Then $C_\lambda(X, E)$ is ultrabornological if and only if every t-bounded subset of νX is contained in $\mathrm{cl}_{\nu X}(K)$ for some $K \in \lambda$.*

This paper is organized as follows: Section 2 contains all preliminary facts and lemmas needed for the proofs of the main results, which are given in Section 3. The last section is devoted to some corollaries and additional results. Let us mention Corollary 4.4 stating that for a Banach space E the space $C_\lambda(X, E)$ is ultrabornological if and only if it is bornological and barrelled, and Corollary 4.6 that realcompactness is preserved by continuous linear quotient maps from $C_k(X, E)$ onto $C_k(Y, F)$ for arbitrary normed spaces E and F.

2. PRELIMINARIES

Recall that a set $W \subseteq C_\lambda(X, E)$ is open if and only if for every $f \in C(X, E)$ there are $K \in \lambda$ and $\varepsilon > 0$ such that $A(f, K, \varepsilon) \subseteq W$, where $A(f, K, \varepsilon) = \{g \in C(X, E) : \|g(x) - f(x)\| < \varepsilon \text{ for all } x \in K\}$. The sets $A(f, K, \varepsilon)$ are not always open in this topology (see [3, 4]), but $\{g \in C(X, E) : \sup\{\|g(x)\| : x \in K\} < \varepsilon, \ K \in \lambda, \ \varepsilon > 0\}$ is a local base at 0 consisting of open convex sets. It is easily seen that $C_\lambda(X, E)$ is a locally convex Hausdorff topological vector space and its topology is determined by the family $\{p_K : K \in \lambda\}$ of semi-norms, where $p_K(f) = \sup\{\|f(x)\| : x \in K\}$. If λ is the closed ideal of all compact (resp. all finite) subsets of X, then we write $C_k(X, E)$ (resp. $C_p(X, E)$) instead of $C_\lambda(X, E)$. By $\nu_E X$, we denote the biggest subset Z of νX such that every $f \in C(X, E)$ has a continuous extension $\bar{f}\colon Z \to E$. There is a closed embedding of $\nu_E X$ in $E^{C(X,E)}$, so $\nu_E X$ is a μ-space. We also adopt the following notation: E and F are equipped with a fixed norm which is denoted by $\|\cdot\|$ and $\||\cdot|\|\colon \beta E \to [0, \infty]$ is the continuous extension of $\|\cdot\|$ considered as a function from E into $[0, \infty)$, where βE is the Čech–Stone compactification of E. $\mathrm{BL}(C_\lambda(X, E), F)$ denotes the set of all bounded linear maps from $C_\lambda(X, E)$ into F.

Let Y be a locally convex topological space. Recall that an absolutely convex set $A \subseteq Y$ is a *bornivore* or *bound absorbing* (resp. a *barrel*) if A absorbs all bounded subsets of Y (resp. A is closed and absorbs all points of Y). We say that Y is *barrelled* (resp. *bornological*) [9] if every barrel (resp. bornivore) in Y is a neighborhood of 0. If every bound-absorbing barrel in Y is a neighborhood of 0, then Y is called *quasi-barrelled.* When every absolutely convex set $A \subseteq Y$ is a neighborhood of 0 in Y, provided it absorbs all bounded, absolutely convex, and sequentially complete subsets of Y, then Y is said to be *ultrabornological.*

Let μ be a linear map from $C(X, E)$ into a linear space. The support of μ is the set $\mathrm{supp}(\mu)$ of all points $x \in \beta X$ satisfying the condition that for every neighborhood U of x in βX there is $f \in C(X, E)$ such that $\mu(f) \neq 0$ and $(\beta f)(\beta X \setminus U) = 0$, where $\beta f\colon \beta X \to \beta E$ is the continuous extension of f. Obviously $\mathrm{supp}(\mu)$ is closed in βX, so it is compact. If $\mathscr{A}$ is a family of linear maps from $C(X, E)$ into Y, then we denote by $\operatorname{supp}(\mathscr{A})$ the union of all $\operatorname{supp}(\mu)$, $\mu \in \mathscr{A}$.

LEMMA 2.1. *Let $\mathscr{A} \subseteq \mathrm{BL}(C_\lambda(X, E), F)$ be a strongly bounded family, i.e., $\bigcup\{\mu(A) : \mu \in \mathscr{A}\}$ is a bounded set in F for every bounded subset A of $C_\lambda(X, E)$. Then $\operatorname{supp}(\mathscr{A})$ is contained in νX and it is a λ-bounded subset of every space $Z \subseteq \nu X$ such that $\nu_\lambda(X) \cup \operatorname{supp}(\mathscr{A}) \subseteq Z$.*

Proof. First let us show that $\operatorname{supp}(\mathscr{A}) \subseteq \nu X$. Suppose there is a point $x_0 \in \operatorname{supp}(\mu^*) \setminus \nu X$ for some $\mu^* \in \mathscr{A}$. Let $\{U_n\}$ be a decreasing sequence of compact neighborhoods of x_0 in βX with $\bigcap\{U_n : n \in \mathbf{N}\}$ is contained in $\beta X \setminus \nu X$. For every n there is $f_n \in C(X, E)$ such that $(\beta f_n)(\beta X \setminus U_n) = 0$ and $\mu^*(f_n) \neq 0$.

Claim. If $g \in C(X, E)$ and $(\beta g)(B) = 0$ for some $B \subseteq \beta X$, then $\beta(cf)(B) = 0$ for every $c \in \mathbf{R}$.

Fix $c \in \mathbf{R}$ and consider the continuous map $\psi \colon \beta E \to \beta E$, $\psi(y) = c.y$ for all $y \in E$. Both $\beta(cg)$ and $\psi \circ \beta g$ are continuous maps from βX into βE with identical restrictions on X, so $\beta(cg) = \psi \circ \beta g$. The last equality implies $\beta(cg)(B) = 0$.

We now proceed to the proof of Lemma 2.1. By the above claim, multiplying f_n by a scalar, if necessary, we can assume that $\|\mu^*(f_n)\| \geq n$ for every n. Fix $K \in \lambda$. Since K is t-bounded in X, $\mathrm{cl}_{\nu X}(K)$ is compact and can meet only finitely many U_n. So $p_K(f_n) = 0$ for almost all n. Thus, $\{f_n\}$ is bounded in $C_\lambda(X, E)$. Consequently, $\{\mu^*(f_n)\}$ is bounded in F, a contradiction.

Suppose Z is a subspace of νX containing $\nu_\lambda(X) \cup \operatorname{supp}(\mathscr{A})$ and $\operatorname{supp}(\mathscr{A})$ is not λ-bounded in Z; i.e., there is a lsc function $\varphi \colon Z \to [0, \infty)$ which is bounded on each $K \in \lambda$ and unbounded on $\operatorname{supp}(\mathscr{A})$. Then for each $n \in \mathbf{N}$ we can find $\mu_n \in \mathscr{A}$ and $x_n \in \varphi^{-1}(n, \infty) \cap \operatorname{supp}(\mu_n)$. Construct a sequence $\{g_n\} \subseteq C(X, E)$ such that:

(1) $g_n(\varphi^{-1}[0, n]) = 0$;

(2) $\|\mu_n(g_n)\| = n + \|b_n\|$, where $b_1 = 0$ and $b_n = \sum\{\mu_n(g_i) : i \leq n - 1\}$ for $n \geq 2$.

