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EXTENSION DIMENSION AND REFINABLE MAPS

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Abstract. Extension dimension is characterized in terms of ω-maps. We apply this result to prove that extension dimension is preserved by refinable maps between metrizable spaces. It is also shown that refinable maps preserve some infinite-dimensional properties.

1. Introduction

The concept of extension dimension was introduced by Dranishnikov [7] (see also [6], [8]). For a normal space X and a space K we write e-dim $X \leqq K$ (the extension dimension of X does not exceed K) if K is an absolute extensor for X. This means that any continuous map $f : A \to K$, defined on a closed subset A of X, admits a continuous extension $\bar{f} : X \to K$. We can enlarge the class of normal spaces X with e-dim $X \leqq K$ by introducing the following notion (see [22, Definition 2.5]): for a space K let $\alpha(K)$ be the class of all normal spaces X such that if $A \subset X$ is closed and $f : A \to K$ is a continuous map, then f can be extended to a map from X into K provided there is a neighborhood U of A and a map $g : U \to K$ extending f. Obviously, $\alpha(K)$ contains all spaces X with e-dim $X \leqq K$ and if K is an absolute neighborhood extensor for X (briefly $K \in \mathrm{ANE}(X)$), then $X \in \alpha(K)$ is equivalent to e-dim $X \leqq K$ (for example, this is true if X is metrizable, or more generally, when X admits a perfect map onto a first countable space [10]).

A space X is said to be $\mathcal{P}$-like [9], where $\mathcal{P}$ is a given class, if for every open locally finite cover ω of X there exists an ω-map from X into a space $Y \in \mathcal{P}$. It is well known that a normal space X satisfies $\dim X \leqq n$ if and only if X is $\mathcal{P}$-like with $\mathcal{P}$ being the class of all simplicial complexes of dimension $\leqq n$. It follows from [9] and [20] that if X is compact and K is a CW-complex, then e-dim $X \leqq K$ iff X is $\mathcal{P}$-like, where $\mathcal{P}$ denotes the class of compact metric spaces Y with e-dim $Y \leqq K$.

In the present note we prove that for a CW-complex K we have $X \in \alpha(K)$ provided X is $\mathcal{P}$-like with respect to the class of paracompact

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spaces from $\alpha(K)$. In particular, if a metrizable space X is $\mathcal{P}$-like with respect to metrizable spaces Y with e-dim $Y \leqq K$, then e-dim $X \leqq K$. It is also shown that, for countable complexes K, this property characterizes the class $\alpha(K)$. We apply the above results to show that if $f : X \to Y$ is a refinable map between metrizable spaces, then e-dim $X \leqq K$ iff e-dim $Y \leqq K$, K is any CW-complex. This generalizes A. Koyama's results from [16], as well as a result of A. Koyama and R. Sher [18]. In the last section we prove that refinable maps preserve S-weakly infinite-dimensionality (resp., finite C-space property). In the class of compact spaces this result was proved earlier by H. Kato [15] and A. Koyma [17] (resp., D. Garity and D. Rohm [12]).

All spaces in this paper are assumed to be normal and all maps are continuous.

2. The class $\alpha(K)$

Lemma 2.1. *Let L be a space homotopy equivalent to a space K. Then $X \in \alpha(L)$ if and only if $X \in \alpha(K)$.*

Proof. There exist maps $\varphi : K \to L$ and $\psi : L \to K$ such that $\varphi \circ \psi \simeq \mathrm{id}_L$ and $\psi \circ \varphi \simeq \mathrm{id}_K$. It suffices to show that $X \in \alpha(L)$ yields $X \in \alpha(K)$. To this end, let $f : A \to K$ be a map such that $A \subset X$ is closed and f can be extended to a map $g : U \to K$, where U is a neighborhood of A in X. Since X is normal, there exists open $V \subset X$ with $A \subset V \subset \overline{V} \subset U$. Then $k = (\varphi \circ g)|\overline{V}$ is a map from $\overline{V}$ into L which is extendable to a map from U into L. Using that $X \in \alpha(L)$ we can find an extension $h : X \to L$ of k. Because $(\psi \circ h)|V \simeq g$, by [13, IV, 2.1], f admits an extension. Hence $X \in \alpha(K)$. □

