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Continuous Selections and Finite C-Spaces

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Abstract. Characterization of paracompact finite C-spaces via continuous selections avoiding Z_σ-sets are given. We apply these results to obtain some properties of finite C-spaces. Factorization theorems and a completion theorem for finite C-spaces are also proved.

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1. Introduction

Finite C-spaces were recently introduced by Borst [6] and classified with the help of transfinite dimension function $\dim_C$ (Borst considered only separable metric spaces, but his definition can be extended for any normal space, see [7]). Every finite C-space has property C in the sense of Addis and Gresham [2] and, in the realm of compact spaces, both classes coincide. On the other hand, every finite C-space is weakly infinite-dimensional in the sense of Smirnov (abbr., S-wid), while C-spaces are weakly infinte-dimensional in the sense of Alexandrov (abbr., A-wid). One of the main problems in infinite dimension theory is whether every weakly infinite-dimensional compact metric space has property C (it is well known that both properties S-wid and A-wid are equivalent for compact spaces). The positive solution of this problem would imply that finite C-spaces and S-wid spaces constitute the same class. But perhaps the most important implication from the positive solution of this problem is a positive solution of another old problem in dimension theory (according to [7] and [21], Pasynkov was who first asked this question in 1972–73): Is any product of two weakly infinite-dimensional compact metric spaces weakly infinite-dimensional?

In the present paper we give other two characterizations of paracompact finite C-spaces in the terms of continuous selections for set-valued maps. The first one is similar to Uspenskij's description of paracompact C-spaces [26, Theorem 1.3] and to the Ščepin and N. Brodskii theorem [24]. To state this result let us agree some notations. All spaces are assumed to be Tychonoff. A set-valued map $\theta\colon X \to 2^Y$,

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where 2^Y is the family of nonempty subsets of Y, is called strongly lower semi-continuous (br., strongly lsc) if for every compact $K \subset Y$ the set $\{x \in X : K \subset \theta(x)\}$ is open in X; θ is said to be m-aspherical if all $\theta(x)$ are C^m-sets (recall that a $A \subset B$ is C^m-embedded, or m-embedded in B, if every continuous image of a k-sphere in A, $k \leqslant m$, is contractible in B; when A is m-embedded in itself, then it is called C^m). We also say that a sequence of maps $\theta_n\colon X \to 2^Y$ is increasing (resp., decreasing) if $\theta_n(x) \subset \theta_{n+1}(x)$ (resp., $\theta_{n+1}(x) \subset \theta_n(x)$) for all $x \in X$ and $n \in \mathbb{N}$; the sequence $\{\theta_n\}$ is aspherical provided it is increasing and each $\theta_n(x)$, $x \in X$, is C^{n-1}-embedded in $\theta_{n+1}(x)$.

THEOREM 1.1. *For a paracompact space X the following are equivalent:*

(a) *X is a finite C-space;*
(b) *For any space Y and any aspherical sequence of strongly lsc maps $\phi_n\colon X \to 2^Y$, there exists m such that ϕ_m admits a continuous selection;*
(c) *For any space Y and any aspherical sequence $\phi_n\colon X \to 2^Y$ of set-valued open-graph maps there exists m such that ϕ_m admits a continuous selection.*

Another characterization of paracompact finite C-spaces is the following selection theorem of Michael's type (see [13, Theorem 1.1] for a similar characterization of paracompact C-spaces).

THEOREM 1.2. *For a paracompact space X the following are equivalent:*

(a) *X is a finite C-space;*
(b) *Suppose that Y is a Banach space and $\phi\colon X \to \mathcal{F}_c(Y)$ is an lsc map. Then, for every decreasing Z-sequence $\{\psi_n\}$ for ϕ consisting of closed-graph maps $\psi_n\colon X \to \mathcal{F}(Y)$, there exist m and a continuous selection for ϕ avoiding the map ψ_m;*
(c) *Let Y be a Banach space and $\phi\colon X \to \mathcal{F}_c(Y)$ be an lsc map. Then, for every decreasing Z-sequence for ϕ consisting of closed sets $F_n \subset Y$, there exists a continuous selection for ϕ avoiding some F_m.*