Since $\varphi^{-1}[0, 1]$ is closed in Z and $x_1 \in \operatorname{supp}(\mu_1) \setminus \varphi^{-1}[0, 1]$, there is $h \in C(X, E)$ with $\mu_1(h) \neq 0$ and $(\beta h)(\varphi^{-1}[0, 1]) = 0$. Set $g_1 = h/\|\mu_1(h)\|$. Assume $g_1, \ldots, g_{n-1}$ are already constructed. Take an open set W in βX such that $W \cap Z = \varphi^{-1}(n, \infty)$. Because $x_n \in W \cap \operatorname{supp}(\mu_n)$, then there exists $f \in C(X, E)$ with $(\beta f)(\beta X \setminus W) = 0$ and $\mu_n(f) \neq 0$. Let $g_n = (n + \|b_n\|)f/\|\mu_n(f)\|$. The inductive construction is completed. We claim that $\{g_n\}$ is bounded in $C_\lambda(X, E)$. Since φ is bounded on the members of λ, for each $K \in \lambda$ an $m_K \in \mathbf{N}$ can be found such that $K \subseteq \varphi^{-1}[0, m_K]$. By (1),

$p_K(g_n) = 0$ for all $n \geq m_K$. Hence $\{g_n\}$ is bounded, which implies that $\bigcup\{\mu_n(g_n)\}$ is a bounded subset of F. This contradicts (2). ▮

COROLLARY 2.2. *If μ is a bounded linear map from $C_\lambda(X, E)$ into F, then* supp(μ) *is a λ-bounded set in νX. If, in addition μ is continuous, then* supp$(\mu) \subseteq \nu_\lambda(X)$.

Proof. The first part of Corollary 2.2 follows from Lemma 2.1. Let μ be continuous on $C_\lambda(X, E)$. There exist $K \in \lambda$ and $\varepsilon > 0$ such that $\|\mu(g)\| \leq 1$ for every $g \in C(X, E)$ with $p_K(g) \leq \varepsilon$. Then $\operatorname{supp}(\mu) \subseteq \operatorname{cl}_{\nu X}(K)$. For if $x \in \operatorname{supp}(\mu) \setminus \operatorname{cl}_{\nu X}(K)$, there is a neighborhood U of x in βX and $f \in C(X, E)$ such that $\beta f|(\beta X \setminus U) = 0$, $U \cap \operatorname{cl}_{\nu X}(K) = \emptyset$ and $\mu(f) \neq 0$. By the Claim following Lemma 2.1, we can assume that $\|\mu(f)\| > 1$. But $(\beta f)(K) = 0$, so $p_K(f) = 0$, which contradicts $\|\mu(f)\| > 1$. ▮

Let $f \in C(X, E)$ and $\varepsilon > 0$. The ε-modification of f is the map $f_\varepsilon \in C(X, E)$, defined by $f_\varepsilon(x) = f(x)$ if $\|f(x)\| \leq \varepsilon$ and by $f_\varepsilon(x) = (\varepsilon f(x))/\|f(x)\|$ otherwise. Obviously $\|f_\varepsilon(x)\| \leq \varepsilon$ for all $x \in X$.

LEMMA 2.3. *Let μ: $C_\lambda(X, E) \to F$ be a bounded linear map. Then $\mu(f) = 0$ for every $f \in C(X, E)$ with $(\beta f)(\operatorname{supp}(\mu)) = 0$.*

Proof.

Step 1. If U is a neighborhood of $\operatorname{supp}(\mu)$ in βX and $f \in C(X, E)$ with $(\beta f)(U) = 0$, then $\mu(f) = 0$.

For $x \notin \operatorname{supp}(\mu)$ take a neighborhood $U(x)$ of x in βX such that $\mu(g) = 0$ provided $g \in C(X, E)$ and $(\beta g)(\beta X \setminus U(x)) = 0$. Without loss of generality, we can suppose that all $U(x)$ and U are regular open in βX and $U(x) \cap \operatorname{supp}(\mu) = \emptyset$, $x \notin \operatorname{supp}(\mu)$. There is a finite cover $\gamma = \{U, U(x_i) : i = 1, \dots, k\}$ of βX and a partition of unity $\{h, h_i : i = 1, \dots, k\}$ subordinated to γ. Set $g_0 = h.f$ and $g_i = h_i.f$, $i = 1, \dots, k$. Since $f = \Sigma\{g_i : i = 0, 1, \dots, k\}$, $\mu(f) = \Sigma\{\mu(g_i) : i = 0, 1, \dots, k\}$. Observe that each g_i, $i = 1, \dots, k$, is 0 on the set $X \setminus U(x_i)$, and because $X \setminus U(x_i)$ is dense in $\beta X \setminus U(x_i)$ we have $(\beta g_i)(\beta X \setminus U(x_i)) = 0$. Hence $\mu(g_i) = 0$ for $i = 1, \dots, k$. Obviously $g_0 \equiv 0$, so $\mu(g_0) = 0$. Therefore, $\mu(f) = 0$.

Step 2. If $f \in C(X, E)$ and $\beta f|\operatorname{supp}(\mu) = 0$, then $\|\mu(f)\| \leq 1$.

Denote by $C_U^*(X, E)$ the space of all bounded continuous maps from X into E with the sup-norm topology. Then μ considered as a linear map on $C_U^*(X, E)$ is bounded, hence continuous. Thus, there is $\varepsilon > 0$ such that $g \in C^*(X, E)$ and $\|g\| = \sup\{\|g(x)\| : x \in X\} \leq \varepsilon$ implies $\|\mu(g)\| \leq 1$. Set $U = \{x \in \beta X : \|\,|\beta f(x)|\,\| < \varepsilon\}$. Observe that $(f - f_\varepsilon) \mid (U \cap X) = 0$, so $\beta(f - f_\varepsilon)(U) = 0$, where f_ε is the ε-modification of f. Since U is a neighborhood of $\operatorname{supp}(\mu)$, by Step 1, $\mu(f - f_\varepsilon) = 0$, i.e., $\mu(f) = \mu(f_\varepsilon)$. Because $f_\varepsilon \in C^*(X, E)$ and $\|f_\varepsilon\| \leq \varepsilon$, we have $\|\mu(f)\| = \|\mu(f_\varepsilon)\| \leq 1$.

Step 3. If $f \in C(X, E)$ and $\beta f \mid \operatorname{supp}(\mu) = 0$, then $\mu(f) = 0$.

Suppose $\mu(f) \neq 0$. Choose $n \in \mathbf{N}$ such that $\|\mu(nf)\| > 1$. Then $\beta(nf)(\operatorname{supp}(\mu)) = 0$ (see the Claim following Lemma 2.1). By Step 2, $\|\mu(nf)\| \leq 1$, a contradiction. ∎

The following lemma is a slight modification of Proposition 3.1 from [11] (see also Proposition 2 from [1] and Proposition 1.2.8 from [4]).

LEMMA 2.4. *Let* $\mathscr{A} \subseteq \mathrm{BL}(C_\lambda(X, E), F)$ *be a pointwise bounded family, i.e.,* $\{\mu(f) : \mu \in \mathscr{A}\}$ *is a bounded subset of* F *for each* $f \in C(X, E)$. *Then* $\operatorname{supp}(\mathscr{A})$ *is a t-bounded set in* νX.

Proof. By Lemma 2.1, $\operatorname{supp}(\mathscr{A}) \subseteq \nu X$. Suppose $\operatorname{supp}(\mathscr{A})$ is not t-bounded in νX. Then there is a continuous real-valued function f on νX and points $x_n \in \bigcup\{\operatorname{supp}(\mu) : \mu \in \mathscr{A}\}$, $n \in \mathbf{N}$, such that $\{f(x_n) : n \in \mathbf{N}\}$ is a discrete and unbounded in $\mathbf{R}$. Embedding $\mathbf{R}$ in E, we can assume that f is a map from νX into E. Then we can find an open family $\{V_n : n \in \mathbf{N}\}$ in νX such that $x_n \in V_n$, $\{f(V_n) : n \in \mathbf{N}\}$ is a discrete family in E and $\sup\{\|f(y_n)\| : n \in \mathbf{N}\} = \infty$ for any sequence $\{y_n\}$ with $y_n \in V_n$. By induction we construct a sequence $\{\mu_k : k \in \mathbf{N}\}$ in $\mathscr{A}$, a subfamily $\{U_k : k \in \mathbf{N}\}$ of $\{V_n : n \in \mathbf{N}\}$, and a subset $\{h_k : k \in \mathbf{N}\}$ of $C(X, E)$ such that:

(1) $(\beta h_k)(\nu X \setminus U_k) = 0$ for every $k \in \mathbf{N}$;

(2) $U_i \neq U_j$, for $i \neq j$;

(3) $\operatorname{supp}\{\mu_1, \dots, \mu_{k-1}\} \cap \operatorname{cl}_{\nu X}(U_k) = \emptyset$ for every $k \geq 2$;

(4) $\|\mu_k(h_k)\| = k + \|b_k\|$ for every $k \in \mathbf{N}$, where $b_k = \Sigma\{\mu_k(h_i) : i < k\}$ for $k \geq 2$ and $b_1 = 0$.