Recall that a map $f : X \to Y$, where X and Y are metrizable spaces, is called uniformly 0-dimensional [14] if there exists a compatible metric on X such that for every $\varepsilon > 0$ and every $y \in f(X)$ there is an open neighborhood U of y such that $f^{-1}(U)$ can be represented as the union of disjoint open sets of diameter $< \varepsilon$. It is well known that every metric space admits a uniformly 0-dimensional map into the Hilbert cube Q.

Theorem 2.2. *Let K be a countable CW-complex and $f : X \to Y$, where $X \in \alpha(K)$ and Y is metrizable. Then there exist metrizable Z and maps $h : X \to Z$ and $g : Z \to Y$ such that* e-dim $Z \leqq K$ *and $f = g \circ h$.*

Proof. Let $k : Y \to Q$ be a uniformly 0-dimensional map. Fix a Polish ANR-space P homotopy equivalent to K. By Lemma 2.1, $X \in \alpha(P)$. Since $P \in \mathrm{ANE}(X)$, we have e-dim $X \leqq P$. Then, according to [6, Proposition 4.9], there exist a separable metric space M with e-dim $M \leqq P$ and maps $\psi : X \to M$, $\varphi : M \to Q$ such that $\varphi \circ \psi = k \circ f$. Applying once more Lemma 2.1, we conclude that e-dim $M \leqq K$. Let Z be the fibered product of M and

Y with respect to φ and k, and let $q : Z \to M$ and $g : Z \to Y$ denote the corresponding projections. Since k is uniformly 0-dimensional, we can find a compatible metric d on Z such that q is uniformly 0-dimensional with respect to d (see [3]). Because e-dim $M \leqq K$, hence e-dim $Z \leqq K$ [19, Theorem 1]. Finally, define $h : X \to Z$ by $h(x) = \big(\psi(x), f(x)\big)$. □

$f : X \to Y$ is called an ω-map, where ω is an open cover of X, if there is an open cover γ of Y such that $f^{-1}(\gamma)$ refines ω.

Corollary 2.3. *Let K be a countable CW-complex and $X \in \alpha(K)$. Then for every locally finite open cover ω of X there is an ω-map from X onto a metrizable space Z with* e-dim $Z \leqq K$.

Proof. Take an ω-map f from X into a metrizable space Y and apply Theorem 2.2 to obtain a metrizable space Z and maps $h : X \to Z$, $g : Z \to Y$ such that e-dim $Z \leqq K$ and $f = g \circ h$. It remains to observe that h is an ω-map because f is such a map. □

It is certainly true that if Theorem 2.2 holds for any CW-complex, then Corollary 2.3 also holds for arbitrary K.

Theorem 2.4. *Let X be a normal space and K be a CW-complex. If for every locally finite open cover ω of X there exists an ω-map into a paracompact space Y with $Y \in \alpha(K)$, then $X \in \alpha(K)$.*

Proof. We follow the ideas from the proof of [9, Lemma 2.1]. There exists a normed space Z and an open subset L of Z homotopy equivalent to K. According to Lemma 2.1, it suffices to show that $X \in \alpha(L)$. To this end, take a map $f : A \to L$, where $A \subset X$ is closed, such that f is extendable to a map from a neighborhood U of A into L. Since X is normal, we can suppose that f is a map from U into L and there is an open set $V \subset X$ containing A such that $\overline{V} \subset U$. To prove that $f|A$ can be extended to a map from X into L, it suffices to find a map $\bar{f} : X \to L$ such that $\bar{f}|V \simeq f|V$ (see [13, IV, 2.1]). Let λ be a locally finite open cover of L such that $\mathrm{St},(z,\lambda)$ is contained in an open ball for every $z \in L$. Let also V' be an open subset of X such that $\overline{V} \subseteqq V' \subseteqq \overline{V'} \subseteqq U$. Then $\omega = \big\{ f^{-1}(W) \cap V' : W \in \lambda \big\} \cup \{X - \overline{V}\}$ is a locally finite open cover of X, so there exists an ω map $p : X \to Y$ into a paracompact space Y with $Y \in \alpha(K)$. Let γ be a locally finite open cover of Y such that $p^{-1}(\gamma)$ refines ω, and let $\{s_G : G \in \gamma\}$ be a partition of unity subordinated to γ. Consider the closure F of $p(V)$ in Y and let $\gamma_F = \{G \in \gamma : G \cap F \neq \emptyset\}$. For every $G \in \gamma_F$ fix a point $x_G \in p^{-1}(G) \cap V$. Denote $a_G = f(x_G)$ if $G \in \gamma_F$ and $a_G = 0$ otherwise, where 0 is the zero-point of Z.