Here, $\mathcal{F}(Y)$ stands for the closed sets $F \subset Y$ and $\mathcal{F}_c(Y)$ denotes the convex elements of $\mathcal{F}(Y)$ (the same notation will be used when Y is a subset of a vector space). A set-valued map $\theta\colon X \to 2^Y$ is lower semi-continuous (br., lsc) if $\theta^{-1}(U) = \{x \in X : \theta(x) \cap U \neq \emptyset\}$ is open in X for every open $U \subset Y$. The term 'continuous selections avoiding a set' is due to E. Michael [19]. If $\phi\colon X \to 2^Y$ is a set-valued map and $F \subset Y$, then a map $f\colon X \to Y$ is a selection for ϕ avoiding F if $f(x) \in \phi(x)\backslash F$ for every $x \in X$. Similarly, if $\psi\colon X \to 2^Y$ is another set-valued map, we say that f is a selection for ϕ avoiding ψ provided $f(x) \in \phi(x)\backslash\psi(x)$ for any $x \in X$. We adopt the following definition for a Z_n-set: if $n \geqslant 0$ we say that $F \in \mathcal{F}(Y)$ is a Z_n-set in Y if the set $C(I^n, Y\backslash F)$ is dense in $C(I^n, Y)$, where $C(I^n, Y)$ stands for the space of all continuous maps from I^n into Y equipped with the compact open topology. When $F = \bigcup\{F_n : n \in \mathbb{N}\}$ such that each F_n is a

Z_n-set in Y, we say that it is a Z_σ-set in Y. The collection of all Z_n-sets in Y is denoted by $\mathcal{Z}_n(Y)$. Obviously, Z_0-sets are precisely the closed and nowhere dense subsets of Y. A sequence of maps $\psi_n\colon X \to 2^Y$ is called a Z-sequence for a given map $\phi\colon X \to 2^Y$ if $\phi(x) \cap \psi_n(x) \in \mathcal{Z}_{n-1}(\phi(x))$ for every $x \in X$ and $n \in \mathbb{N}$.

The definition and some general properties of C-spaces and finite C-spaces are given in Section 2. Theorem 1.1 and Theorem 1.2 are proved in Section 3 and Section 4, respectively. Section 5 is devoted to some applications. The idea to characterize dimension like properties by removing maps from ‘small’ sets goes back to the Alexandroff characterization of n-dimensional compacta in Euclidean spaces $\mathbb{R}^k$ as the compacta which are removable from any $(k-n-1)$-dimensional polyhedron [3, 11] (see [9] for more general treatment of removable maps in Euclidean spaces). Following this general idea, we apply Theorem 1.2 to characterize finite C-spaces in terms of removable sequences of maps (see Proposition 5.1). This characterization is then used to obtain that finite C-space property is preserved by approximately invertible maps (an analogue to Ancel’s result [1]). In the final Section 6 we prove a factorization theorem for finite C-spaces when the image space is metrizable and apply this theorem to obtain that any metrizable finite C-space has a completion with the same property.

2. Some Preliminary Results

This section is devoted to the definition and general properties of C-spaces and finite C-spaces.

C-space property was introduced by Haver [15] for compact metric spaces. Addis and Gresham [2] reformulated Haver’s definition so that it has meaning for any space: A space X has property C (or X is a C-space) if for any sequence $\{\omega_n\}$ of open covers of X there exists a sequence $\{\gamma_n\}$ of open disjoint families in X such that each γ_n refines ω_n and $\bigcup\{\gamma_n : n \in \mathbb{N}\}$ covers X. The family $\{\gamma_n\}$ is called C-refinement for $\{\omega_n\}$. We now adopt the following definition: X is a C-space if every sequence $\{\omega_n\}$ of locally finite and functionally open covers of X has a C-refinement $\{\gamma_n\}$ such that $\bigcup\{\gamma_n : n \in \mathbb{N}\}$ is a locally finite and functionally open cover of X. In the realm of paracompact spaces, this definition coincides with the Addis and Gresham one. Let us note that countable-dimensional metric spaces have property C [2]. Since the countable sum theorem for property C holds in the class of paracompact spaces [13], any strongly countable-dimensional (a countable union of closed finite-dimensional subspaces) paracompact is a C-space. On the other hand, there exists a metric C-compactum which is not countable-dimensional [23].

