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Universal metric spaces and extension dimension

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Abstract

For any countable CW-complex K and a cardinal number $\tau \geqslant \omega$ we construct a completely metrizable space $X(K,\tau)$ of weight τ with the following properties: $\text{e-dim}\, X(K,\tau) \leqslant K$, $X(K,\tau)$ is an absolute extensor for all normal spaces Y with $\text{e-dim}\, Y \leqslant K$, and for any completely metrizable space Z of weight $\leqslant \tau$ and $\text{e-dim}\, Z \leqslant K$ the set of closed embeddings $Z \to X(K,\tau)$ is dense in the space $C(Z, X(K,\tau))$ of all continuous maps from Z into $X(K,\tau)$ endowed with the limitation topology. This result is applied to prove the existence of universal spaces for all metrizable spaces of given weight and with a given cohomological dimension.

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The existence of universal separable metric spaces for extension dimension with respect to countable CW-complexes was proved by Olszewski in [12]. In the class of all metric spaces of a given weight this problem was recently solved by Levin [11]. In the present note we show the existence of universal metric spaces having some extra properties (see Theorem 1 below). The concept of extension dimension was introduced by Dranishnikov [5] (see also [2,6]). For a normal space X and a CW-complex K we write $\text{e-dim}\, X \leqslant K$ (the extension dimension of X does not exceed K) if K is an absolute extensor for X. This means that any continuous map $f: A \to K$, defined on a closed subset A of X, admits a continuous extension $\bar{f}: X \to K$. Since not every CW-complex is an absolute neighborhood extensor for normal spaces, we can enlarge the class of normal spaces X with $\text{e-dim}\, X \leqslant K$ (K is a CW-complex) by introducing the following notion

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(see [14, Definition 2.5]): A normal space X is in the class $\alpha(K)$ if every continuous map from a closed $A \subset X$ to K which extends to a map of a neighborhood of A to K can be extended to a map of X to K. Obviously, if $K \in ANE(X)$ (this, for example, holds for every X admitting a perfect map onto a first countable paracompact space [8]) then $X \in \alpha(K)$ if and only if e-dim $X \leqslant K$. We also adopt the following definition: a continuous map $f : X \to Y$ is called K-soft (respectively, K-soft with respect to metrizable spaces) if for any normal (respectively, metrizable) space Z with $Z \in \alpha(K)$, any closed $Z_0 \subset Z$, and any two maps $g : Z_0 \to X$, $h : Z \to Y$ with $f \circ g = h|Z_0$, there exists a map $k : Z \to X$ such that $f \circ k = h$ and $k|Z_0 = g$.

For any CW-complex K and any cardinal number $\tau \geqslant \omega$ let $\mathcal{M}(K, \tau)$ be the class of all completely metrizable spaces X of weight τ with e-dim $X \leqslant K$. The following theorem is our main result:

Theorem 1. *For any countable CW-complex K and a cardinal number $\tau \geqslant \omega$ there exists a completely metrizable space $X(K, \tau)$ and a K-soft map $f(K, \tau) : X(K, \tau) \to l_2(\tau)$ satisfying the following properties*:

(a) $X(K, \tau) \in \mathcal{M}(K, \tau)$.

(b) *$X(K, \tau)$ is an absolute extensor for all normal spaces Y with $Y \in \alpha(K)$.*

(c) *$f(K, \tau)$ is strongly (K, τ)-universal, i.e., for any open cover $\mathcal{U}$ of $X(K, \tau)$, any* (*complete*) *metric space Z of weight $\leqslant \tau$ with* e-dim $Z \leqslant K$ *and any map $g : Z \to X(K, \tau)$ there exists a* (*closed*) *embedding $h : Z \to X(K, \tau)$ $\mathcal{U}$-close to g with $f(K, \tau) \circ g = f(K, \tau) \circ h$.*

For the case $K = S^n$, $n \in \mathbb{N}$, Theorem 1 was proved in [4, Theorem 2.7]. Our proof of Theorem 1 is based on the next few lemmas and the techniques developed in [3,4].