Claim. *For every $y \in F$ there is a ball $B \subset L$ with $f\big(p^{-1}\,\mathrm{St}\,(y,\gamma)\big) \subset B$.*

If $\mathrm{St}\,(y,\gamma) = \{G_i : i = 1, 2, \dots, n\}$ and $G_y = \cap\{G_i : i = 1, \dots, n\}$, then G_y meets $p(V)$, so we can find $x \in p^{-1}(G_y) \cap V$. Since each $p^{-1}(G_i)$ is contained in a element of ω and meets V, we have $p^{-1}\big(\mathrm{St}\,(y,\gamma)\big) \subset \mathrm{St}\,\big(x, f^{-1}(\lambda)\big)$.

Hence $f(p^{-1}(\mathrm{St}(y,\gamma)) \subset \mathrm{St}\big(f(x),\lambda\big)$. Finally, choose an open ball $B \subset L$ with $\mathrm{St}\big(f(x),\lambda\big) \subset B$.

Define a map $g : F \to L$ by $g(y) = \sum\big\{ s_G(y) \cdot a_G : G \in \gamma_F\big\}$. According to the Claim, this definition is correct. Moreover, again by the Claim, for every $x \in V$ there is an open ball in L containing both $f(x)$ and $g\big(p(x)\big)$. Therefore $f|V \simeq (g \circ p)|V$. Observe that the formula $\bar{g} = \sum\big\{ s_G(y) \cdot a_G : G \in \gamma\big\}$ defines a map from Y into Z which extends g. Obviously $O = \bar{g}^{-1}(L)$ is an open subset of Y and g has an extension to a map from O into L. Since $Y \in \alpha(K)$, by Lemma 2.1, $Y \in \alpha(L)$. Hence g can be extended to a map $q : Y \to L$. Let $\bar{f} = q \circ p$. Then $\bar{f}|V = (q \circ p)|V = (g \circ p)|V \simeq f|V$. □

Combining Corollary 2.3 and Theorem 2.4 we obtain the following characterization of the class $\alpha(K)$.

COROLLARY 2.5. *Let K be a countable CW-complex. Then a normal space X belongs to $\alpha(K)$ if and only if for every locally finite open cover ω of X there exists an ω-map into a metrizable space Y with* e-dim $Y \leqq K$.

3. Extension dimension

A surjective map $r : X \to Y$ is called refinable [16] if for any locally finite open cover ω of X and any open cover γ of Y there exists a surjective ω-map $f : X \to Y$ such that r and f are γ-close (i.e., for every $x \in X$ there is an element of γ containing both points $r(x)$ and $f(x)$). The map f is called an (ω,γ)-refinement of r. When there exists a closed (ω,γ)-refinement for r we say that r is c-refinable. For compact metric spaces this definition coincides with the original one given by J. Ford and J. Rogers [11].

Koyama proved that $\dim X = \dim Y$ provided X and Y are metric spaces and r is refinable [16, Theorem 2]. Under the same hypotheses, A. Koyama and R. Sher showed [18] that $\dim_G X = \dim_G Y$ for any finitely generated Abelian group G. Here $\dim_G X$ stands for the cohomological dimension of X with respect to G. In case r is c-refinable we have $K \in AE(X)$ if and only if $K \in AE(Y)$ for any simplicial complex K (see [16, Theorem 1]). In the present section we generalize all these results by proving that if r is a refinable map between metric spaces X and Y, then e-dim $X \leqq K$ if and only if e-dim $Y \leqq K$ for any CW-complex K.