In [6], Borst originally defined finite C property for separable metric spaces: X is a finite C-space if for any sequence $\{\omega_n\}$ of finite open covers of X there exists a finite sequence $\{\gamma_n\}_{n=1}^k$ of disjoint open families in X such that each γ_n refines ω_n and $\bigcup\{\gamma_n : n = 1, \dots, k\}$ covers X. The sequence $\{\gamma_n\}_{n=1}^k$ is called a finite C-refinement for $\{\omega_n\}$. Obviously, the existence of a finite C-refinement for $\{\omega_n\}$ is equivalent to the existence of a finite C-refinement consisting of finite families. As above, we reformulate this definition for arbitrary spaces: X is a finite C-space if any sequence $\{\omega_n\}$ of finite functionally open covers of X has a finite C-refinement $\{\gamma_n\}_{n=1}^{k}$ such that each γ_n is a finite functionally open and disjoint family. Note that, in the realm of normal spaces we can omit the requirement ω_n and γ_n to be functionally open.

Since any locally finite family of open sets in a pseudocompact space is finite, we have the following

PROPOSITION 2.1. *Property C and finite C-space property are equivalent for any pseudocompact space.*

The proof of our first observation follows from the well known fact that if $Y \subset X$ is C^*-embedded (i.e., every bounded function on Y can be continuously extended to a bounded function on X), then for any finite functionally open cover ω of Y there exists a finite functionally open cover γ of X such that ω is refined by $\gamma \cap Y$.

PROPOSITION 2.2. *For any space X we have:*

(a) *If X is a finite C-space and $Y \subset X$ is C^*-embedded, then Y has finite C-space property;*
(b) *X is a finite C-space if and only if βX is a C-space.*

Let us note that, in case X is normal, Proposition 2.2(b) was proved by Chatyrko [7] (he proved somewhat more, that $\dim_C X = \dim_C \beta X$). Next lemma is well known in case of metrizable spaces (see [6]). The same proof remains valid in our situation.

LEMMA 2.3. *Every normal finite C-space is S-weakly infinite-dimensional.*

Recall that a space X satisfies the condition (K) [22] if there is a compact set $K \subset X$ such that every closed in X set which is disjoint from K has finite covering dimension dim. Borst [6, Theorem 3.8] proved that his definition of finite C-spaces is equivalent to the following one: Any sequence of open covers has a finite C-refinement. This implies that, in the realm of metric spaces, every finite C-space has property C. Next theorem shows that the same is true for any space.

THEOREM 2.4. *For a space X the following condition are equivalent:*

(a) *X is a finite C-space;*
(b) *Any sequence $\{\omega_n\}$ of locally finite functionally open covers of X has a finite C-refinement $\{\gamma_n\}_{n=1}^{s}$ such that $\{\gamma_n : n = 1, 2, \ldots, s\}$ is a locally finite functionally open cover of X.*