Lemma 2. *For any countable CW-complex K and any separable* (*completely*) *metrizable space X there exists a separable* (*completely*) *metrizable space Y_X with* e-dim $Y_X \leqslant K$ *and a K-soft map $f : Y_X \to X$.*

Proof. Let P be a Polish ANR homotopically equivalent to K and $\varphi : X \to P$ and $\psi : P \to X$ be two maps such that $\psi \circ \varphi$ is homotopic to id_P and $\psi\varphi$ is homotopic to id_K. For extension dimension with respect to P this lemma was proved in [2, Proposition 5.9]. So, for a given (complete) separable metric space X there is a (complete) separable metric space Y_X with e-dim $Y_X \leqslant P$ and a P-soft map $f : Y_X \to X$. According to next claim, f is K-soft.

Claim. *If $Z \in \alpha(K)$ is normal, then $Z \in \alpha(P)$.*

Proof. Suppose $Z \in \alpha(K)$ is a normal space. Since every Polish ANR is an ANE for normal spaces, $Z \in \alpha(P)$ is equivalent to $P \in AE(Z)$. Take a map $g : A \to P$, where $A \subset Z$ is closed and consider the map $\psi \circ g : A \to K$. Because g can be extended to a map from a neighborhood U of A into P, $\psi \circ g$ can be extended to a map from U to K.

Since $Z \in \alpha(K)$, there is an extension $h : Z \to K$ of $\psi \circ g$. Then the restriction $(\varphi h)|A$ is homotopic to g. Finally, using that the Homotopy Extension Theorem holds for normal spaces and Polish ANRs, we conclude that g is extendable to a map from Z into P. Hence $Z \in \alpha(P)$. □

It remains only to show that e-dim $Y_X \leqslant K$. And this follows from e-dim $Y_X \leqslant P$ and the fact that the Homotopy Extension Theorem holds for metric spaces and CW-complexes [8]. □

Lemma 3 [11]. *Let $f : X \to Y$ be a uniformly* 0*-dimensional map of metrizable spaces X and Y. Then* e-dim $X \leqslant$ e-dim Y.

Recall that a map $f : X \to Y$, where X and Y are metrizable spaces, is called uniformly 0-dimensional [10] if there exists a compatible metric on X such that for every $\varepsilon > 0$ and every $y \in f(X)$ there is an open neighborhood U of y such that $f^{-1}(U)$ can be represented as the union of disjoint open sets of diam $< \varepsilon$. It is well known [10] that every metric space admits a uniformly 0-dimensional map into Hilbert cube Q.

Lemma 4. *For any countable CW-complex K and a* (*completely*) *metrizable space Y of weight τ there exist a* (*completely*) *metrizable space X of weight τ and a K-soft map $f : X \to Y$ such that* e-dim $X \leqslant K$.

Proof. It suffices to prove this corollary when Y is the space $l_2(\tau)$. Fix a compatible metric d_1 on $l_2(\tau)$ and an uniformly 0-dimensional surjection (with respect to d_1) $g : l_2(\tau) \to A$ with A a separable metric space. By Lemma 2, there exists a separable metric space Z with e-dim $Z \leqslant K$ and a K-soft map $h : Z \to A$. Let X be the fibered product of $l_2(\tau)$ and Z with respect to g and h, and let $f : X \to l_2(\tau)$ and $p : X \to Z$ denote the corresponding projections of this fibered product. If d_2 is any metric on Z, then p is uniformly 0-dimensional with respect to the metric $(d_1^2 + d_2^2)^{1/2}$ (see [1]). Hence, by Lemma 3, e-dim $Z \leqslant K$. The K-softness of h implies that f is also K-soft.

It remains only to show that X is completely metrizable. To this end let B_X be the space obtained from βX by making the points of $\beta X - X$ isolated. According to [13, Lemma 2], B_X is paracompact, and obviously, B_X is first countable.

Claim. e-dim $B_X \leqslant K$.