LEMMA 3.1. *Let r be a refinable map from a normal space X onto a paracompact space Y and L a locally convex subset of a normed space Z. Suppose $A \subset W \subset \overline{W} \subset W_1 \subset \overline{W_1} \subset F \subset W_2$, where W, W_1 and W_2 (resp., A and F) are open (resp., closed) subsets of Y. Further, let $H_1 = r^{-1}(\overline{W_1})$ and $H_2 = r^{-1}(\overline{W_2})$. If $g : \overline{W_1} \to L$ and $h : H_2 \to L$ are continuous with $h|H_1 = (g \circ r)|H_1$, then $g|A$ can be extended to a continuous map from F into L.*

PROOF. Obviously, L is an ANR as a set having a base of convex subsets. By [13, IV, 2.1], $g|A$ is extendable over F provided there exists a map $p : F \to L$ such that $p|D \simeq g|D$ for some neighborhood D of A. We shall construct a map p satisfying this condition with $D = W$.

Let λ be a locally finite open cover of L such that $\mathrm{St}\,(G, \lambda)$ is contained in a convex subset of L for every $G \in \lambda$. Take an open set $U \subset Y$ such that $F \subset U \subset \overline{U} \subset W_2$ and denote $\omega = \left\{ h^{-1}(G) \cap r^{-1}(W_2) : G \in \lambda \right\} \cup \left\{ X - r^{-1}(\overline{U}) \right\}$. Take a locally finite open cover γ of Y refining the cover $\left\{ g^{-1}(G) \cap W_1 : G \in \lambda \right\} \cup \{ Y - \overline{W} \}$ such that $\mathrm{St}\,(\overline{W}, \gamma) \subset W_1$ and $\mathrm{St}\,(F, \gamma) \subset U$. There exists a (ω, γ)-refinement $f : X \to Y$ for r, and let β be an open star-refinement of γ such that $f^{-1}(\beta)$ refines ω. Since f is surjective, each $f^{-1}(V) \neq \emptyset$, $V \in \beta$. Let $\beta_1 = \{ V \in \beta : V \cap F \neq \emptyset \}$ and for every $V \in \beta_1$ pick a point $x_V \in f^{-1}(V \cap F)$ such that $x_V \in f^{-1}(W)$ when V meets W.

CLAIM. *$f^{-1}(W) \subset H_1$ and $f^{-1}(F) \subset r^{-1}(U)$.*

If $z \in f^{-1}(W)$, then there exists $O \in \gamma$ such that $f(z), r(z) \in O$ (recall that f is γ-close to r). So, $O \cap W \neq \emptyset$, i.e. $O \subset \mathrm{St}\,(\overline{W}, \gamma)$. Since $\mathrm{St}\,(\overline{W}, \gamma) \subset W_1$, $O \subset W_1$. Hence, $z \in r^{-1}(O) \subset H_1$. Similarly, using the inclusion $\mathrm{St}\,(F, \gamma) \subset U$, we can show that $f^{-1}(F) \subset r^{-1}(U)$.

Since $f^{-1}(F) \subset r^{-1}(U)$ (see the claim above), $f^{-1}(V)$ meets $r^{-1}(U)$ for every $V \in \beta_1$. On the other hand, $f^{-1}(V)$ is contained in an element of ω, hence each $f^{-1}(V)$, $V \in \beta_1$, is contained in $r^{-1}(W_2)$. Therefore, the points $a_V = h(x_V)$, $V \in \beta_1$, are determined. Consider also the points $y_V = f(x_V)$ for $V \in \beta_1$. Take a partition of unity $\{ s_V : V \in \beta \}$ subordinated to β and define the map $p : F \to Z$ by $p(y) = \sum \left\{ s_V(y) \cdot a_V : V \in \beta_1 \right\}$.

