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# THE EXTENSION DIMENSION AND $C$-SPACES

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Abstract

Some generalizations of the classical Hurewicz formula are obtained for the extension dimension and $C$-spaces. A characterization is also given of the class of metrizable spaces that are absolute neighborhood extensors for all metrizable $C$-spaces.

## 1. *Introduction*

The dimension-lowering Hurewicz theorem states that if $f : X \to Y$ is a closed map, then $\dim X \leqslant \dim f + \dim Y$, where $\dim f = \sup\{\dim f^{-1}(y) : y \in Y\}$. (This was first proved by Hurewicz [**18**] for metric compacta, and was later extended [**24**] for paracompact spaces; see also [**20**, **21**].) In the present paper we prove a version of Hurewicz's theorem for the extension dimension *e-dim* (a precise definition of this concept is given in Section 2). The 'dimensional scale' corresponding to the extension dimension is much finer than the usual integer-valued one. Roughly speaking, the extension dimension of a space is (determined by) a complex. For instance, the inequality $\dim X \leqslant n$ is equivalent to $\text{e-dim}\, X \leqslant \mathbb{S}^n$, and the inequality $\dim_G X \leqslant n$ is equivalent to $\text{e-dim}\, X \leqslant K(G, n)$. (Here, $K(G, n)$ denotes the corresponding Eilenberg–MacLane complex.) The extension dimension allows us to detect new properties of the spaces generated by the new scale. Moreover, a variety of known facts can now be viewed from a more general point of view.

One of the first such generalizations of the classical Hurewicz inequality was obtained in [**12**] as follows: if $f : X \to Y$ is a light map between compact spaces (that is, if $\dim f = 0$), then $\text{e-dim}\, X \leqslant \text{e-dim}\, Y$. This observation, combined with a result of Pasynkov [**22**], yields another generalization of the Hurewicz formula: if $\dim f \leqslant n$ and $X$ and $Y$ are finite-dimensional metric compacta, then $\text{e-dim}\, X \leqslant \text{e-dim}(Y \times \mathbb{I}^n)$. The most general extension of the Hurewicz formula was obtained recently in [**13**]: if $\text{e-dim}(Y \times f^{-1}(y)) \leqslant K$ for every $y \in Y$, then $\text{e-dim}\, X \leqslant K$, provided that $X$ and $Y$ are finite-dimensional metric compacta with $Y$ dimensionally full-valued and $K$ a countable CW-complex.

It is clear now that the Hurewicz formula can be generalized in several possible directions. One of them is to replace the inequality $\dim f \leqslant n$ by $\text{e-dim}\, f^{-1}(y) \leqslant K$ for every $y \in Y$, and to leave $Y$ to be finite-dimensional, say $\dim Y = m$. Note that the inequality $\dim X \leqslant n + m$ from the Hurewicz formula is equivalent to $\text{e-dim}\, X \leqslant \mathbb{S}^{n+m}$ and $\mathbb{S}^{n+m} = \Sigma^m \mathbb{S}^n$, where $\Sigma^m \mathbb{S}^n$ denotes the $m$-iterated suspension of $\mathbb{S}^n$. Consequently, $\dim X \leqslant n + m$ if and only if $\text{e-dim}\, X \leqslant \Sigma^m \mathbb{S}^n$. These observations 'justify' our first results as 'extensional analogues' of the Hurewicz formula.

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(For simplicity, they are not given in their most general forms; complete versions are recorded below as Theorem 2.4 and Corollary 2.7.)

THEOREM. *Let $f: X \to Y$ be a closed surjection of metrizable spaces, and let* $\dim Y \leqslant m$. *If $K$ is a CW-complex such that* $\text{e-dim}(\mathbb{I}^m \times f^{-1}(y)) \leqslant K$ *for any $y \in Y$, then* $\text{e-dim}\, X \leqslant K$.

COROLLARY. *Let $f: X \to Y$, and let the spaces $X$ and $Y$ be as in the above theorem. Then* $\text{e-dim}\, X \leqslant \Sigma^m K$, *provided that* $\text{e-dim}\, f^{-1}(y) \leqslant K$ *for any $y \in Y$.*