Proof. (a) $\Rightarrow$ (b) Let $\omega_n = \{U_\alpha^n : \alpha \in A_n\}$ be a sequence of locally finite functionally open covers of X. Then there exists a map $f\colon X \to Y$ onto a metrizable space Y and a sequence $\eta_n = \{W_\alpha^n : \alpha \in A_n\}$ of open covers of Y such that $f^{-1}(W_\alpha^n) \subset U_\alpha^n$ for every n and $\alpha \in A_n$ (see [12, Exercise 5.1.J(b)]). Let $Z = (\beta f)^{-1}(Y)$, where $\beta f\colon \beta X \to \beta Y$ is the extension of f, and $\overline{\omega_n} = (\beta f)^{-1}(\eta_n)$. Then each $\overline{\omega_n}$ is a functionally open cover of Z and its restriction on X refines ω_n. By Proposition 2.2(b), Z is a finite C-space. Hence, by Lemma 2.3, Z is S-wid. Because every paracompact S-wid space satisfies the condition (K) (see [4]), so does Z. The set $K \subset Z$ has property C (as a compact subset of a finite C-space). Therefore, there exists a finite C-refinement $\{\overline{\gamma_n}\}_{n=1}^k$ of $\{\overline{\omega_n} \cap K\}$ consisting of finite, disjoint and functionally open in K families. Since K is compact, we can find finite, disjoint and functionally open in Z families β_n, $n = 1, \ldots, k$, such that every β_n refines $\overline{\omega_n}$ and $V = \bigcup\{G \in \beta_n : n = 1, \ldots, k\}$ covers K. Next, take an open set $W \subset Z$ such that $Z \backslash \overline{W}$ is functionally open in Z and $K \subset W \subset \overline{W} \subset V$. Then $Z \backslash W$, being closed in Z and disjoint from K, is finite-dimensional. So is $Z \backslash \overline{W}$ as an F_σ-subset of $Z \backslash W$. Let $\dim Z \backslash \overline{W} = m$. According to [2, Proposition 2.12], there exists a finite C-refinement $\{\beta_n\}_{n=k+1}^{k+m+2}$ of $\{\overline{\omega_n} \cap (Z \backslash \overline{W})\}_{n=k+1}^{k+m+2}$. Since $Z \backslash \overline{W}$ is paracompact, we can assume that $\{\beta_n : n = k+1, \ldots, k+m+2\}$ is a locally finite and functionally open cover of $Z \backslash \overline{W}$. We finally obtain that $\{\beta_n\}_{n=1}^{k+m+2}$ is a finite C-refinement for the sequence $\{\overline{\omega_n}\}$ such that $\{\beta_n : n = 1, \ldots, k + m + 2\}$ is a locally finite and functionally open cover of Z. Then $\{\beta_n \cap X\}_{n=1}^{k+m+1}$ is the required finite C-refinement of $\{\omega_n\}$.

(b) $\Rightarrow$ (a) Let $\omega = \{\omega_n\}$ be a sequence of finite functionally open covers of X. According to our assumption, there is a finite C-refinement $\{\beta_n\}_{n=1}^k$ for ω such that $\{\beta_n : n = 1, \ldots, k\}$ is a locally finite and functionally open cover of X. For fixed m let $\omega_m = \{U_i : i = 1, \ldots, s\}$ and $\beta_m = \{V_\alpha : \alpha \in A\}$. We define disjoint sets $A_i \subset A$, $i = 1, 2, \ldots, s$, by $A_1 = \{\alpha : V_\alpha \subset U_1\}$ and $A_{i+1} = \{\alpha \in A \backslash \bigcup_{j=1}^i A_j : V_\alpha \subset U_{i+1}\}$. Since the union of any locally finite family consisting of functionally open sets is again functionally open, every $W_i = \bigcup\{V_\alpha : \alpha \in A_i\}$ is functionally open. Moreover, $\gamma_m = \{W_i : i = 1, \ldots, s\}$ is a disjoint family refining ω_m. Then $\{\gamma_n\}_{n=1}^k$ is a finite C-refinement of ω and consists of finite functionally open families. □

COROLLARY 2.5. *Every finite C-space has property C.*

Next corollary was actually proved in Theorem 2.4.

COROLLARY 2.6. *If X is paracompact, then X is a finite C-space if and only if X satisfies condition (K) such that $K \subset X$ has property C.*

3. Proof of Theorem 1.1

(a) $\Rightarrow$ (b) Our proof is a slight modification of the proof of [26, Theorem 2.1]. Suppose X is a finite C-space and let $\{\phi_n\}$ be an aspherical sequence of strongly lsc maps from X into Y. We shall construct by induction the following objects (for simplicity, every abstract simplicial complex is identified with its polyhedron equipped with the Whitehead topology):

(1) a sequence of Λ_n of pairwise disjoint sets;

(2) two sequences $\{\omega_n\}$ and $\{\gamma_n\}$ of open locally finite covers of X such that $\omega_n = \{U_\alpha : \alpha \in \Lambda_n\}$, $\gamma_n = \{V_\alpha : \alpha \in \Lambda_n\}$ and $\overline{U_\alpha} \subset V_\alpha$ for every $\alpha \in \Lambda_n$;