Let us show that p is a map from F into L. If $y \in F$ and $x \in X$ with $y = f(x)$, then $f^{-1}\big(\mathrm{St}\,(y, \beta)\big) \subset \mathrm{St}\,(x, \omega) \subset r^{-1}(W_2)$. So, $h\big(f^{-1}\big(\mathrm{St}\,(y, \beta)\big)\big) \subset \mathrm{St}\,\big(h(x), \lambda\big)$. Consequently, $\mathrm{St}\,\big(h(x), \lambda\big)$ contains all points a_V with $y \in V$. Since there exists a convex set $B \subset L$ containing $\mathrm{St}\,\big(h(x), \lambda\big)$, we obtain that $p(y) \in B$.

It remains to prove that $p|W \simeq g|W$. Since β is a star-refinement of γ, for any $y \in W$ there exists $G_y \in \lambda$ with $\mathrm{St}\,(y, \beta) \subset g^{-1}(G_y)$. Fix $y \in W$ and $x \in X$ with $y = f(x)$. Let $\left\{ V(1), V(2), \dots, V(n) \right\}$ be the set of all $V \in \beta_1$ containing y. According to the choice of the points x_V, each $x_{V(i)} \in f^{-1}(W)$, so y and $y_{V(i)}$, $i = 1, \dots, n$, belong to W. Now, since f, r are γ-close, we can find $G_i \in \lambda$, $i = 1, \dots, n$ such that $y_{V(i)}, r(x_{V(i)}) \in g^{-1}(G_i)$. Therefore $y \in g^{-1}(G_y)$ and $y_{V(i)} \in g^{-1}(G_y \cap G_i)$. Hence $\Big\{ g(y), g\Big(r\big(x_{V(i)}\big)\Big) : i = 1, \dots, n \Big\}$ is a subset of $\mathrm{St}\,(G_y, \lambda)$. The last set is contained in a convex set $B_1 \subset L$. Because $x_{V(i)} \in f^{-1}(W) \subset H_1$, $i = 1, \dots, n$, we have $g\big(r\big(x_{V(i)}\big)\big) = h(x_{V(i)}) = a_{V(i)}$, $i = 1, \dots, n$. We finally obtain that B_1 contains $g(y)$ and all a_V with $y \in V$. Therefore B_1 contains both $p(y)$ and $g(y)$. So, $p|W \simeq g|W$. □

Proposition 3.2. *Let r be a refinable map from a normal space X onto a paracompact space Y. If K is any CW-complex and $X \in \alpha(K)$, then $Y \in \alpha(K)$.*

Proof. As in the proof of Theorem 2.4, let L be an open subset of a normed space Z homotopy equivalent to K. It suffices to show that $Y \in \alpha(L)$. Towards this end, let $g : A \to L$ be a map with $A \subset Y$ closed and such that g is extendable to a map $\bar{g} : U \to L$, where U is a neighborhood of A. Choose sets W and W_1 open in Y such that $A \subset W \subset \overline{W} \subset W_1 \subset \overline{W_1} \subset U$ and let $H_1 = r^{-1}(\overline{W_1})$. Since, by Lemma 2.1, $X \in \alpha(L)$, and the map $(\bar{g} \circ r)|H_1$ is extendable to a map from a neighborhood of H_1 into L (the map $\bar{g} \circ r : r^{-1}(U) \to L$ can serve as such an extension), there exists a map $h : X \to L$ extending $(\bar{g} \circ r)|H_1$. Now, we apply Lemma 3.1 (with F replaced by Y and H_2 by X) to conclude that g is extendable to a map from Y into L. Hence, $Y \in \alpha(L)$. □

Proposition 3.3. *Let K be a CW-complex and r a refinable map from a normal space X onto a paracompact space Y with $Y \in \alpha(K)$. Then $X \in \alpha(K)$.*

Proof. Since for every locally finite open cover ω of X there exists an ω-map from X onto Y, the proof follows from Theorem 2.4. □

Proposition 3.2 and Proposition 3.3 imply the following general result.