Let $\mu_1 \in \mathscr{A}$ be such that $x_1 \in \operatorname{supp}(\mu_1)$ and $U_1 = V_1$. Then there is $h \in C(X, E)$ such that $(\beta h)(\nu X \setminus U_1) = 0$ and $\mu_1(h) \neq 0$. Let $q = 1/\|\mu_1(h)\|$ and $h_1 = q.h$. Then $(\beta h_1)(\nu X \setminus U_1) = 0$ (see the Claim following Lemma 2.1) and $\|\mu_1(h_1)\| = 1$.

Let $k \geq 2$ and suppose we have found $\mu_1, \dots, \mu_{k-1}$, $U_1, \dots, U_{k-1}$, and $h_1, \dots, h_{k-1}$ satisfying (1)–(4). Since $H_k = \operatorname{supp}\{\mu_1, \dots, \mu_{k-1}\}$ is a compact subset of νX (by Lemma 2.1), there is $n \in \mathbf{N}$ such that $\operatorname{cl}_{\nu X}(V_n) \cap H_k = \emptyset$. Take $\mu_k \in \mathscr{A}$ with $x_n \in \operatorname{supp}(\mu_k)$. Let $U_k = V_n$. Since U_k is a neighborhood of x_n and since $x_n \in \operatorname{supp}(\mu_k)$, there is $h \in C(X, E)$ such that $(\beta h)(\nu X \setminus U_k) = 0$ and $\mu_k(h) \neq 0$. Put $c = (k + \|b_k\|)/\|\mu_k(h)\|$ and $h_k = c.h$. Then $(\beta h_k)(\nu X \setminus U_k) = 0$ and $\|\mu_k(h_k)\| = k + \|b_k\|$. Observe that by (3) and the fact that $x_n \in \operatorname{supp}(\mu_k) \cap U_k$ we have $U_i \neq U_j$ for $i \neq j$, with $i, j \in \{1, 2, \dots, k\}$. This completes the inductive construction.

Since $\{U_k : k \in \mathbf{N}\}$ is a subfamily of $\{V_n : n \in \mathbf{N}\}$, by (2), $\{U_k : k \in \mathbf{N}\}$ is a discrete family in νX. Let $g = \Sigma\{h_k : k \in \mathbf{N}\}$. By (1), every $x \in X$ has a neighborhood $U(x)$ such that $g|U(x)$ is a finite sum. Thus, $g \in C(X, E)$. For every $k \in \mathbf{N}$ we set $g_k = \Sigma\{h_i : i \leq k\}$, $A_k = \operatorname{cl}_{\nu X}[(\beta h_k)^{-1}(\beta E \setminus \{0\})$

$\cap\, \nu X]$ and $W_k = \nu X \setminus \bigcup\{A_j : k \le j-1\}$. By (1), $A_j \subseteq \mathrm{cl}_{\nu X}(U_j)$ for every $j \in \mathbf{N}$. Hence, $\{A_j : j \in \mathbf{N}\}$ is a discrete family in νX, so W_k is open in νX. By (3), $\mathrm{supp}(\mu_k) \cap \mathrm{cl}_{\nu X}(U_j) = \emptyset$ for $j \ge k+1$. This implies $\mathrm{supp}(\mu_k) \cap A_j = \emptyset$, so $\mathrm{supp}(\mu_k) \subseteq W_k$. For $j \ge k+1$, h_j is 0 on $W_k \cap X$. Hence, $g \mid (W_k \cap X) = g_k \mid (W_k \cap X)$, so $\beta(g - g_k) \mid W_k = 0$. Since W_k is a neighborhood of $\mathrm{supp}(\mu_k)$ in νX, we have, by Lemma 2.3, that $\mu_k(g) = \mu_k(g_k)$. But $\mu_k(g_k) = b_k + \mu_k(h_k)$. So that, by (4), $\|\mu_k(g_k)\| \ge \|\mu_j(h_k)\| - \|b_k\| = k + \|b_k\| - \|b_k\| = k$. Thus, $\|\mu_k(g)\| \ge k$ for every $k \in \mathbf{N}$. Consequently, $\mathscr{A}$ is not pointwise bounded. This gives us a contradiction. ▮

Let V be a subset of $C(X,E)$. Then the support of V is the set $\mathrm{supp}(V)$ of all $x \in \beta X$ satisfying the condition that for every neighborhood U of x in βX there is $f \in C(X,E)$ such that $(\beta f)(\beta X/U) = 0$ and $f \notin V$. We have to note that a similar notion was defined in [5]: A closed set $B \subseteq \beta X$ is a support set of V if $f \in V$ whenever $\beta f|B = 0$, and the intersection of all support sets is the support of V. Obviously, $\mathrm{supp}(V)$ is closed in βX and $V_1 \subseteq V_2$ implies $\mathrm{supp}(V_2) \subseteq \mathrm{supp}(V_1)$. We also have $\mathrm{supp}(C(X,E)) = \emptyset$. The converse implication follows from the following:

LEMMA 2.5. *Let $V \subseteq C(X,E)$ be an absolutely convex set.*

(i) *If U is a neighborhood of* $\mathrm{supp}(V)$ *in βX, then $f \in V$ provided $f \in C(X,E)$ and $\beta f|U = 0$.*

(ii) *If $\{g \in C^*(X,E) : \|g\| \le \eta\} \subseteq V$ for some $\eta > 0$, then there is $\varepsilon > 0$ with $\{g \in C(X,E) : \|\,|\beta g(x)|\,\| < \varepsilon$ for all $x \in$* $\mathrm{supp}(V)\} \subseteq V$.

Proof. (i) Let U be a neighborhood of $\mathrm{supp}(V)$ and $f \in C(X,E)$ with $(\beta f)(U) = 0$. For every $x \notin \mathrm{supp}(V)$ take a neighborhood $U(x)$ of x in βX such that $U(x) \cap \mathrm{supp}(V) = \emptyset$ and $g \in V$ for every $g \in C(X,E)$ with $(\beta g)(\beta X \setminus U(x)) = 0$. Without loss of generality, we can suppose that all $U(x)$ and U are regular open in βX. There is a finite cover $\gamma = \{U, U(x_i) : i = 1,\dots,k-1\}$ of βX and a partition of unity $\{h, h_i : i = 1,\dots,k-1\}$ subordinated to γ. Set $g_0 = h.f$ and $g_i = h_i.f$, $i = 1,\dots,k-1$. Since V is convex and $f = \sum\{(1/k)(k.g_i) : i = 0,1,\dots,k-1\}$, then it is enough to show that each $kg_i \in V$. As in Lemma 2.3, Step 1, we can prove that $(\beta g_i)(\beta X \setminus U(x_i)) = 0$ for $i = 1,\dots,k-1$. Then $\beta(kg_i)(\beta X \setminus U(x_i)) = 0$ (see the Claim following Lemma 2.1), so $kg_i \in V$ for $i = 1,\dots,k-1$. Since $kg_0 \equiv 0$ and since V is absolutely convex, we have $kg_0 \in V$.