The second part of this paper deals with $C$-spaces (see [**1**]; see also [**16**]), predominantly with the class $\mathscr{C}$ of all metrizable $C$-spaces. It is well known that $\mathscr{C}$ contains (strongly) countable-dimensional metrizable spaces—that is, metrizable spaces that are countable unions of (closed) finite-dimensional subsets—but there exists a metric $C$-compactum that is not countable-dimensional [**23**]. A Hurewicz-type theorem is known to be true [**17**] for paracompact $C$-spaces (that is, if $f: X \to Y$ is a closed surjection between paracompact spaces, and if $Y$ and all fibers $f^{-1}(y)$, where $y \in Y$, are $C$-spaces, then $X$ is also a $C$-space). The extensional properties of $X$ in such a situation are discussed in Theorem 3.2. In particular, we conclude (see Corollary 3.3) that absolute extensors for the class $\mathscr{C}$, denoted by AE($\mathscr{C}$), are precisely aspherical absolute neighborhood extensors for the same class (ANE($\mathscr{C}$)). Moreover, in Theorem 3.6 we present a description of ANE($\mathscr{C}$)-spaces, and we provide an answer to a related question posed by F. Ancel [**3**, Question 5.13(c)]. Another implication of Theorem 3.6 is that any subclass of $\mathscr{C}$ that contains strongly countable-dimensional spaces has the same absolute (neighborhood) extensors as the class $\mathscr{C}$. In particular, if $\mathscr{M}$ is such a proper subclass of $\mathscr{C}$, then $\mathscr{M}$ cannot be distinguished by the existence of a metric space $K$ such that $\text{e-dim}\, X \leqslant K$ if and only if $X \in \mathscr{M}$. (J. Dijkstra [**8**] arrived at the same observation for the classes $\mathscr{M}_\alpha$ of all metrizable spaces with transfinite inductive dimension $\leqslant \alpha$, where $\alpha$ is an infinite ordinal.)

## 2. *Generalized Hurewicz theorems for the extension dimension*

All the spaces considered in this paper are at least completely regular, and all the single-valued maps are continuous.

A space $K$ is called an *absolute* (*neighborhood*) *extensor of* $X$ (notation: $K \in$ A(N)E($X$)) if every map $f: A \to K$, where defined on a closed subspace $A$ of $X$, admits an extension over the whole of $X$ or, respectively, over a neighborhood of $A$ in $X$. (In these notes we follow the standard definition of the concept of an absolute (neighborhood) extensor. It should be noted, however, that in certain situations this definition is not satisfactory, and requires a modification. Such an approach is developed in [**4**, **5**, **6**].) Next let us introduce a relation '$\leqslant$' for CW-complexes. Following [**10**] (see also [**11**] and [**6**]), we say that $L \leqslant K$ if for each space $X$ the condition $L \in \text{AE}(X)$ implies that the condition $K \in \text{AE}(X)$. Equivalence classes of CW-complexes with respect to this relation are called *extension types*. The relation defined above creates a partial order in a collection of extension types of complexes. This partial order is still denoted by $\leqslant$, and the extension type $[K]$ of a complex $K$ is still denoted, for simplicity, by $K$. Note that under these definitions a collection of the extension types of all the complexes has both maximal and minimal elements. The minimal element is the extension type of the 0-dimensional sphere $\mathbb{S}^0$ (that is, the two-point discrete space), and the maximal element is obviously the extension type of the one-point space (or, equivalently, of any contractible CW-complex). Finally, the extension dimension of a space $X$ is the minimum of the extension types of the complexes $K$ satisfying the relation $K \in \mathrm{AE}(X)$: $\text{e-dim}\, X = \min\{[K] : K \in \mathrm{AE}(X)\}$. For simplicity, we write $\text{e-dim}\, X \leqslant K$ instead of $\text{e-dim}\, X \leqslant [K]$.

The cone of a space $X$ (notation: $\mathrm{Cone}(X)$) is the quotient set $X \times [0,1]/(X \times \{1\})$ with the following topology: $U$ is open in $\mathrm{Cone}(X)$ if and only if $U \cap (X \times [0,1))$ is open in $X \times [0,1)$ with the product topology and, if the vertex $v$ belongs to $U$, then $X \times (t,1) \subset U$ for some $0 < t < 1$. We need the following result of Dydak [**14**]: if $K$ is a space with at least two points, then $K \in \mathrm{ANE}(X)$ if and only if $\text{e-dim}\, X \leqslant \mathrm{Cone}(K)$.

LEMMA 2.1. *Let $H \subset X$ be a zero-set in $X$, and let $\text{e-dim}\, X \leqslant K$. Then every map $f : H \to \mathrm{Cone}(K)\backslash\{v\}$ extends to a map from $X$ into $\mathrm{Cone}(K)\backslash\{v\}$.*

*Proof.* Let $\pi_1 : \mathrm{Cone}(K)\backslash\{v\} \to K$ and $\pi_2 : \mathrm{Cone}(K) \to [0,1]$ be the natural projections. Then $f = (f_1, f_2)$ with $f_i = \pi_i \circ f$, $i = 1, 2$. Since $\text{e-dim}\, X \leqslant K$ implies that $\text{e-dim}\, X \leqslant \mathrm{Cone}(K)$ (by the Dydak result mentioned above), there exists a map $g : X \to \mathrm{Cone}(K)$ extending $f$. Then $p = \pi_2 \circ g$ extends $f_2$. Fix a function $q : X \to [0,1]$ such that $H = q^{-1}(0)$, and define $s : X \to [0,1)$ by $s(x) = (1 - q(x))p(x)$. Since $\text{e-dim}\, X \leqslant K$, $f_1$ can be extended to a map $h : X \to K$. Then $\overline{f} = (h, s) : X \to \mathrm{Cone}(K)\backslash\{v\}$ is the required extension of $f$. □