Corollary 3.4. *For every CW-complex K and a refinable map from a normal space X onto a paracompact space Y we have $X \in \alpha(K)$ if and only if $Y \in \alpha(K)$.*

Since every CW-complex is an ANE for the class of all metrizable spaces, we have

Corollary 3.5. *If r is a refinable map between the metric spaces X and Y, and K is a CW-complex, then* $\text{e-dim}\, X \leqq K$ *if and only if* $\text{e-dim}\, Y \leqq K$.

4. Infinite-dimensional spaces

The preservation of infinite-dimensional properties under refinable maps is widely treated by different authors. H. Kato [15] has shown that refinable maps between compact metric spaces preserve weakly infinite-dimensionality and A. Koyama [17] extended this result by proving that S-weakly infinite-dimensionality is preserved by c-refinable maps between normal spaces. The analogous question concerning property C was settled by D. Garity and D. Rohm [12] for compact metric spaces. F. Ancel [2] introduced approximately invertible maps and established that any such map with compact fibres and metric domain and range preserves property C. Because every refinable map between compact metric spaces is approximately invertible, Ancel's result implies that of Garity and Rohm. We shall prove in this section that refinable maps with normal domain and paracompact range preserve S-weakly infinite-dimensionality and finite C-property. Let us note that in the class of compact spaces weakly infinite-dimensionality and S-weakly infinite-dimensionality, as well as C-space property and finite C-space property, are equivalent.

Recall that a space X is called A-weakly infinite-dimensional [3] (resp. S-weakly infinite-dimensional) if for any sequence $\{A_i, B_i\}$ of disjoint pairs of closed sets in X there exist closed separators C_i between A_i and B_i such that $\cap_{i=1}^{\infty} C_i = \emptyset$ (resp., $\cap_{i=1}^{n} C_i = \emptyset$ for some n). Usually, A-weakly infinite-dimensional spaces are called weakly infinite-dimensional.

Another type infinite-dimensional property is the following one: X is said to be a C-space [1] if for any sequence of open covers $\{\omega_i\}$ of X there exists a sequence of disjoint open families $\{\gamma_i\}$ such that γ_i refines ω_i and $\cup_{i=1}^{\infty} \gamma_i$ covers X. The sequence $\{\gamma_i\}$ is called a C-refinement of $\{\omega_i\}$. If, in the above definition, the sequence $\{\gamma_i\}$ is finite (and satisfying all other properties), then X is called a finite C-space [4]. Every finite C-space is S-weakly infinite-dimensional and every C-space is weakly infinite-dimensional. Moreover, X is S-weakly infinite dimensional (resp., a finite C-space) iff βX is weakly infinite-dimensional (resp., a C-space), see [3] and [5]. Recall that every countable dimensional (a countable union of finite-dimensional subspaces) metric space has property C, but there exists a metrizable C-compactum which is not countable dimensional [21].

Let us note that, even for the class of compact metric spaces, there is no CW-complex K such that X is weakly infinite-dimensional (resp., X has property C) if and only if $\text{e-dim}\, X \leqq K$. Otherwise, since the Hilbert cube Q is the inverse limit of a sequence of finite-dimensional spaces, by [23], we would have that Q is weakly infinite-dimensional (resp., C-space). Therefore, we can not apply the results in Section 2 to conclude that weakly infinite-dimensionality and the property C are preserved by refinable maps.

THEOREM 4.1. *S-weakly infinite-dimensionality is preserved by refinable maps with normal domains and paracompact ranges.*

PROOF. Let r be a refinable map from X onto Y, where X is normal S-weakly infinite-dimensional and Y paracompact. We need the following characterization of S-weakly infinite-dimensionality [3]: a space Z is S-weakly infinite-dimensional iff for every map $f : Z \to Q$ there is n such that $\pi_n \circ f$ is an inessential map; here each $\pi_n : Q \to I^n$ is the projection from Q onto its n-dimensional face I^n generated by the first n coordinates. So, fix $f : Y \to Q$. Then, by the above characterization, there exists n with $h_1 = \pi_n \circ f \circ r$ inessential. Let $f_n = \pi_n \circ f$ and $A = f_n^{-1}(S^{n-1})$, S^{n-1} being the boundary of I^n. We are going to show that f_n is inessential, i.e. $f_n|A$ can be extended to a map from Y into S^{n-1}.