(ii) Let $I(\eta) = \{g \in C^*(X,E) : \|g\| \le \eta\} \subseteq V$ for some $\eta > 0$. Let $\varepsilon = \eta/2$ and $g \in C(X,E)$ be such that $\|\,|\beta g(x)|\,\| < \varepsilon$ for all $x \in \mathrm{supp}(V)$. Then $\sup\{\|\,|\beta g(x)|\,\| : x \in \mathrm{supp}(V)\} < \delta$ for some positive $\delta < \varepsilon$, because $\mathrm{supp}(V)$ is compact. Consider the neighborhood U_δ of $\mathrm{supp}(V)$ in βX, $U_\delta = \{x \in \beta X : \|\,|\beta g(x)|\,\| < \delta\}$, and the δ-modification g_δ of g. We have $2(g - g_\delta)(x) = 0$ for all $x \in U_\delta \cap X$, so $\beta(2(g - g_\delta))(U_\delta) = 0$, because $U_\delta \cap X$ is dense in U_δ. By (i), $2(g - g_\delta) \in V$. But $2g_\delta \in I(\eta)$, so $2g_\delta \in V$. Since g can be represented in the form $\frac{1}{2}(2(g - g_\delta)) + \frac{1}{2}(2g_\delta)$, we finally obtain $g \in V$. ▮

LEMMA 2.6. *Let V be an absolutely convex set in $C(X, E)$ absorbing all segments $I_f = \{g \in C(X, E) : \|g(x)\| \le \|f(x)\|$ for all $x \in X\}$, $f \in C(X, E)$. Then* $\operatorname{supp}(V) \subseteq \nu X$.

Proof. Our proof follows the argument from [5, p. 102]. Suppose there is $x_0 \in \operatorname{supp}(V) \setminus \nu X$. Let $\{U_n\}$ be a decreasing sequence of compact neighborhoods of x_0 in βX such that their intersection is contained in $\beta X \setminus \nu X$. For every n there is $f_n \in C(X, E)$ with $(\beta f_n)(\beta X \setminus U_n) = 0$ and $f_n \notin V$. Define $\varphi \colon X \to \mathbf{R}$ and $f \colon X \to E$ by $\varphi(x) = \sup\{n.\|f_n(x)\| : n \in \mathbf{N}\}$ and $f(x) = \varphi(x).e$, where $e \in E$ and $\|e\| = 1$. Observe that $\varphi(x) = \sup\{i.\|f_i(x)\| : i = 1, \dots, n\}$ for every $x \in X \setminus U_n$, so φ is continuous on each set $X \setminus U_n$. Because $\{X \setminus U_n : n \in \mathbf{N}\}$ is an increasing sequence of open sets and their union covers X, φ is continuous and so is f. Since V absorbs I_f, there exists $m \in \mathbf{N}$ with $I_f \subseteq mV$. But $m.f_m \in I_f$; hence $m.f_m \in m.V$. Thus we obtain $f_m \in V$, which of course is a contradiction. ▮

COROLLARY 2.7. *Let X be a realcompact space and $V \subseteq C(X, E)$ be absolutely convex. Then V is a bornivore in $C_k(X, E)$ if and only if V absorbs all segments $I_f = \{g \in C(X, E) : \|g(x)\| \le \|f(x)\|$ for all $x \in X\}$, $f \in C(X, E)$.*

Proof. Since $X = \nu X$, by Lemma 2.6, $\operatorname{supp}(V)$ is a compact subset of X. Because each I_f is a bounded set in $C_k(X, E)$, we only have to show that V is a bornivore provided it absorbs all segments I_f. To this end, suppose V absorbs all I_f, $f \in C(X, E)$. According to Lemma 2.5(ii), there is $\varepsilon > 0$ such that V contains the set $J(\varepsilon) = \{g \in C(X, E) : \|g(x)\| < \varepsilon$ for all $x \in \operatorname{supp}(V)\}$. Because $J(\varepsilon)$ is a neighborhood of 0 in $C_k(X, E)$, it absorbs all bounded sets in $C_k(X, E)$; hence so does V. Therefore, V is a bornivore in $C_k(X, E)$. ▮

LEMMA 2.8. *Let B be a t-bounded set in νX and let $\varepsilon > 0$. Then the set $V_B(\varepsilon) = \{f \in C(X, E) : \sup\{\| \,|\beta f(x)|\, \| : x \in B\} \le \varepsilon\}$ is absolutely convex, sequentially closed, and absorbing in $C_\lambda(X, E)$. If, in addition, B is λ-bounded, then $V_B(\varepsilon)$ is a bornivore in $C_\lambda(X, E)$.*

Proof. (a) $V_B(\varepsilon)$ is absolutely convex. If $f_1, f_2 \in V_B(\varepsilon)$ and $q \in [0, 1]$, let $U_i = \{x \in \beta X : \| \,|\beta f_i(x)|\, \| < \eta\}$, where $\eta > \varepsilon$ and $i = 1, 2$. Then $U = U_1 \cap U_2$ is open in βX and contains B. Since $U \cap X$ is dense in U and $\|qf_1(x) + (1 - q)f_2(x)\| < \eta$ for all $x \in U \cap X$, $\sup\{\| \,|\beta[qf_1 + (1 - q)f_2](x)|\, \| : x \in B\} \le \eta$. Observe that the last inequality is true for any $\eta > \varepsilon$, which implies $qf_1 + (1 - q)f_2 \in V_B(\varepsilon)$. Hence $V_B(\varepsilon)$ is convex.

The same arguments with f_2 replaced by 0 show that $V_B(\varepsilon)$ is absolutely convex.

(b) $V_B(\varepsilon)$ is absorbing. For a fixed $f \in C(X, E)$, $\varphi_f(x) = \|f(x)\|$, $x \in X$, defines a continuous real-valued function on X. Obviously, $\overline{\varphi}_f\colon \nu X \to \mathbf{R}$, $\overline{\varphi}_f(x) = \|\,|\beta f(x)|\,\|$, is a continuous extension of φ_f. Because B is t-bounded in νX, there is $n \in \mathbf{N}$ such that $\sup\{\|\,|\beta f(x)|\,\| : x \in B\} \le n.\varepsilon$. Since $\overline{\varphi}_{(1/n)f} = (1/n)\overline{\varphi}_f$, the last inequality is equivalent to $\sup\{\|\,|\beta((1/n)f)(x)|\,\| : x \in B\} \le \varepsilon$. So $(1/n)f \in V_B(\varepsilon)$, i.e., $f \in n.V_B(\varepsilon)$. Thus $V_B(\varepsilon)$ absorbs f.

(c) $V_B(\varepsilon)$ is sequentially closed. Let $\{f_n\} \subseteq V_B$ be a convergent sequence in $C_\lambda(X, E)$ and let $f = \lim f_n$. We show that $f \in V_B(\varepsilon)$. Assuming this is not the case, there are $y \in B$ and $\eta > \varepsilon$ such that $\|\,|\beta f(y)|\,\| > \eta$. Set $W_n = \{x \in \nu X : |\,\|\,|\beta f(x)|\,\| - \|\,|\beta f_n(x)|\,\|\,| > \frac{\eta - \varepsilon}{2}\}$. Since $\|\,|\beta f_n(y)|\,\| \le \varepsilon$, each W_n is a neighborhood of y in νX. Then $W = \bigcap\{W_n : n \in \mathbf{N}\}$ is a G_δ-set in νX containing y, so there is $x^* \in W \cap X$. Because $\{f_n\}$ converges to f implies $\{\|f_n(x^*)\|\}$ converges to $\|f(x^*)\|$, we have $|\,\|f(x^*) - \|f_k(x^*)\|\,| < \frac{\eta - \varepsilon}{2}$ for some $k \in \mathbf{N}$. The last inequality contradicts $x^* \in W_k$. Therefore $f \in V_B(\varepsilon)$.