Proof. Let $s : F \to K$ be an arbitrary map, where $F \subset B_X$ is closed. Since e-dim $X \leqslant K$, there exists an extension $s_1 : F \cup X \to K$ of s. Now we need the following result [8, Theorem 11.2]: any contractible CW-complex is an absolute extensor for all spaces admitting a perfect map onto a first countable paracompact space. This statement implies that Cone(K) is an absolute extensor for B_X. Therefore, there exists an extension $s_2 : B_X \to$ Cone(K) of s_1. Let $H = s_2^{-1}(\text{Cone}(K) - \{b\})$, where $b \in$ Cone$(K) - K$. Fix a retraction $r :$ Cone$(K) - \{b\} \to K$. Since H is clopen in B_X, it follows that $r \circ s_2$

can be extended to a map $s_3 : B_X \to K$. Then s_3 is an extension of s and, consequently, e-dim $B_X \leqslant K$. □

Now let us go back to the proof of the completeness of X. Considering X as a closed subset of B_X and using the fact that $l_2(\tau)$ is an absolute extensor for paracompact spaces, we can find a map $q : B_X \to l_2(\tau)$ such that $q|X = f$. Then, since f is K-soft and e-dim $B_X \leqslant K$, there exists a retraction from B_X onto X. Finally, applying the argument from [13, the proof of Lemma 2], we conclude that X is complete. □

Proof of Theorem 1. We will construct an inverse sequence $S = \{X_n, p_n^{n+1}, n \in \mathbb{N}\}$ such that:

(1) $X_1 = l_2(\tau)$ and $X_n \in \mathcal{M}(K, \tau)$ for each $n \geqslant 1$;

(2) each $p_n^{n+1} : X_{n+1} \to X_n$ is a K-soft map such that for any completely metrizable space Z of weight $\leqslant \tau$ with e-dim $Z \leqslant K$ and any map $g : Z \to X_n$ there exists a closed embedding $h : Z \to X_{n+1}$ with $p_n^{n+1} \circ h = g$.

If X_i and p_{i-1}^i have already been constructed for $i = 1, 2, \ldots, n$, then, by Lemma 4, there exist a completely metrizable space X_{n+1} of weight τ and a K-soft map $h_{n+1} : X_{n+1} \to X_n \times l_2(\tau)$ such that e-dim $X_{n+1} \leqslant K$. Let $p_n^{n+1} = \pi_n \circ h_{n+1}$, where $\pi_n : X_n \times l_2(\tau) \to X_n$ is the natural projection. Denote by $X(K, \tau)$ the limit space of S and by $f(K, \tau)$ the limit projection $p_1 : X(K, \tau) \to X_1$. Obviously, $X(K, \tau)$ is a completely metrizable space of weight τ and $f(K, \tau)$ is K-soft. Following the proof of Lemma 2.6 from [4] one can show that $f(K, \tau)$ is strongly (K, τ)-universal. Finally, by the limit theorem of Rubin–Schapiro [15], e-dim $X(K, \tau) \leqslant K$. Observe that $X(K, \tau)$ is an absolute extensor for all normal spaces Y with e-dim $Y \leqslant K$ because $l_2(\tau)$ is an absolute extensor for normal spaces and $f(K, \tau)$ is K-soft.

We can apply Theorem 1 to obtain universal spaces for all metrizable spaces with a given cohomological dimension and a given weight. Recall that, for any abelian group G and a natural number n, the cohomological dimension $\dim_G X$ of X with a coefficient group G is $\leqslant n$ iff e-dim $X \leqslant K(G, n)$, where X is a normal space and $K(G, n)$ is the Eilenberg–MacLane complex. Let us agree the following notations: a map f is called (G, n)-soft iff it is $K(G, n)$-soft, and f is strongly (G, n, τ)-universal iff f is strongly $(K(G, n), \tau)$-universal.

Corollary 5. *Let G be a countable* (*respectively, torsion*) *abelian group. Then for every $n \in \mathbb{N}$ and $\tau \geqslant \omega$ there exists a completely metrizable space $X_\tau(G, n)$ of weight τ and a map $f_\tau(G, n) : X_\tau(G, n) \to l_2(\tau)$ such that*:

(a) $\dim_G X_\tau(G, n) = n$.