To this end, fix an interior point a from I^n and let $U = f_n^{-1}(L)$ with $L = I^n - \{a\}$. Take open sets W and W_1 in Y with $A \subset W \subset \overline{W} \subset W_1 \subset \overline{W_1} \subset U$ and denote $H_1 = r^{-1}(\overline{W_1})$. Since h_1 is inessential and $h_1(H_1) \subset L$, there exists a map $h : X \to L$ extending $h_1|H_1$. By Lemma 3.1, $f_n|A$ admits an extension $g : Y \to L$, and that suffices to find an extension of $f_n|A$ from Y into S^{n-1}. □

The next proposition extends the result of D. Garity and D. Rohm [12] that property C is preserved by refinable maps between compact metric spaces.

Proposition 4.2. *Let $r : X \to Y$ be refinable with X normal and Y paracompact. If X is a finite C-space, then $r(K)$ has property C for every compact $K \subset X$.*

Proof. Let $\exp(Q)$ be the space of all closed subsets of Q with the Vietoris topology and let $\mathcal{Z}(Q) \subset \exp(Q)$ consist of all Z-sets in Q. We need the following result of Uspenskij [24, Theorem 1.4]: A compact space Z has property C if and only if for any map $\phi : Z \to \mathcal{Z}(Q)$ there exists a map $g : Z \to Q$ such that $g(z) \notin \phi(z)$ for every $z \in Z$.

So, fix a map $\phi : r(K) \to \mathcal{Z}(Q)$ and take an extension $\Phi : Y \to \exp(Q)$ of ϕ (such an extension exists because $\exp(Q)$ is an AR). Next, let $X_1 = r_1^{-1}(Y)$ and $K_1 = r_1^{-1}\big(r(K)\big)$, where $r_1 = \beta r : \beta X \to \beta Y$. Since βX is a C-space, so is K_1 (as a closed subset of βX). Consider the map $\Psi : X_1 \to \exp(Q)$, $\Psi = \Phi \circ r_1$. Then, by the above mentioned result of Uspenskij, there exists a map $h : K_1 \to Q$ with $h(x) \notin \Psi(x)$ for every $x \in K_1$. Take $\varepsilon > 0$ such that $d\big(h(x), \Psi(x)\big) > \varepsilon$ for all $x \in K_1$, where d is the metric in Q, and extend h to a map $g : U \to Q$, where U is a neighborhood of K_1 in X_1, satisfying $d\big(g(x), \Psi(x)\big) > \varepsilon$ for every $x \in U$. The next step is to find a neighborhood W of $r(K)$ in Y with $r_1^{-1}(W) \subset r_1^{-1}(\overline{W}) \subset U$ and to choose an open cover γ of Y such that $\mathrm{St}\,\big(r(K), \gamma\big) \subset W$ and $d_H\big(\Phi(y), \Phi(z)\big) < 2^{-1}\varepsilon$ for any two γ-close points $y, z \in Y$, where d_H is the Hausdorff metric on $\exp(Q)$. Let λ be a finite open cover of Q such that each $\mathrm{St}\,(q, \lambda)$, $q \in Q$, is contained in a convex set of diameter $\leqq 2^{-1}\varepsilon$. Then $\omega = \big\{ g^{-1}(G) \cap X : G \in \lambda \big\} \cup \big\{ X - r_1^{-1}(\overline{W}) \big\}$ covers X. So, there exists an (ω, γ)-refinement $f : X \to Y$ of r and a locally finite open cover α of Y with $f^{-1}(\alpha)$ refining ω. Since $\mathrm{St}\,\big(r(K), \gamma\big) \subset W$, we have $f^{-1}\big(r(K)\big) \subset r^{-1}(W)$ (see the claim from Lemma 3.1). Hence, if $\alpha_1 = \big\{ V \in \alpha : V \cap r(K) \neq \emptyset \big\}$, then we can choose a point $x_V \in f^{-1}(V) \cap r^{-1}(W)$ for every $V \in \alpha_1$. Finally, let $\{s_V : V \in \alpha\}$ be a partition of unity subordinated to α and define the map $p : r(K) \to Q$ by $p(y) = \sum \big\{ s_V(y) \cdot a_V : V \in \alpha_1 \big\}$, where $a_V = g(x_V)$ for every $V \in \alpha_1$.