(3) an increasing sequence of simplicial complexes $\{\mathcal{K}_n\}$ such that $\dim\mathcal{K}_n \leqslant n-1$ and $\bigcup_{k=1}^{k=n} \Lambda_k$ is the set of vertices of $\mathcal{K}_n$;

(4) finite subcomplexes P_α of $\mathcal{K}_n$ for every $\alpha \in \Lambda_n$ and $n \in \mathbb{N}$;

(5) continuous maps $g_n\colon \mathcal{K}_n \to Y$ such that g_n extends g_m whenever $m < n$ and $g_n(P_\alpha) \subset \phi_n(x)$ for every $x \in V_\alpha \in \gamma_n$;

(6) if $\bigcap_{j=1}^{j=k} U_{\alpha(j)} \neq \emptyset$, where $\alpha(j) \in \Lambda_{n(j)}$, $j = 1, \dots, k$, and $n(1) < \cdots < n(k)$, then the set $\{\alpha(1), \dots, \alpha(k)\}$ is a simplex of the complex $P_{\alpha(k)}$.

Let us describe the first step of the induction. For every $x \in X$ fix a point $c(x) \in \phi_1(x)$. Since ϕ_1 is strongly lsc, we can find neighborhoods $O(x)$ in X such that $z \in O(x)$ yields $c(x) \subset \phi_1(z)$. Let $\gamma_1 = \{V_\alpha : \alpha \in \Lambda_1\}$ be a locally finite open refinement of of the cover $\{O(x) : x \in X\}$ and $\omega_1 = \{U_\alpha : \alpha \in \Lambda_1\}$ an index closure refinement of γ_1. Let $\mathcal{K}_1$ be the 0-dimensional complex with the set of vertices Λ_1 and every P_α, $\alpha \in \Lambda_1$ be the subcomplex of $\mathcal{K}_1$ with one vertex α. Choose the points $x_\alpha \in X$ such that $V_\alpha \subset O(x_\alpha)$ and define $g_1\colon \mathcal{K}_1 \to Y$ by $g_1(\alpha) = c(x_\alpha)$. Following the Uspenskij arguments from the proof of [26, Theorem 2.1] and using that each $\phi_n(x)$ is C^{n-1}-embedded in $\phi_{n+1}(x)$, one can complete the construction.

As in the proof of [26, Theorem 2.1], we can see that property (6) holds. We set $\mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$ and $g\colon \mathcal{K} \to Y$ is the continuous map extending all maps g_n. Since X is a paracompact finite C-space, by Theorem 2.4, there exists a finite sequence $\{\lambda_k\}_{k=1}^{k=m}$ of open disjoint families such that λ_k refines ω_k and $\lambda = \bigcup_{k=1}^{k=m}$ is a locally finite cover of X. We can suppose that each $\lambda_k = \{W_\alpha : \alpha \in \Lambda_k\}$ is an index refinement of ω_k, so $\lambda = \{W_\alpha : \alpha \in \Gamma\}$, where $\Gamma = \bigcup_{k=1}^{k=m} \Lambda_k$. Let $\{h_\alpha : \alpha \in \Gamma\}$ be a locally finite partition of unity subordinated to λ. For any $x \in X$ let $s(x) = \{\alpha \in \Gamma : h_\alpha(x) > 0\}$. The sets $s(x)$ are finite and since each λ_k is disjoint, we have $|s(x) \cap \Lambda_k| \leqslant 1$ for $k \leqslant m$. Hence $s(x) = \{\alpha_1, \dots, \alpha_j\}$ such that $\alpha_k \in \Lambda_{i(k)}$ for each $k = 1, \dots, j$ and $i(1) < \cdots < i(j) \leqslant m$. Observe that $\bigcap_{k=1}^{k=j} U_{\alpha_k} \neq \emptyset$ (it contains x). Therefore, by (6), $s(x)$ is a simplex of P_{α_j}. Then the formula $h(x) = \sum\{h_\alpha(x)\alpha : \alpha \in \Gamma\}$ defines a continuous map $h\colon X \to \mathcal{K}_m$ such that $h(x) \in P_\delta$ for some $\delta \in s(x)$. But $\delta \in s(x)$ implies $W_\delta \subset U_\delta \subset V_\delta$. Consequently, by (5), $g_m(h(x)) \in g_m(P_\delta) \subset \phi_m(x)$. We finally obtain that the composition $g_m \circ h$ is a selection for ϕ_m.