In what follows, $C(X, M)$ denotes the space of all continuous maps from $X$ into $M$ equipped with the compact-open topology. A set-valued map $\phi : X \to 2^Y$ is called *strongly lower semi-continuous* (abbreviated as 'strongly lsc') if, for any $x \in X$ and a compact set $P \subset \phi(x)$, there exists a neighborhood $U$ of $x$ such that $P \subset \phi(z)$ for every $z \in U$. Here, $2^Y$ stands for the family of all nonempty subsets of $Y$. We also write '$X$ is $C^n$' to indicate that every continuous image of a k-sphere in $X$, where $k \leqslant n$, is contractible in $X$.

PROPOSITION 2.2. *Suppose that $f : X \to Y$ is a closed surjection such that $X$ is a k-space, that $K \in \mathrm{ANE}(\mathbb{I}^m \times X)$, and that $\text{e-dim}(\mathbb{I}^m \times f^{-1}(y)) \leqslant K$ for any $y \in Y$. Let $M$ be the cone of $K$ with a vertex $v$, and let $h : A \to K$ be a map with $A \subset X$ a zero-set. Then the set-valued map $\phi : Y \to 2^{C(X,M)}$, where*

$$\phi(y) = \{g \in C(X, M) : g(f^{-1}(y)) \subset M\backslash\{v\} \text{ and } g(x) = h(x) \text{ for all } x \in A\},$$

*is strongly lsc, and each $\phi(y)$ is $C^{m-1}$.*

*Proof.* *Claim* 1. *$\phi(y) \neq \emptyset$ for each $y \in Y$.*

Observe first that $K \in \mathrm{ANE}(\mathbb{I}^m \times X)$ implies that $M \in \mathrm{AE}(\mathbb{I}^m \times X)$; in particular, $M \in \mathrm{AE}(X)$. For fixed $y \in Y$, extend $h|(f^{-1}(y) \cap A)$ to a map $g_1 : f^{-1}(y) \to K$ (such an extension exists because $f^{-1}(y)$ is a closed subset of $\mathbb{I}^m \times f^{-1}(y)$), so $\text{e-dim}\, f^{-1}(y) \leqslant K$). Then $g_1$ and $h$ define a map from $f^{-1}(y) \cup A$ into $K$ which is extendable to a map $g : X \to M$. Obviously, $g(f^{-1}(y)) \subset K$ and $g|A = h|A$, so $g \in \phi(y)$.

*Claim* 2. *$\phi$ is strongly lsc.*

Let $y_0 \in Y$, and let $P \subset \phi(y_0)$ be compact. We have to find a neighborhood $V$ of $y_0$ in $Y$ such that $P \subset \phi(y)$ for every $y \in V$. Let $P(x) = \{g(x) : g \in P\}$, $x \in X$.

Since $P \subset C(X, M)$ is compact and $X$ is a k-space, by the Ascoli theorem, each $P(x)$ is compact and $P$ is evenly continuous. This easily implies that the set $W = \{x \in X : P(x) \subset M \backslash \{v\}\}$ is open in $X$ and, obviously, $f^{-1}(y_0) \subset W$. Because $f$ is closed, there exists a neighborhood $V$ of $y_0$ in $Y$ with $f^{-1}(V) \subset W$. Then, according to the choice of $W$ and the definition of $\phi$, $P \subset \phi(y)$ for every $y \in V$.