(d) $V_B(\varepsilon)$ is a bornivore in $C_\lambda(X, E)$ provided B is λ-bounded. We need to show that $V_B(\varepsilon)$ absorbs all bounded subsets of $C_\lambda(X, E)$. To this end, let $A \subseteq C_\lambda(X, E)$ be an arbitrary bounded set. Define a function $\varphi_A\colon \nu X \to [0, \infty]$ by $\varphi_A(x) = \sup\{\|\,|\beta g(x)|\,\| : g \in A\}$. It is clear that $G_a = \varphi_A^{-1}((a, \infty])$ is open in νX for every $a \in (0, \infty)$. If $\varphi_A(z) = \infty$ for some $z \in \nu X$, then $G = \bigcap\{G_n : n \in \mathbf{N}\}$ is a non-empty G_δ-set in νX. Hence $x \in G \cap X$ for some x, which implies $\varphi_A(x) = \infty$. This contradicts the fact that A is bounded in $C_\lambda(X, E)$. Thus, φ_A is a lsc function from νX into $[0, \infty)$. Since A is bounded, φ_A is bounded on each $K \in \lambda$. Then, according to the λ-boundedness of B, there is $m \in \mathbf{N}$ with $\varphi_A(B) \subseteq [0, m]$. This implies $\overline{\varphi}_g(B) \subseteq [0, m]$ for every $g \in A$, where $\overline{\varphi}_g$ is the function defined in (b). Proceeding as in (b) we can show that $A \subseteq m.V_B(\varepsilon)$. Thus, $V_B(\varepsilon)$ absorbs all bounded sets in $C_\lambda(X, E)$. ▮

COROLLARY 2.9. *Let $B \subseteq \nu_\lambda(X)$ be a t-bounded set in νX and let $\varepsilon > 0$. Then the set $V_B(\varepsilon)$ is a barrel in $C_\lambda(X, E)$. If, in addition, B is λ-bounded, then $V_B(\varepsilon)$ is a bound-absorbing barrel in $C_\lambda(X, E)$.*

Proof. There exists a linear map $u\colon C(X, E) \to C(\nu_E X, E)$ such that $u(g)|X = g$ for every $g \in C(X, E)$. We also have $\nu_\lambda(X) \subseteq \nu_E X$ because $\nu_E X$ is a μ-space containing X. Hence $\beta g(x) = u(g)(x)$ for $x \in \nu_E X$ and $g \in C(X, E)$. By Lemma 2.8, we have to prove only that $V_B(\varepsilon)$ is closed in $C_\lambda(X, E)$. To this end, let $\{f_\alpha : \alpha \in \Lambda\} \subseteq V_B(\varepsilon)$ be a net in $C_\lambda(X, E)$ converging to $f \in C_\lambda(X, E)$. Assuming $f \notin V_B(\varepsilon)$, there are $x_0 \in B$ and $\eta > \varepsilon$ such that $\|\,|\beta f(x_0)|\,\| > \eta$. Then $x_0 \in \mathrm{cl}_{\nu X}(K)$ for some $K \in \lambda$ because $B \subseteq \nu_\lambda(X)$. Since $\lim p_K(f - f_\alpha) = 0$, we can find $\gamma \in \Lambda$ with

$p_K(f - f_\gamma) < \frac{\eta - \varepsilon}{2}$. Let $W_1 = \{x \in \upsilon_E X : \|u(f)(x)\| > \eta\}$ and $W_2 = \{x \in \upsilon_E X : \|u(f_\gamma)(x)\| < \frac{\eta + \varepsilon}{2}\}$. Obviously, $W = W_1 \cap W_2$ is open in $\upsilon_E X$ and contains X_0, so there is $x^* \in W \cap K$. Then $x^* \in K$ implies $\|f(x^*) - f_\gamma(x^*)\| < \frac{\eta - \varepsilon}{2}$. Since $x^* \in W_2$, we have $\|f(x^*)\| \leq \|f(x^*) - f_\gamma(x^*)\| + \|f_\gamma(x^*)\| < \frac{\eta - \varepsilon}{2} + \frac{\varepsilon + \eta}{2} = \eta$. On the other hand, $\|f(x^*)\| = \|u(f)(x^*)\| > \eta$, because $x^* \in W_1$, a contradiction. ▮

If E and F are normed spaces, then $[C(X, E), \mathrm{BL}(C_\lambda(X, E), F)]$ is a dual pair, so we can define polar sets. If $A \subseteq C(X, E)$ recall that $A^\circ = \{\mu \in \mathrm{BL}(C_\lambda(X, E), F) : \|\mu(f)\| \leq 1$ for all $f \in A\}$ is the polar of A and $B^\circ = \{f \in C(X, E) : \|\mu(f)\| \leq 1$ for all $\mu \in B\}$ is the polar of $B \subseteq \mathrm{BL}(C_\lambda(X, E), F)$. We also consider the polar A°_C of A relative to the space $\mathrm{CL}(C_\lambda(X, E), F)$ of all continuous linear maps from $C_\lambda(X, E)$ into F, i.e., $A^\circ_C = \{\mu \in \mathrm{CL}(C_\lambda(X, E), F) : \|\mu(f)\| \leq 1$ for all $f \in A\}$. An easy check shows that $A \subseteq (A^\circ_C)^\circ \subseteq A^{\circ\circ}$, and if A is a closed absolutely convex set in $C_\lambda(X, E)$, then $A = A^{\circ\circ}$.

LEMMA 2.10. *Let V be an absolutely convex subset of $C(X, E)$. Then* $\mathrm{supp}(V^\circ_C) \subseteq \mathrm{supp}(V^\circ) \subseteq \mathrm{supp}(V^{\circ\circ}) \subseteq \mathrm{supp}(V)$ *and* $\mathrm{supp}(V^\circ)$ *is dense in* $\mathrm{supp}(V^{\circ\circ})$. *If, in addition, V is closed in $C_\lambda(X, E)$, then* $\mathrm{supp}(V^\circ_C)$ *is dense in* $\mathrm{supp}(V)$.

Proof. Obviously, $V^\circ_C \subseteq V^\circ$ and $V \subseteq V^{\circ\circ}$ imply $\mathrm{supp}(V^\circ_C) \subseteq \mathrm{supp}(V^\circ)$ and $\mathrm{supp}(V^{\circ\circ}) \subseteq \mathrm{supp}(V)$, respectively. Let us show first that $\mathrm{supp}(V^\circ)$ is a dense subset of $\mathrm{supp}(V^{\circ\circ})$. First we prove that $\mathrm{supp}(V^\circ) \subseteq \mathrm{supp}(V^{\circ\circ})$. Suppose there is $x \in \mathrm{supp}(V^\circ) \setminus \mathrm{supp}(V^{\circ\circ})$. Take disjoint neighborhoods $U(x)$ and U of x and $\mathrm{supp}(V^{\circ\circ})$, respectively, in βX, and $\mu \in V^\circ$ with $x \in \mathrm{supp}(\mu)$. Then there exists $f \in C(X, E)$ such that $\beta f | (\beta X \setminus U(x)) = 0$ and $\mu(f) \neq 0$. According to the Claim following Lemma 2.1, we can assume that $\|\mu(f)\| > 1$. Since $V^{\circ\circ}$ is an absolutely convex subset of $C(X, E)$ and $(\beta f)(U) = 0$, then by Lemma 2.5(i), $f \in V^{\circ\circ}$. Thus, $\|\mu(f)\| \leq 1$, because $\mu \in V^\circ$, a contradiction. To show that $\mathrm{supp}(V^\circ)$ is dense in $\mathrm{supp}(V^{\circ\circ})$ let $y \in \mathrm{supp}(V^{\circ\circ}) \setminus \mathrm{cl}_{\beta X}(\mathrm{supp}(V^\circ))$ for some $y \in \beta X$. Then there exists a neighborhood $U(y)$ of y in βX and $g \in C(X, E)$ such that $U(y) \cap \mathrm{cl}_{\beta X}(\mathrm{supp}(V^\circ)) = \emptyset$, $g \notin V^{\circ\circ}$ and βg is 0 outside $U(y)$. Thus, $\beta g \mid (\mathrm{supp}(V^\circ)) = 0$. So, by Lemma 2.3, $\mu(g) = 0$ for every $\mu \in V^\circ$. The last yields $g \in V^{\circ\circ}$, a contradiction.