(b) *$X_\tau(G, n)$ is an absolute extensor for all normal* (*respectively, metrizable*) *spaces Y with $\dim_G Y \leqslant n$.*

(c) *$f_\tau(G, n)$ is strongly (G, n, τ)-universal and (G, n)-soft* (*respectively, (G, n)-soft with respect to metrizable spaces*).

Proof. If G is countable, the proof follows directly from Theorem 1 with $K = K(G, n)$. If G is torsion, by [7, Theorem B(a)], there exists a countable family $\sigma(G)$ of countable groups such that $\dim_G Y = \max\{\dim_H Y\colon\ H \in \sigma(G)\}$ for any metrizable space Y. Then, according to [9, Lemma 2.4], for each $n \in \mathbb{N}$ there is a countable complex K_n with $\dim_G Y \leqslant n$ if and only if $\text{e-dim}\, Y \leqslant K_n$, Y is any metrizable space. Finally, apply Theorem 1 to K_n. □

Similarly, using Theorem 1 and [7, Theorem B(a),(b) and (f)], we can obtain the following

Corollary 6. *Let G be an arbitrary abelian group. Then for every $n \in \mathbb{N}$ and $\tau \geqslant \omega$ there exists a completely metrizable space $Y_\tau(G,n)$ of weight τ and a map $g_\tau(G,n) : Y_\tau(G,n) \to l_2(\tau)$ such that*:

(a) $\dim_G Y_\tau(G,n) \leqslant n+1$,
(b) *$Y_\tau(G,n)$ is an absolute extensor for all metrizable spaces Z with $\dim_G Z \leqslant n$,*
(c) *$g_\tau(G,n)$ is strongly (G,n,τ)-universal and (G,n)-soft with respect to metrizable spaces.*

For $\tau = \omega$ weaker versions of Corollaries 5 and 6 were proved in [12].

References

[1] P. Alexandroff, B. Pasynkov, Introduction to Dimension Theory, Nauka, Moscow, 1973.
[2] A. Chigogidze, Cohomological dimension of Tychonov spaces, Topology Appl. 79 (1997) 197–228.
[3] A. Chigogidze, n-soft maps on n-dimensional spaces, Mat. Notes 46 (1989) 88–95 (in Russian).
[4] A. Chigogidze, V. Valov, Universal maps and surjective characterizations of completely metrizable LC^n-spaces, Proc. Amer. Math. Soc. 109 (1990) 1125–1133.
[5] A. Dranishnikov, The Eilenberg–Borsuk theorem for mappings into an arbitrary complex, Russian Acad. Sci. Sb. Math. 81 (1995) 467–475.
[6] A. Dranishnikov, J. Dydak, Extension dimension and extension types, Preprint.
[7] J. Dydak, Cohomological dimension and metrizable spaces, Trans. Amer. Math. Soc. 337 (1) (1993) 219–234.
[8] J. Dydak, Extension theory: The interface between set-theoretic and algebraic topology, Preprint.
[9] J. Dydak, Cohomological dimension and metrizable spaces, II, Trans. Amer. Math. Soc. 348 (4) (1996) 1647–1661.
[10] M. Katetov, On the dimension of non-separable spaces I, Czechoslovak Math. J. 2 (77) (1952) 333–368 (in Russian).
[11] M. Levin, On extension dimension of metrizable spaces, Preprint.
[12] W. Olszewski, Universal separable metrizable spaces of given cohomological dimension, Topology Appl. 61 (1995) 293–299.
[13] T. Przymusinski, Collectionwise normality and absolute retracts, Fund. Math. 98 (1978) 61–73.
[14] L. Rubin, Characterizing cohomological dimension: The cohomological dimension of $A \cup B$, Topology Appl. 40 (1991) 233–263.
[15] L. Rubin, P. Schapiro, Limit theorem for inverse sequences of metric spaces in extension theory, Preprint.