*Claim* 3. *Each $\phi(y)$ is $C^{m-1}$.*

For a fixed $y \in Y$, take an arbitrary map $u: \mathbb{S}^{n-1} \to \phi(y)$, where $n \leqslant m$. We shall show that $u$ can be extended continuously to a map from $\mathbb{I}^n$ into $\phi(y)$ (we identify $\mathbb{S}^{n-1}$ with the boundary of $\mathbb{I}^n$). Since $\mathbb{S}^{n-1} \times X$ is a k-space (as a product of a compact space and a k-space), the map $u_1: \mathbb{S}^{n-1} \times X \to M$, $u_1(z, x) = u(z)(x)$, is continuous (see [**15**]). Because $u_1(z, x) = h(x)$ for every $(z, x) \in \mathbb{S}^{n-1} \times A$, we can extend $u_1|(\mathbb{S}^{n-1} \times A)$ to a map $u_2: \mathbb{I}^n \times A \to K$, $u_2(z, x) = h(x)$. Then we have a closed subset $H = (\mathbb{S}^{n-1} \times f^{-1}(y)) \cup (\mathbb{I}^n \times (f^{-1}(y) \cap A))$ of $\mathbb{I}^n \times f^{-1}(y)$ and a map $u_3: H \to M \backslash \{v\}$ defined by $u_3|(\mathbb{S}^{n-1} \times f^{-1}(y)) = u_1|(\mathbb{S}^{n-1} \times f^{-1}(y))$ and $u_3|(\mathbb{I}^n \times (f^{-1}(y) \cap A)) = u_2|(\mathbb{I}^n \times (f^{-1}(y) \cap A))$. Since $\mathbb{S}^{n-1}$ and $f^{-1}(y) \cap A$ are zero-sets in $\mathbb{I}^n$ and $f^{-1}(y)$, respectively, both $\mathbb{S}^{n-1} \times f^{-1}(y)$ and $\mathbb{I}^n \times (f^{-1}(y) \cap A)$ are zero-sets in $\mathbb{I}^n \times f^{-1}(y)$, and so is $H$. Note that e-dim$(\mathbb{I}^n \times f^{-1}(y)) \leqslant K$ because $\mathbb{I}^n \times f^{-1}(y)$ is closed in $\mathbb{I}^m \times f^{-1}(y)$. Therefore, by Lemma 2.1, $u_3$ extends to a map $u_4: \mathbb{I}^n \times f^{-1}(y) \to M \backslash \{v\}$. Now, let $F$ be the union of the sets $F_1 = \mathbb{I}^n \times f^{-1}(y)$, $F_2 = \mathbb{I}^n \times A$ and $F_3 = \mathbb{S}^{n-1} \times X$. We define the map $p: F \to M$ by $p|F_1 = u_4$, $p|F_2 = u_2$ and $p|F_3 = u_1$. Obviously, $F$ is closed in $\mathbb{I}^n \times X$. Since $M \in \mathrm{AE}(\mathbb{I}^n \times X)$, there exists an extension $q: \mathbb{I}^n \times X \to M$ of $p$. To finish the proof of Claim 3, observe that $q$ generates the map $\overline{u}: \mathbb{I}^n \to C(X, M)$, $\overline{u}(z)(x) = q(z, x)$. Moreover, $q(z, x) = h(x)$ for any $(z, x) \in \mathbb{I}^n \times A$ and $q(\mathbb{I}^n \times f^{-1}(y)) \subset M \backslash \{v\}$. So $\overline{u}$ is a map from $\mathbb{I}^n$ to $\phi(y)$ that extends $u$. □

Now we need the following result of E. Michael [**25**, Remark 2].

PROPOSITION 2.3. *Let $X$ be paracompact with* $\dim X \leqslant m$ *and $Y$ an arbitrary space. Then every strongly lsc mapping $\varphi: X \to 2^Y$ has a continuous selection, provided that $\varphi(x)$ is $C^{m-1}$ for each $x \in X$.*

THEOREM 2.4. *Let $f: X \to Y$ be a closed surjection with $X$ a k-space and $Y$ paracompact of dimension* $\dim Y \leqslant m$. *If $K$ is any space such that $K \in \mathrm{ANE}(\mathbb{I}^m \times X)$ and* e-dim$(\mathbb{I}^m \times f^{-1}(y)) \leqslant K$ *for any $y \in Y$, then* e-dim $X \leqslant K$.

*Proof.* Suppose that $A \subset X$ is closed, and that $h: A \to K$ is a map. We have to find a continuous extension $\overline{h}: X \to K$ of $h$. Let $M$ be the cone of $K$ with a vertex $v$. Since $M \in \mathrm{AE}(X)$, there exists a map $q: X \to M$ extending $h$. Then $q^{-1}(K)$ is a zero-set in $X$ (because $K$ is such a set in $M$) containing $A$. Therefore, we can assume that $A$ is a zero-set in $X$. Next, define the set-valued map $\phi: Y \to 2^{C(X,M)}$, $\phi(y) = \{g \in C(X, M) : g(f^{-1}(y)) \subset M \backslash \{v\}$ and $g(x) = h(x)$ for all $x \in A\}$ (a similar idea was earlier used by V. Gutev and V. Valov). By Proposition 2.2, $\phi: Y \to C(X, M)$ is a strongly lsc map, with each $\phi(y)$ being a $C^{m-1}$-set. Since $\dim Y \leqslant m$, we can apply Proposition 2.3 to obtain a continuous selection $t: Y \to C(X, M)$ for $\phi$. Then $g: X \to M$, defined by $g(x) = t(f(x))(x)$, is continuous on every compact subset of $X$, and because $X$ is a k-space, $g$ is continuous. Since $t(f(x)) \in \phi(f(x))$, we have $g(x) = h(x)$ for all $x \in A$ and $g(x) \in M \backslash \{v\}$, $x \in X$. Finally, if $\pi_1: M \backslash \{v\} \to K$ denotes the natural retraction, then $\overline{h} = \pi \circ g: X \to K$ is the required continuous extension of $h$. □

A k-space $X$ is called a *cw-space* [**11**] if every contractible CW-complex is an AE($X$). In particular, if $X$ is a cw-space and $K$ is any CW-complex, then Cone$(K) \in$ AE($X$). Any metrizable space (or, more generally, every space admitting a perfect map onto a first countable paracompact space) is *cw* [**14**].