(b) $\Rightarrow$ (c) This implication is trivial because every set-valued map with an open graph is strongly lsc.

(c) $\Rightarrow$ (a) We shall prove that Theorem 1.1(c) implies Theorem 1.2(b), and then the proof of the present implication will follow from Theorem 1.2.

We agree the following notations: for any metric space Y, a point $y \in Y$ and a positive number δ the open (resp., closed) ball in Y with center y and radius δ is denoted by $B(y, \delta)$ (resp., $\overline{B}(y, \delta)$). If Y is a normed space, then d stands for the metric generated by the norm of Y.

Now we need some notations from [13]. Let Y be a metric space. For every pair of set-valued maps $\phi\colon X \to 2^Y$ and $\psi\colon X \to 2^Y$ a set-valued map $\Delta_{(\phi,\psi)}\colon X \times Y \to 2^{\mathbb{R}} \cup \{\emptyset\}$ is associated such that, for $(x, y) \in X \times Y$, the value $\Delta_{(\phi,\psi)}(x, y)$ consists of all $\delta > 0$ for which there exists a neighborhood U_δ of x with $B(y, \delta) \cap \phi(z) \neq \emptyset$ and $B(y, \delta) \cap \psi(z) = \emptyset$ whenever $z \in U_\delta$. We define a set-valued map

$$\Phi_{(\phi,\psi)}\colon X \to 2^Y \cup \{\emptyset\} \text{by} \Phi_{(\phi,\psi)}(x) = \{y \in Y : \Delta_{(\phi,\psi)}(x, y) \neq \emptyset\}.$$

Consider the functions $u_{(\phi,\psi)}, l_{(\phi,\psi)}\colon G(\Phi_{(\phi,\psi)}) \to \mathbb{R}$, such that $u_{(\phi,\psi)}(x, y) = \inf \Delta_{(\phi,\psi)}(x, y)$ and $l_{(\phi,\psi)}(x, y) = \sup \Delta_{(\phi,\psi)}(x, y)$, where $G(\Phi_{(\phi,\psi)}) = \{(x, y) \in X \times Y : y \in \Phi_{(\phi,\psi)}(x)\}$ is the graph of $\Phi_{(\phi,\psi)}$.

LEMMA 3.1 ([13]). *Let Y be a metric space, and let $\phi\colon X \to 2^Y$ and $\psi\colon X \to 2^Y$ be set-valued maps such that ϕ is lsc, ψ has a closed graph and $\phi(x)\backslash\psi(x) \neq \emptyset$ for every $x \in X$. Then,*

(a) $\phi(x)\backslash\psi(x) \subset \Phi_{(\phi,\psi)}(x)$ *for every $x \in X$;*
(b) *The graph of $\Phi_{(\phi,\psi)}$ is open in $X \times Y$;*
(c) $u_{(\phi,\psi)}$ *and $l_{(\phi,\psi)}$ are, respectively, usc and lsc functions.*

PROPOSITION 3.2. *Theorem* 1.1(c) *yields Theorem* 1.2(b).

Proof. Suppose that Y is a Banach space, $\phi\colon X \to \mathcal{F}_c(Y)$ is lsc and $\{\psi_n\}$ a decreasing sequence of maps $\psi_n\colon X \to \mathcal{F}(Y)$ such that each ψ_n has a closed graph and $\{\psi_n\}$ is a Z-sequence for ϕ. Consider the maps $\Phi_n = \Phi_{(\phi,\psi_n)}$. Observe that each $\phi(x)\backslash\psi_n(x) \neq \emptyset$ because $\psi_n(x) \cap \phi(x)$ is a Z_{n-1}-set in $\phi(x)$. Then, by Lemma 3.1, each Φ_n has an open graph and $\phi(x)\backslash\psi_n(x) \subset \Phi_n(x)$ for all $x \in X$ and n. Since $\psi_{n+1}(x) \subset \psi_n(x)$, $\Delta_{(\phi,\psi_n)}(x, y)$ is a subset of $\Delta_{(\phi,\psi_{n+1})}(x, y)$ for every n and $(x, y) \in X \times Y$. Consquently, $\{\Phi_n\}$ is an increasing sequence.