Suppose V is closed in $C_\lambda(X, E)$ and $\mathrm{supp}(V^\circ_C)$ is not dense in $\mathrm{supp}(V)$. Repeating the same arguments from the proof that $\mathrm{supp}(V^\circ)$ is dense in $\mathrm{supp}(V^{\circ\circ})$ we can find $g \in C(X, E)$ such that $g \notin V$ and $\mu(g) = 0$ for all $\mu \in V^\circ_C$. Then $g \in (V^\circ_C)^\circ$. Finally, since $(V^\circ_C)^\circ = V$, we arrive at the contradiction. ▮

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1. Suppose $C_\lambda(X, E)$ is barrelled and let B be an arbitrary t-bounded subset of $\nu_\lambda(X)$. By Corollary 2.9, the set $V_B(1) = \{f \in C(X, E) : \sup\{\| \mid \beta f(x) \mid \| : x \in B\} \leq 1\}$ is a barrel in $C_\lambda(X, E)$. Therefore, $V_B(1)$ contains a neighborhood of 0 in $C_\lambda(X, E)$. So, there are $K \in \lambda$ and $\varepsilon > 0$ such that $f \in C_\lambda(X, E)$ and $p_K(f) < \varepsilon$ imply $f \in V_B(1)$. Then $B \subseteq \mathrm{cl}_{\nu X}(K)$. Otherwise, since $\mathrm{cl}_{\nu X}(K)$ is compact, there would be $x \in B \setminus \mathrm{cl}_{\nu X}(K)$ and $g \in C(\nu X, E)$ such that $g \mid \mathrm{cl}_{\nu X}(K) = 0$ and $\|g(x)\| = 2$, i.e., $h = g|X \notin V_B(1)$, which contradicts $p_K(h) < \varepsilon$.

Now, assume that each t-bounded subset of $\nu_\lambda(X)$ is contained in $\mathrm{cl}_{\nu X}(K)$ for some $K \in \lambda$. Let V be a barrel in $C_\lambda(X, E)$ and let V_C° be its polar in $\mathrm{CL}(C_\lambda(X, E), E)$.

Claim. $\mathrm{supp}(V_C^\circ)$ is a t-bounded subset of $\nu_\lambda(X)$.

Since V is absorbing in $C_\lambda(X, E)$, V_C° is pointwise bounded. Then, by Lemma 2.4, $\mathrm{supp}(V_C^\circ)$ is t-bounded in νX. Finally, by Corollary 2.2, $\mathrm{supp}(V_C^\circ) \subseteq \nu_\lambda(X)$. Obviously, $\mathrm{supp}(V_C^\circ)$, being t-bounded in νX, is t-bounded in $\nu_\lambda(X)$.

Proceeding to the proof of Theorem 1 and applying the Claim, we can find $K \in \lambda$ such that $\mathrm{supp}(V_C^\circ) \subseteq \mathrm{cl}_{\nu X}(K)$. Because V is closed in $C_\lambda(X, E)$, by Lemma 2.10, $\mathrm{supp}(V_C^\circ)$ is dense in $\mathrm{supp}(V)$. So $\mathrm{supp}(V) \subseteq \mathrm{cl}_{\nu X}(K)$ ($\mathrm{cl}_{\nu X}(K)$ is compact). Since $C_U^*(X, E)$ is barrelled (as a Banach space) and $V \cap C_U^*(X, E)$ is a barrel in this space, V contains $\{g \in C^*(X, E) : \|g\| \leq \eta\}$ for some $\eta > 0$. It follows from Lemma 2.5(ii) that there is $\varepsilon > 0$ with $V(\varepsilon) = \{g \in C(X, E) : \sup\{\| \mid \beta g(x) \mid \| : x \in \mathrm{supp}(V)\} \leq \varepsilon\}$ contained in V. Because $\mathrm{supp}(V) \subseteq \mathrm{cl}_{\nu X}(K)$, the last inclusion implies $\{f \in C(X, E) : p_K(f) < \varepsilon\} \subseteq V(\varepsilon) \subseteq V$. Hence, V is a neighborhood of 0 in $C_\lambda(X, E)$. ▮

Proof of Theorem 2. We follow very closely the proof of Theorem 1. Suppose $C_\lambda(X, E)$ is quasi-barrelled and let B be an arbitrary λ-bounded subset of $\nu_\lambda(X)$. By Corollary 2.9, the set $V_B(1) = \{f \in C(X, E) : \sup\{\| \mid \beta f(x) \mid \| : x \in B\} \leq 1\}$ is a bound-absorbing barrel in $C_\lambda(X, E)$. Therefore, $V_B(1)$ contains a neighborhood of 0 in $C_\lambda(X, E)$. So there are $K \in \lambda$ and $\varepsilon > 0$ such that $f \in C_\lambda(X, E)$ and $p_K(f) < \varepsilon$ imply $f \in V_B(1)$. Then, as in the proof of Theorem 1, $B \subseteq \mathrm{cl}_{\nu X}(K)$.

To prove the other implication, assume that each λ-bounded subset of $\nu_\lambda(X)$ is contained in $\mathrm{cl}_{\nu X}(K)$ for some $K \in \lambda$. Let V be a bound-absorbing barrel in $C_\lambda(X, E)$ and let V_C° be its polar in $\mathrm{CL}(C_\lambda(X, E), E)$. Since V absorbs all bounded sets in $C_\lambda(X, E)$, V_C° is strongly bounded. Then, by Lemma 2.1 and Corollary 2.2, $\mathrm{supp}(V_C^\circ)$ is λ-bounded in $\nu_\lambda(X)$. Hence, there is $K \in \lambda$ such that $\mathrm{supp}(V_C^\circ) \subseteq \mathrm{cl}_{\nu X}(K)$. The closedness of V in $C_\lambda(X, E)$ and Lemma 2.10 yield $\mathrm{supp}(V_C^\circ)$ is dense in $\mathrm{supp}(V)$. Thus,

$\operatorname{supp}(V) \subseteq \operatorname{cl}_{\nu X}(K)$. Since $C_U^*(X,E)$ is quasi-barrelled (as metrizable) and $V \cap C_U^*(X,E)$ is a bounded absorbing barrel in this space, V contains a sphere $\{g \in C^*(X,E): \|g\| \le \eta\}$ for some $\eta > 0$. Then, applying Lemma 2.5(ii), we can conclude that there exists $\varepsilon > 0$ with $\{f \in C(X,E): p_K(f) < \varepsilon\} \subseteq V$. Hence, V is a neighborhood of 0 in $C_\lambda(X,E)$. ▮

Proof of Theorem 3. Suppose $C_\lambda(X,E)$ is bornological and let B be an arbitrary λ-bounded subset of νX. By Lemma 2.8, the set $V_B(1) = \{f \in C(X,E): \sup\{\| |\beta f(x)| \|: x \in B\} \le 1\}$ is a bornivore in $C_\lambda(X,E)$. Therefore, $V_B(1)$ contains a neighborhood of 0 in $C_\lambda(X,E)$. So there are $K \in \lambda$ and $\varepsilon > 0$ such that $f \in C_\lambda(X,E)$ and $p_K(f) < \varepsilon$ imply $f \in V_B(1)$. Then, as in the proof of Theorem 1, $B \subseteq \operatorname{cl}_{\nu X}(K)$.

To prove sufficiency, assume that each λ-bounded subset of νX is contained in $\operatorname{cl}_{\nu X}(K)$ for some $K \in \lambda$. To establish that $C_\lambda(X,E)$ is bornological it suffices to show that any bounded linear map μ from $C_\lambda(X,E)$ into a locally convex linear topological space F is continuous. Since F is isomorphic to a subspace of a product of normed spaces, we can assume that F is itself a normed space. Then $V = \{f \in C(X,E): \|\mu(f)\| \le 1\}$ is a bornivore in $C_\lambda(X,E)$ and μ belongs to the polar V° in $\operatorname{BL}(C_\lambda(X,E),F)$. By Lemma 2.10, $\operatorname{supp}(\mu)$ is a subset of $\operatorname{supp}(V)$. We claim that $\operatorname{supp}(\mu) = \operatorname{supp}(V)$. If there is $x \in \operatorname{supp}(V) \setminus \operatorname{supp}(\mu)$, then we can find a neighborhood U of x in βX disjoint from $\operatorname{supp}(\mu)$ and $f \in C(X,E)$ such that $\|\mu(f)\| > 1$ and βf is 0 outside U. Then $\mu(f) = 0$, because $\beta f|\operatorname{supp}(\mu) = 0$ (Lemma 2.3), a contradiction. Thus, $\operatorname{supp}(\mu) = \operatorname{supp}(V)$. But $\operatorname{supp}(\mu)$ is λ-bounded in νX (by Corollary 2.2), so there exists $K \in \lambda$ with $\operatorname{supp}(V) \subseteq \operatorname{cl}_{\nu X}(K)$. Since $C_U^*(X,E)$ is a normed space and since $V \cap C_U^*(X,E)$ is a bornivore in this space, then V contains a sphere $\{g \in C^*(X,E): \|g\| \le \eta\}$ for some $\eta > 0$. Proceeding as in the proof of Theorem 1, we conclude that V is a neighborhood of 0 in $C_\lambda(X,E)$. Therefore, μ is continuous. ▮