Corollary 2.5. *Let $f: X \to Y$ be a closed surjection, where $Y$ is paracompact with* $\dim Y \leqslant m$ *and $\mathbb{I}^m \times X$ is a cw-space. If $K$ is a CW-complex such that* e-dim$(\mathbb{I}^m \times f^{-1}(y)) \leqslant K$ *for every $y \in Y$, then* e-dim $X \leqslant K$.

*Proof.* Since $X$ is a k-space and $K \in$ ANE$(\mathbb{I}^m \times X)$, we can apply Theorem 2.4. □

Lemma 2.6. *If* e-dim $X \leqslant K$, *where $X \times \mathbb{I}$ is a paracompact cw-space and $K$ is a CW-complex, then* e-dim$(X \times \mathbb{I}) \leqslant \Sigma K$.

*Proof.* This lemma was proved by Dranishnikov [**9**] for metric spaces $X$. His proof, coupled with [**11**, Propositions 1.17 and 1.18], works in our situation too. □

Corollary 2.7. *Let $X \times \mathbb{I}^m$ be a paracompact cw-space, let $K$ be a CW-complex, and let $f: X \to Y$ be a closed surjection with* $\dim Y \leqslant m$. *If* e-dim $f^{-1}(y) \leqslant K$ *for every $y \in Y$, then* e-dim $X \leqslant \Sigma^m K$.

*Proof.* Observe first that $Y$ is paracompact as a closed image of the paracompact $X$. By Lemma 2.6, e-dim$(\mathbb{I}^m \times f^{-1}(y)) \leqslant \Sigma^m K$ for any $y \in Y$. Then the proof follows from Corollary 2.5, with $K$ replaced by $\Sigma^m K$. □

## 3. *C-spaces*

Recall that $X$ is a $C$-space [**1**] if for any sequence $\{\omega_n\}$ of open covers of $X$ there exists a sequence $\{\gamma_n\}$ of open disjoint families in $X$ such that each $\gamma_n$ refines $\omega_n$ and $\bigcup\{\gamma_n : n \in \mathbb{N}\}$ covers $X$. Property $C$ is a dimensional-type property, and it admits a characterization similar to that of finite-dimensional spaces (see Proposition 2.3). (Everywhere below a space is said to be *aspherical* if it is $C^n$ for all $n$.)

Proposition 3.1 [**20**]. *A paracompact $X$ is a $C$-space if and only if every strongly lsc map $\phi: X \to 2^Y$ with aspherical images $\phi(x)$, $x \in X$, where $Y$ is an arbitrary space, has a continuous selection.*

Theorem 3.2. *Let $f: X \to Y$ be a closed surjection with $X$ a k-space and $Y$ a paracompact $C$-space. If $K$ is a space satisfying both conditions $K \in$ ANE$(\mathbb{I}^m \times X)$ and $K \in$ AE$(\mathbb{I}^m \times f^{-1}(y))$ for any $m \in \mathbb{N}$ and any $y \in Y$, then $K \in$ AE$(X)$.*

*Proof.* We follow the proof of Theorem 2.4. Maintaining the same notations and now applying Proposition 3.1 (instead of Proposition 2.3), it suffices to show that if $A$ is a zero-set in $X$, then the formula

$$\phi(y) = \{g \in C(X, M) : g(f^{-1}(y)) \subset M \backslash \{v\} \text{ and } g(x) = h(x) \text{ for all } x \in A\}$$

defines a set-valued map $\phi: Y \to 2^{C(X,M)}$ which is strongly lsc, and each $\phi(y)$ is aspherical. This follows directly from Proposition 2.2. □

Theorem 3.2 is of no interest to us when $K$ is a CW-complex. Indeed, $K \in \mathrm{AE}(\mathbb{I}^m \times f^{-1}(y))$ for all $m$ implies that every homotopy group of $K$ is trivial. So $K$ is contractible, and therefore it is an absolute extensor for any cw-space. On the other hand, the Borsuk example of a contractible and locally contractible compact metric space which is not an AE for the class of all metrizable spaces shows that Theorem 3.2 has a meaning for general spaces $K$.

Let $\mathscr{C}$ denote the class of all metrizable $C$-spaces. We write $K \in \mathrm{A(N)E}(\mathscr{C})$ if $K \in \mathrm{A(N)E}(X)$ for any $X \in \mathscr{C}$; when the class of all metrizable spaces is considered, we simply write $K \in \mathrm{A(N)E}$.