CLAIM. *Each Φ_n is $(n-2)$-aspherical.*

This claim can be proved by making use of the arguments from the proof of [13, Claim 1, Proposition 3.1].

We now proceed to the rest of the proof. We already proved that $\{\Phi_n\}$ satisfies the hypotheses of Theorem 1.1(c). Hence, there exists $m \in \mathbb{N}$ such that Φ_m admits a continuous selection $g\colon X \to Y$. Relying once again on Lemma 3.1(c) and the results of [8, 10, 16], there exists a continuous selection $\delta\colon X \to \mathbb{R}$ for the map $\Delta_{(\phi,\psi_m)}(x, g(x))$, $x \in X$. Then $d(g(x), \phi(x)) < \delta(x) < d(g(x), \psi_m(x))$ for every $x \in X$. Therefore, $F(x) = \overline{B(g(x), \delta(x)) \cap \phi(x)}$ defines an lsc map $F\colon X \to \mathcal{F}_c(Y)$. Hence, applying Michael's convex-valued selection theorem [18, Theorem 3.2], we get a continuous selection f for F. Since $F(x) \subset \phi(x)\backslash\psi_m(x)$, f is as required.

4. Proof of Theorem 1.2

First, note that implication (a) $\Rightarrow$ (b) follows from Theorem 1.1, implication (a) $\Rightarrow$ (c) and Proposition 3.2. Implication (b) $\Rightarrow$ (c) is trivial, so it remains only to prove the implication (c) $\Rightarrow$ (a).

(c) $\Rightarrow$ (a) Take a sequence $\{\omega_n : n = 1, 2, \dots\}$ of finite open covers of X. By definition, we must prove that there exists a finite sequence $\{\gamma_n : n = 1, 2, \dots, m\}$ of disjoint open families in X such that each γ_n refines ω_n and the union $\bigcup\{\gamma_n : n = 1, 2, \dots, m\}$ is a cover of X. To this end, we proceed just like in [26, Theorem 1.3, implication $S_2 \Rightarrow C$], (see also [13, Theorem 1.1, implication (c) $\Rightarrow$ (a)]) with a few modifications. Note that, for every n, there exists an ω_n-map $f_n\colon X \to Q_n$ of X to a compact polyhedron Q_n. Considering Q_n as a subspace of a Euclidean space $\mathbb{R}^{k(n)}$ and multiplying, if necessary, f_n by a constant, we may assume that Q_n is the cube $[0, k(n)]^{k(n)}$ and that

(7) $\{f_n^{-1}(U) : U \text{is an open ball of radius 2 in} (Q_n, d_n)\}$ is an open cover of X refining ω_n.

Here we consider the metric $d_n((x_j), (y_j)) = \max |x_j - y_j|$ on $\mathbb{R}^{k(n)}$. Let A_n be the union of all sets $B(j, s) = \{(x_m) \in Q_n : x_j = s\}$, where j, s are natural numbers with $1 \leqslant j \leqslant k(n)$ and $1 \leqslant s \leqslant k(n) - 1$. Obviously, $Q = \prod_{n=1}^{\infty} Q_n$ is homeomorphic to the Hilbert cube. Next, let Z be the linear subset of $E = \prod_{n=1}^{\infty} \mathbb{R}^{k(n)}$ defined as

$$Z = \left\{ (y_{k(n)}) \in E : \sum_{n=1}^{\infty} \frac{\|y_{k(n)}\|}{2^{k(n)}} < +\infty \right\}.$$

The space Z is equipped with the norm

$$\|(y_{k(n)})\|_{\omega} = \sum_{n=1}^{\infty} \frac{\|y_{k(n)}\|}{2^{k(n)}},$$

where $\|y_{k(n)}\| = \max |y_{k(n),j}|$, $y_{k(n),j}$ being the jth coordinate of $y_{n(k)}$, is the norm on $\mathbb{R}^{k(n)}$. Observe that the topology of $(Z, \|\cdot\|_{\omega})$ coincides with the topology of Z as a subspace of E, and let Y be the completion of Z.