Proof of Theorem 4. Suppose $C_\lambda(X,E)$ is ultrabornological and B is a t-bounded subset of νX. We need the following external characterization of ultrabornological spaces (see [5, p. 115]): A locally convex space is ultrabornological if and only if it is an inductive limit of Banach spaces. Thus, there exists a family of Banach space $\{E_\gamma : \gamma \in \Gamma\}$ and continuous linear maps h_γ: $E_\gamma \to C_\lambda(X,E)$ such that the linear span of $\bigcup\{h_\gamma(E_\gamma): \gamma \in \Gamma\}$ is $C_\lambda(X,E)$ and a set $U \subseteq C(X,E)$ is a neighborhood of 0 in $C_\lambda(X,E)$ if and only if $h_\gamma^{-1}(U)$ is a neighborhood of 0 in E_γ for each $\gamma \in \Gamma$. By Lemma 2.8, $V_B(1) = \{f \in C(X,E): \sup\{\| |\beta f(x)| \|: x \in B\} \le 1\}$ is a sequentially closed, absolutely convex, and absorbing set in $C_\lambda(X,E)$. This yields that each $h_\gamma^{-1}(V_B(1))$ is sequentially closed, absolutely convex, and absorbing in E_γ. Since E_γ is metrizable, then $h_\gamma^{-1}(V_B(1))$ is closed in

E_γ; i.e., it is a barrel in E_γ. Thus, each $h_\gamma^{-1}(V_B(1))$ is a neighborhood of 0 in $C_\lambda(X, E)$. So, there is $K \in \lambda$ with $B \subseteq \mathrm{cl}_{\nu X}(K)$.

Suppose now that each t-bounded set in νX is contained in $\mathrm{cl}_{\nu X}(K)$ for some $K \in \lambda$. Then $\nu_\lambda(X) = \nu X$ and since $\nu_\lambda(X) \subseteq \nu_E(X)$, we have $\nu_\lambda(X) = \nu_E(X) = \nu X$. The last implies that the restriction map $\pi\colon C(\nu X, E) \to C(X, E)$ is both surjective and injective. Clearly, π, considered as a map from $C_k(\nu X, E)$ onto $C_\lambda(X, E)$, is continuous. Because every compact subset of νX is contained in some $\mathrm{cl}_{\nu X}(K)$, $K \in \lambda$, π^{-1} is also continuous. Hence, $C_\lambda(X, E)$ is linearly homeomorphic to $C_k(\nu X, E)$. So it suffices to show that $C_k(\nu X, E)$ is ultrabornological. To prove that we follow the arguments from [5, Exercise 2.1, p. 115]. For every $f \in C(\nu X, E)$, let I_f be the set $\{g \in C(\nu X, E) : \|g(x)\| \leq \|f(x)\| \text{ for all } x \in \nu X\}$ and let Y_f be the subspace of $C(\nu X, E)$ spanned on I_f. Obviously, I_f is absolutely convex in $C(\nu X, E)$ and absorbs all $g \in Y_f$. So, its Minkowski functional $p_f(g) = \inf\{a > 0 : g \in a.I_f\}$ is a semi-norm on Y_f. Since $\|g(x)\| \leq p_f(g).\|f(x)\|$ for every $g \in Y_f$ and $x \in \nu X$, then p_f is a norm on Y_f. Using the completeness of E one can show that (Y_f, p_f) is a Banach space, $f \in C(\nu X, E)$. We claim that $C_k(\nu X, E)$ is an inductive limit of the Banach spaces (Y_f, p_f), $f \in C(\nu X, E)$. Because $C(\nu X, E) = \bigcup\{Y_f : f \in C(\nu X, E)\}$, it remains only to prove that an absolutely convex set $V \subseteq C(\nu X, E)$ is a neighborhood of 0 in $C_k(\nu X, E)$ provided $V \cap Y_f$ is a neighborhood of 0 in each (Y_f, p_f). It is clear that I_f is the unit sphere in (Y_f, p_f), which yields that $V \cap Y_f$ absorbs I_f. Thus, V absorbs all I_f, $f \in C(\nu X, E)$. According to Corollary 2.7, V is a bornivore in $C_k(\nu X, E)$. Finally, since by Theorem 3, $C_k(\nu X, E)$ is bornological, V is a neighborhood of 0 in $C_k(\nu X, E)$. Therefore, $C_k(\nu X, E)$ is ultrabornological as an inductive limit of Banach spaces. ▮

4. SOME MORE RESULTS

Let us note first the following corollaries from the main results:

COROLLARY 4.1. *Let E be a Banach space and let λ consist of compact sets. Then $C_\lambda(X, E)$ is barrelled if and only if X is a μ-space and every compact subset of X is contained in some member of λ.*

We say that X is a c-space if every c-bounded closed subset of X is compact, where c is the closed ideal of all compact sets in X.

COROLLARY 4.2. *Let E be a normed space and let λ consist of compact sets. Then $C_\lambda(X, E)$ is quasi-barrelled if and only if X is a c-space and every λ-bounded subset of X is contained in some member of λ.*

COROLLARY 4.3. *If X is a μ-space and E is a Banach space, then $C_k(X, E)$ is barrelled if and only if it is quasi-barrelled.*

COROLLARY 4.4. *For a Banach space E the space $C_\lambda(X, E)$ is ultrabornological if and only if it is bornological and barrelled.*

Obviously, every ultrabornological space is bornological. By Theorem 4, we have that $X = \nu X$ if λ consists of compact sets and $C_\lambda(X, E)$ is bornological. But for a realcompact space X a set $K \subseteq X$ is t-bounded in X if and only if it is c-bounded. So we have

COROLLARY 4.5. *Let E be a normed space. Then the following conditions are equivalent*:

(1) *X is realcompact*;

(2) *$C_k(X, E)$ is bornological*;

If E is a Banach space, then (1) *and* (2) *are equivalent to "$C_k(X, E)$ is ultrabornological."*

In the case when E is the real line, Corollaries 4.4 and 4.5 were obtained, respectively, in [12] and [5].

COROLLARY 4.6. *Suppose u is a continuous linear quotient map from $C_k(X, E)$ onto $C_k(Y, F)$, where E and F are normed spaces. Then Y is realcompact provided X is realcompact.*

Proof. Let X be realcompact. If u is a quotient continuous linear map from $C_k(X, E)$ onto $C_k(Y, F)$, then $C_k(Y, F)$ is bornological as a quotient image of the bornological space $C_k(X, E)$. Hence, by Corollary 4.5, Y is realcompact. ∎

The conclusion of Corollary 4.6 also holds if E and F are Banach spaces and u is a linear homeomorphism between $C_p(X, E)$ and $C_p(Y, F)$. In this case u can be extended to a linear homeomorphism between $C_k(X, E)$ and $C_k(Y, F)$ (see Proposition 3.6 and Corollary 3.9(ii) from [11]). Let us note that realcompactness is preserved by homeomorphisms (not necessary linear) between $C_p(X)$ and $C_p(Y)$ (see [2, 10]). We do not know if the same is true for the spaces $C_p(X, E)$ and $C_p(Y, E)$, where E is a Banach space.

By $\mathrm{CL}_p(C_k(X, E), F)$ we denote the set $\mathrm{CL}(C_k(X, E), F)$ with the pointwise topology.