Corollary 3.3. *A space $K \in \mathrm{ANE}(\mathscr{C})$ is an $\mathrm{AE}(\mathscr{C})$ if and only if $K$ is aspherical.*

*Proof.* Any $\mathrm{AE}(\mathscr{C})$ is aspherical (because the class $\mathscr{C}$ contains all finite-dimensional spaces) and an $\mathrm{ANE}(\mathscr{C})$. Suppose that $K \in \mathrm{ANE}(\mathscr{C})$ is aspherical, and that $X \in \mathscr{C}$. We apply Theorem 3.2 in the special case when $X = Y$ and $f$ is the identity map. In this special case, Proposition 2.2 is true if $\text{e-dim}(\mathbb{I}^m \times f^{-1}(y)) \leqslant K$ is replaced by $K \in C^{m-1}$. Indeed, Claim 1 becomes trivial; to prove Claim 2 we do not need to apply Lemma 2.1 because the set $H$ is homeomorphic either to $\mathbb{I}^n$ if $y \in A$ or to $\mathbb{S}^{n-1}$ otherwise. We need to show that any map from $\mathbb{S}^{n-1}$ into $K$ is extendable to a map from $\mathbb{I}^n$ into $K$, $n \leqslant m$. In order to apply Theorem 3.2, it remains only to check that $K \in \mathrm{ANE}(X \times \mathbb{I}^m)$ for all $m$, and that is true because $X \times \mathbb{I}^m \in \mathscr{C}$ [**17**]. □

Let us now discuss some sufficient (and necessary) conditions for a metric space to be an $\mathrm{ANE}(\mathscr{C})$. Let $\mathscr{P}$ be a topological property. We say that $X \subset E$ is a $UV(\mathscr{P})$ subset of $E$ if each neighborhood $U$ of $X$ in $E$ contains a neighborhood $V$ of $X$ in $E$ such that any map $h: Z \to V$, where $Z \in \mathscr{P}$, extends to a map $\bar{h}: \mathrm{Cone}(Z) \to U$. A closed surjection $f: X \to Y$ is called $UV(\mathscr{P})$ if each of its point inverses is a $UV(\mathscr{P})$ subset of $X$. Recall that if, in the above definition, $V$ is contractible in $U$, then $X$ is called $UV^\infty$; a cell-like space is a compact metric space $X$ such that $X$ is a $UV^\infty$ set in every ANE-space $E$ in which it is embedded as a closed subset (see, for example, [**3**]). In the existing terminology, a $UV^\infty$ map is a perfect map with $UV^\infty$ preimages, and a cell-like map is a perfect map with cell-like preimages. Obviously, every $UV^\infty$ map is $UV(\mathscr{P})$ for any property $\mathscr{P}$.

Proposition 3.4. *Either of the following two conditions is sufficient for a metrizable space $Y$ to be an* $\mathrm{ANE}(\mathscr{C})$.

(a) *$Y$ is locally contractible* (*or, more generally, there exists a metrizable space $X$ and a $UV^\infty$ map from $X$ onto $Y$*).

(b) *$Y$ has a base of open aspherical sets.*

*Proof.* The first condition was proved by Ancel [**3**, Theorem C.5.9]; see also [**1**] for the case of local contractibility. Condition (b) can be obtained by using the arguments of Ageev and Repovš [**2**, proof of Theorem 1.3]. □

Not every metrizable $\mathrm{ANE}(\mathscr{C})$-space is locally contractible. J. van Mill provided an example of a cell-like image of the Hilbert cube such that no nonempty open subset is contractible in that space [**19**]. At the same time, by [**3**], this example is an $\mathrm{ANE}(\mathscr{C})$. In view of the above-mentioned result of Ancel [**3**, Theorem C.5.9], it is interesting to establish whether any metrizable $\mathrm{ANE}(\mathscr{C})$ is a $UV^\infty$ image of a metrizable space. In such a case, the class of metrizable ANE($\mathscr{C}$) spaces would be precisely the class of all $UV^{\infty}$ images of metrizable spaces. We can provide a similar characterization ANE($\mathscr{C}$) in terms of $UV$(s.c.d.) maps, where 's.c.d.' denotes the property *strong countable-dimensionality*.

PROPOSITION 3.5. *Let $f: M \to X$ be a surjective map between metrizable spaces. If, for any $x \in X$ and its neighborhood $U(x)$ in $X$, there exists another neighborhood $V(x)$ of $x$ in $X$ such that $\overline{V}(x) = f^{-1}(V(x))$ is contractible in $\overline{U}(x) = f^{-1}(U(x))$, then $X \in \mathrm{ANE}(\mathscr{C})$.*

*Proof.* The first step is to show that $X$ is an approximate absolute neighborhood extensor for the class $\mathscr{C}$; that is, if $H$ is a metrizable $C$-space, if $A \subset H$ is closed and if $h: A \to X$ is a map, then for every open cover $\gamma$ of $X$ there is a neighborhood $W_A$ of $A$ in $H$ and a map $\overline{h}: W_A \to X$ such that $\overline{h}|A$ is $\gamma$-close to $h$. We follow the construction from the proof of [**2**, Theorem 4.3, first part]. For every $x \in X$ and $n \geqslant 0$, we fix points $z(x) \in f^{-1}(x)$ and neighborhoods $V_n(x) \subset U_n(x)$ of $x$ in $X$ such that:

(1) $\overline{V}_n(x)$ contracts in $\overline{U}_n(x)$ to $z(x)$ for all $n \geqslant 0$ and $x \in X$;
(2) the cover $\alpha_0 = \{U_0(x) : x \in X\}$ refines $\gamma$;
(3) the cover $\alpha_n = \{U_n(x) : x \in X\}$ star-refines $\beta_{n-1} = \{V_{n-1}(x) : x \in X\}$ for any $n \geqslant 1$; that is, $\{\mathrm{St}(U, \alpha_n) : U \in \alpha_n\}$ refines $\beta_{n-1}$.

Observe that we have corresponding covers $\overline{\gamma} = f^{-1}(\gamma)$, $\overline{\alpha}_n = \{\overline{U}_n(x) : x \in X\}$ and $\overline{\beta}_n = \{\overline{V}_n(x) : x \in X\}$ of $M$ such that $\overline{\alpha}_0$ refines $\overline{\gamma}$ and $\overline{\alpha}_n$ star-refines $\overline{\beta}_{n-1}$, $n \geqslant 1$. For every $n \geqslant 0$ and $x \in X$, we fix a contraction map $F^{x,n}: \overline{V}_n(x) \times [0,1] \to \overline{U}_n(x)$ with $F^{x,n}(z,1) = z(x)$. Since $A$ is a $C$-space (as a closed subset of $H$), there is a sequence of disjoint open families $\{\mu_n : n = 1,2,\ldots\}$ in $H$ such that the restriction of each $\mu_n$ on $A$ refines $h^{-1}(\beta_n)$ and $\mu = \bigcup\{\mu_n : n = 1,2,\ldots\}$ covers $A$. Further, let $\mathscr{K}$ be the nerve of $\mu$, and let $\theta: W_A = \cup\{W : W \in \mu\} \to |\mathscr{K}|$ be a barycentric map. We define a map $g: |\mathscr{K}| \to M$ such that the family $\{g(\theta(y)) \cup f^{-1}(h(y)) : y \in A\}$ refines $\overline{\gamma}$. Then the map $\overline{h} = f \circ g \circ \theta$ will be the required $\gamma$-approximation of $h$. Any simplex $(W_0, W_1, \ldots, W_k)$ from $\mathscr{K}$, where $W_i \in \mu_{n(i)}$, can be ordered such that $n(0) < n(1) < \ldots < n(k)$. (This is possible because $\cap\{W_i : i = 1,2,\ldots,k\} \neq \emptyset$, so the numbers $n(i)$ are different.) By (3), for any $W \in \mu_n$ there exists $x(W) \in X$ with $\mathrm{St}(h(W \cap A), \alpha_n) \subset V_{n-1}(x(W))$. We define $g_0: |\mathscr{K}^0| \to M$ by $g_0(W) = z(x(W))$, $W \in \mu$. Using the contractions $F^{x,n}$, as in [**2**, proof of Theorem 4.3], we can define by induction, maps $g_n: |\mathscr{K}^n| \to M$ such that the restriction of $g_n$ on $|\mathscr{K}^i|$ is $g_i$, $i \leqslant n$, and for any simplex $\triangle^n = (W_0, W_1, \ldots, W_n) \in |\mathscr{K}^n|$ we have:

(4) $f^{-1}(h(W_0 \cap A)) \cup g_n(\triangle^n) \subset \overline{U}_{n_0-1}(x(W_0))$.

We thus obtain a map $g: |\mathscr{K}| \to M$ and, by (4), $\overline{h}|A$ and $h$ are $\gamma$-close, where $\overline{h} = f \circ g \circ \theta$. Indeed, if $y \in A$ and $\theta(y) \in \triangle^n$ for some simplex $\triangle^n = (W_0, W_1, \ldots, W_n)$, then $\overline{h}(y) \in f(g_n(\triangle^n))$ and $h(y) \in h(W_0 \cap A)$. According to (4), the last two inclusions imply that both $\overline{h}(y)$ and $h(y)$ belong to $U_{n_0-1}(x(W_0))$. So, $\overline{h}(y)$ and $h(y)$ are $\alpha_{n_0-1}$-close and, since $n_0 - 1 \geqslant 0$, they are also $\gamma$-close. Therefore, $X$ is an approximate absolute neighborhood extensor for the class $\mathscr{C}$.