It remains to show that $p(y) \notin \phi(y)$ for every $y \in r(K)$. Fix $y \in r(K)$ and $x \in X$ with $y = f(x)$. Then $f^{-1}\big(\mathrm{St}\,(y, \alpha)\big) \cap r^{-1}(W)$ contains x and all x_V with $y \in V$. Moreover, $f^{-1}\big(\mathrm{St}\,(y, \alpha)\big) \cap r^{-1}(W)$ is a subset of $\mathrm{St}\,(x, \omega)$

$\cap r^{-1}(W)$. So, $g\big(f^{-1}\big(\mathrm{St}\,(y,\alpha)\big)\cap r^{-1}(W)\big)\subset \mathrm{St}\,\big(g(x),\lambda\big)$. Take a convex set $B\subset Q$ of diameter $\leqq 2^{-1}\varepsilon$ which contains $\mathrm{St}\,\big(g(x),\lambda\big)$. Then, $p(y),g(x)\in B$ and, because $d\big(g(x),\Psi(x)\big)>\varepsilon$, $d\big(p(y),\Psi(x)\big)>2^{-1}\varepsilon$. On the other hand, $y=f(x)$ and $r(x)$ are γ-close, so $d_H\big(\phi(y),\Psi(x)\big)<2^{-1}\varepsilon$. Therefore, $p(y)\notin\phi(y)$. □

THEOREM 4.3. *The finite C-space property is preserved by refinable maps between paracompact spaces.*

PROOF. Suppose X and Y are paracompact, X is a finite C-space and $r: X\to Y$ is refinable. By [4], there exists a compact C-space $K\subset X$ such that every closed set $H\subset X$ which is disjoint from K has a finite dimension. Because this property characterizes finite C-spaces [4], it suffices to show that every closed set in Y disjoint from $r(K)$ is finite-dimensional and $r(K)$ has property C. By Proposition 4.2, $r(K)$ is a C-space. So, it remains only to show that all closed sets in Y outside $r(K)$ are finite-dimensional.

Suppose $B\subset Y$ is closed and $B\cap r(K)=\emptyset$. Take an open $W_2\subset Y$ containing B with $\overline{W_2}$ disjoint from $r(K)$. Then $H_2=r^{-1}(\overline{W_2})$ is closed in X and disjoint from K. Hence, $\dim H_2=n$ is finite. We shall prove that $\dim B\leqq n$.

Let $A\subset B$ be closed and $q: A\to S^n$. Our intention is to extend q to a map from B into S^n. Take an open set $U\subset Y$ such that $B\subset U\subset\overline{U}\subset W_2$ and denote $F=\overline{U}$. The proof will be completed if we can find an extension $\bar q: F\to L$ of q, where L stands for the cube I^{n+1} with a deleted interior point. Towards this end, extend q to a map $g:\overline{W_1}\to S^n$, where $W_1\subset Y$ is open with $A\subset W_1\subset\overline{W_1}\subset U$, and then choose any open set $W\subset Y$ satisfying $A\subset W\subset\overline{W}\subset W_1$. Then $g\circ r$ is a map from $H_1=r^{-1}(\overline{W_1})$ to S^n and, since $\dim H_2\leqq n$, there exists an extension $h: H_2\to L$ of $g\circ r$. Finally, apply Lemma 3.1 to find an extension $f: F\to L$ of g and observe that $f|A=q$. □

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