THEOREM 4.7. *If E is a Banach space, then the following conditions are equivalent*:

(1) *$C_k(X, E)$ is barrelled*;

(2) *X is a μ-space*;

(3) *$\mathrm{CL}_p(C_k(X, E), F)$ is a μ-space for every normed space F.*

Proof. The equivalence of (1) and (2) follows from Corollary 4.1, while the equivalence of (2) and (3) follows from [11, Proposition 3.7 and Corollary 3.9]. ∎

Here is another result similar to the theorem above. Recall that a space is *Dieudonné complete* [7] if it can be embedded as a closed subset of a product of metrizable spaces.

THEOREM 4.8. *If* $C_\lambda(X, E)$ *is bornological, then* $\mathrm{CL}_p(C_\lambda(X, E), F)$ *is Dieudonné complete for any normed space* F.

Proof. Suppose $C_\lambda(X, E)$ is bornological and F is a normed space. There is a natural embedding of $\mathrm{CL}_p(C_\lambda(X, E), F)$ in the product $F^{C(X,E)}$. Consider the G_δ-closure G of $\mathrm{CL}_p(C_\lambda(X, E), F)$ in $F^{C(X,E)}$; i.e., the set of all $\mu \in F^{C(X,E)}$ such that any G_δ-subset of $F^{C(X,E)}$ containing μ meets $\mathrm{CL}_p(C_\lambda(X, E), F)$. Then G is Dieudonné complete (see [6, 7]). So it is enough to show that $G = \mathrm{CL}_p(C_\lambda(X, E), F)$. Let $\mu \in G$. Obviously, μ is a linear map from $C(X, E)$ into F. Because $C_\lambda(X, E)$ is bornological, our proof is reduced to showing that μ is bounded on $C_\lambda(X, E)$. Assuming this is not the case, there is a bounded sequence $\{f_n\} \subseteq C_\lambda(X, E)$ such that $\{\mu(f_n)\}$ is unbounded in F. Since $\{q \in F^{C(X,E)} : q(f_n) = \mu(f_n),\ n \in \mathbf{N}\}$ is a G_δ-subset of $F^{C(X,E)}$ containing μ, there exists $\mu^* \in \mathrm{CL}_p(C_\lambda(X, E), F)$ with $\mu^*(f_n) = \mu(f_n)$ for each n. Thus, $\{\mu^*(f_n)\}$ is unbounded in F, which contradicts the fact that μ^* is continuous and $\{f_n\}$ is bounded in $C_\lambda(X, E)$. ∎

Veličko [12] proved that if X is a k_λ-*space* (i.e., a set $A \subseteq X$ is closed if and only if $A \cap K$ is closed in K for every $K \in \lambda$), then $C_\lambda(X)$ is bornological if and only if it is ultrabornological. The same is true for $C_\lambda(X, E)$, when E is a Banach space.

LEMMA 4.9. *Let* X *be a* k_λ-*space. Then every* t-*bounded subset of* $\nu_\lambda(X)$ *is* λ-*bounded in* $\nu_\lambda(X)$.

Proof. Suppose $H \subseteq \nu_\lambda(X)$ is t-bounded but not λ-bounded. Then there is a lsc function $\varphi\colon \nu_\lambda(X) \to [0, \infty)$, which is bounded on each $K \in \lambda$ and unbounded on H. So for every n we can choose $x_n \in H \cap \varphi^{-1}(n, \infty)$. Construct by induction a sequence $\{g_n\} \subseteq C(\nu_\lambda(X))$ such that:

(1) $g_n(\varphi^{-1}[0, n]) = 0$;

(2) $|g_n(x_n)| = n + |b_n|$, where $b_1 = 0$ and $b_n = \Sigma\{g_i(x_n) : i \leq n - 1\}$ for $n \geq 2$;

(3) $g_i(x_j) = 0$ for all $j < i$.

For every $K \in \lambda$ there is $n_K \in \mathbf{N}$ such that $K \subseteq \varphi^{-1}([0, n_K])$, so by (1), $g_n(K) = 0$ for each $n \geq n_K + 1$. The last implies that the function $f(x) =$

$\Sigma\{g_n(x): n \in \mathbf{N}\}$, $x \in X$ is well defined and continuous on X, because $f|K = \Sigma\{g_i|K : i \leq n_K\}$ is continuous on K, $K \in \lambda$. Let h: $\nu_\lambda(X) \to \mathbf{R}$ be the continuous extension of f. To get a contradiction it is enough to show that h is unbounded on H. To this end fix $n \in \mathbf{N}$ and consider $K \in \lambda$ such that $x_n \in \mathrm{cl}_{\nu X}(K)$. Then the functions h and $\Sigma\{g_i : i \leq n_K\}$ have the same restrictions on K, hence their restrictions on $\mathrm{cl}_{\nu X}(K)$ are also equal. Therefore, $h(x_n) = \Sigma\{g_i(x_n): i \leq n_K\}$. Because $\varphi(x_n) > n$ and $\varphi(K)$ is contained in $[0, n_K]$, we have $n < n_K$. Finally, by (2) and (3), $h(x_n) = \Sigma\{g_i(x_n): i \leq n\} = g_n(x_n) + b_n$. Thus, $|h(x_n)| \geq |g_n(x_n)| - |b_n| = n$; i.e., h is unbounded on H. ∎

THEOREM 4.10. *Let E be a Banach space and let X be a k_λ-space. Then*:

(i) *$C_\lambda(X, E)$ is barrelled if and only if it is quasi-barrelled*;

(ii) *$C_\lambda(X, E)$ is bornological if and only if it is ultrabornological.*

Proof. Since every λ-bounded subset of $\nu_\lambda(X)$ is t-bounded, (i) follows from Lemma 4.9 and Theorems 1 and 2. For (ii), we have to show only that $C_\lambda(X, E)$ is ultrabornological provided it is bornological. If $C_\lambda(X, E)$ is bornological, by Theorem 3, $\nu X = \nu_\lambda(X)$ and every λ-bounded subset of νX is contained in $\mathrm{cl}_{\nu X}(K)$ for some $K \in \lambda$. Then Lemma 4.9 and Theorem 4 imply that $C_\lambda(X, E)$ is ultrabornological. ∎

COROLLARY 4.11. *Let X be a realcompact k-space and E be a Banach space. Then barrelledness, quasi-barrelledness, bornologity, and ultrabornologity of $C_k(X, E)$ are equivalent.*

REFERENCES

1. A. V. Arhangel'skii, On linear homeomorphisms of function spaces, *Soviet Math. Dokl.* **25** (1982), 852–855.
2. A. V. Arhangel'skii, "Topological Function Spaces," Kluwer Academic, Dordrecht, 1992.
3. P. Powers, Limitation topologies on function spaces, *Trans. Amer. Math. Soc.* **314**, No. 1 (1989), 421–431.
4. J. Baars and J. de Groot, "On Topological and Linear Equivalences of Certain Function Spaces," CWI Tract 86, Center for Mathematics and Computer Science, Amsterdam, 1992.
5. E. Beckenstein, L. Narici, and C. Suffel, "Topological Algebras," North-Holland Mathematics Studies, Vol. 24, North-Holland, Amsterdam, 1977.
6. J. Dieudonné, Sur les espaces uniformes completes, *Ann. Sci. École Norm. Sup.* **56** (1939), 277–291.
7. R. Engelking, "General Topology," PWN, Warsaw, 1977.
8. H. J. K. Junnila, J. C. Smith, and R. Telgárski, Closure-preserving covers by small sets, *Topology Appl.* **23** (1986), 237–262.
9. J. Kelly and I. Namioka, "Linear Topological Spaces," Graduate Texts in Mathematics, Vol. 36, Springer-Verlag, Berlin/New York, 1976.

10. V. Uspenskii, A characterization of realcompactness in terms of the topology of pointwise convergence on the function spaces, *Comment. Math. Univ. Carolin.* **24**, No. 1 (1983), 121–126.
11. V. Valov, Linear mappings between function spaces, preprint.
12. N. Veličko, On theory of spaces of continuous functions, *Trudy Mat. Inst. Ross. Akad. Nauk.* **193** (1992), 553–559. (in Russian).