To complete the proof, we state the following result, which was actually proved in [**2**] but not explicitly formulated there. If $\mathscr{M}$ is a class of metrizable spaces such that $Y \times [0,1) \in \mathscr{M}$ for every $Y \in \mathscr{M}$, then any approximate absolute neighborhood extensor for $\mathscr{M}$ is an ANE($\mathscr{M}$). Since $\mathscr{C}$ is closed with respect to multiplication by $[0,1)$, we have $X \in \mathrm{ANE}(\mathscr{C})$. □

THEOREM 3.6. *For a metrizable space $X$, the following conditions are equivalent.*

(a) *$X$ is an* ANE *for the class of metrizable* (*strongly*) *countable-dimensional spaces.*

(b) *$X$ is a $UV$*(s.c.d.) *image of a metrizable space.*

(c) *$X$ is an* ANE($\mathscr{C}$).

*Proof.* Since every metrizable (strongly) countable-dimensional space has the property $C$, condition (c) implies condition (a). Standard arguments show that every metrizable $X$ that is an ANE for the class of metrizable (strongly) countable-dimensional spaces has the following property.

($*$) For every $x \in X$ and its neighborhood $U(x)$ in $X$, there is a neighborhood $V(x) \subset U(x)$ such that any map from a closed subset of a (strongly) countable-dimensional metrizable space $Z$ into $V(x)$ extends to a map from $Z$ into $U(x)$.

Hence, condition (a) implies that the identity map of $X$ is $UV$(s.c.d.). It thus remains to prove that condition (b) implies condition (c). Let $f: Y \to X$ be a $UV$(s.c.d.) map with $Y$ metrizable. We need the following result of M. Zarichnyi [**26**]. There exists an $\omega$-soft map from a $\sigma$-compact strongly countable-dimensional metrizable space onto the Hilbert cube. Here, a map $g: M \to H$ is called *$\omega$-soft* if for every strongly countable-dimensional metrizable space $Z$, its closed subset $B \subset Z$, and any two maps $\phi: Z \to H$ and $\psi: B \to M$ such that $g \circ \psi = \phi|B$, there exists a map $\Phi: Z \to M$ extending $\psi$ with $g \circ \Phi = \phi$. Using the Zarichnyi result, for every cardinal $\tau$ we can construct a strongly countable-dimensional metrizable space $M(\tau)$ of weight $\tau$ and an $\omega$-soft map $g: M(\tau) \to l_2(\tau)$, where $l_2(\tau)$ denotes the Hilbert space of weight $\tau$ (see [**7**] for a similar reduction). Embedding $Y$ into $l_2(\tau)$ for some $\tau$ and considering the restriction $g_Y$ of $g$ onto $M_Y = g^{-1}(Y)$, we obtain a strongly countable-dimensional metrizable space $M_Y$ and an $\omega$-soft map $g_Y: M_Y \to Y$. Let $q = f \circ g_Y$. We shall show that $q: M_Y \to X$ satisfies the hypotheses of Proposition 3.5. To this end, let $U(x)$ be a neighborhood of $x \in X$. Since $f$ is $UV$(s.c.d.), there exists a neighborhood $W(x) \subset f^{-1}(U(x))$ such that every map from a strongly countable-dimensional metrizable space $Z$ into $W(x)$ extends to a map from Cone($Z$) into $f^{-1}(U(x))$. Then $f^{-1}(V(x)) \subset W(x)$ for some neighborhood $V(x)$ of $x$ in $X$ because $f$ is closed. Now consider $\overline{V}(x) = q^{-1}(V(x))$ and $\overline{U}(x) = q^{-1}(U(x))$. Since $\overline{V}(x)$ is strongly countable-dimensional, there exists a map $\phi: \mathrm{Cone}(\overline{V}(x)) \to f^{-1}(U(x))$ extending the restriction $g_Y|\overline{V}(x)$. Finally, using the fact that $g_Y$ is $\omega$-soft, we can lift $\phi$ to a map $\Phi: \mathrm{Cone}(\overline{V}(x)) \to \overline{U}(x)$ such that $\Phi|\overline{V}(x)$ is the identity. Therefore, $\overline{V}(x)$ is contractible in $\overline{U}(x)$ and, by Proposition 3.5, $X \in$ ANE($\mathscr{C}$). □

The equivalence of conditions (*a*) and (*c*) from Theorem 3.6 yields the following observation: if $\mathscr{M}$ is a subclass of $\mathscr{C}$ containing all strongly countable-dimensional spaces, then ANE($\mathscr{M}$) coincides with ANE($\mathscr{C}$) in the realm of metrizable spaces. Consequently, since every $K \in$ AE($\mathscr{M}$) is aspherical, the above observation, combined with Corollary 3.3, implies also that $\mathscr{M}$ and $\mathscr{C}$ have the same metrizable AE-spaces. Finally, we would like to point out that Theorem 3.6 provides an answer to the question [**3**, Question 5.13(c)] of whether a metrizable space $X$ is an ANE for the class of countable-dimensional spaces if $X$ has the property ($*$) mentioned in the proof of Theorem 3.6.

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