It suffices to find a finite sequence $g_n\colon X \to Q_n$, $n = 1, \dots, m$, of continuous maps such that

(8) $d_n(f_n(x), g_n(x)) \leqslant 1$ for all $x \in X$ and $n \leqslant m$;

(9) For every $x \in X$ there is $n \leqslant m$ with $g_n(x) \notin A_n$.

Indeed, let λ_n be the family of all components of $Q_n \backslash A_n$ which is disjoint and consists of open sets in Q_n with diameter $\leqslant 1$. Then (8) yields that each of the disjoint families $\gamma_n = g_n^{-1}(\lambda_n)$ refines ω_n (see [26, proof of Theorem 1.3]), and (9) implies that $\bigcup_{n=1}^{n=m} \gamma_n$ covers X. Let us observe also that condition (9) is equivalent to the existence of a continuous map $G\colon X \to Q(m) = \prod_{n=1}^{m} Q_n$ avoiding the set $A(m) = \prod_{n=1}^{m} A_n$.

We shall prove that, under the hypotheses of (c), such g_n's exist. For every n consider the map $\Phi_n\colon X \to \mathcal{F}_c(\mathbb{R}^{k(n)})$ defined by

$$\Phi_n(x) = \{y \in Q_n : d_n(f_n(x), y) \leqslant 1\}, \quad x \in X$$

and define $\Phi\colon X \to \mathcal{F}_c(Y)$ by

$$\Phi(x) = \prod\{\Phi_n(x) : n = 1, 2, \ldots\}, \quad x \in X.$$

It is easily seen that Φ is lsc and $\Phi(x) \subset Q \subset Z$ for every $x \in X$.

We proceed to the final step in this proof. Thus, we have a Banach space Y and an lsc map $\Phi\colon X \to \mathcal{F}_c(Y)$. Further, if $A = \prod_{n=1}^{\infty} A_n \subset Q$, then for every $x \in X$, $\Phi(x) \cap A$ is the product $\prod_{n=1}^{\infty} A_n \cap \Phi_n(x)$. Since each $A_n \cap \Phi_n(x)$ is closed and nowhere dense in $\Phi_n(x)$, by [5, Corollary 2], $\prod_{n=1}^{n=k} A_n \cap \Phi_n(x)$ is a Z_{k-1}-set in $\prod_{n=1}^{n=k} \Phi_n(x)$ for every k. This yields that $\Phi(x) \cap F_k$ is a Z_{k-1}-set in $\Phi(x)$, $x \in X$, where $F_k = A(k) \times \prod_{n=k+1}^{\infty} Q_n$. Therefore the map Φ and the decreasing sequence $\{F_k\}$ satisfy the hypotheses of (c). Hence, there is m and a continuous selection G_1 for Φ avoiding F_m. Let π_m be the projection from Q onto $Q(m)$ and $G = \pi_m \circ G_1$. Then $G = (g_1, \ldots, g_m)$, where each g_n is a continuous map from X into Q_n. It is easily seen that the maps g_n satisfy (8) and (9). Hence, the proof is complete.

5. Some Corollaries and Applications

Let $(Y_n, \|\cdot\|_n)$ be a sequence of normed spaces. We denote by Y the product $\prod_{n=1}^{\infty} Y_n$ and for every m let $\pi_m\colon Y \to Y(m)$ be the natural projection, where $Y(m) = \prod_{n=1}^{m} Y_n$. We say that a sequence of maps $f_n\colon X \to Y_n$ is finitely removable from a set $A \subset Y$ if for every sequence $\{\epsilon_n\}$ of positive real numbers there exists a finite sequence of maps $g_n\colon X \to Y_n$, $n = 1, \ldots, m$, such that $g(X) \cap \pi_m(A) = \emptyset$ and $\|(f_n(x) - g_n(x)\|_n \leqslant \epsilon_n$ for every n and $x \in X$, where $g = (g_n)\colon X \to Y(m)$. A sequence of maps $f_n\colon X \to Y_n$ is called uniformly bounded if there exists a common bound for all diameters $\operatorname{diam}(f_n(X))$, $n \in \mathbb{N}$.

Our first application of Theorem 1.2 is the following characterization of finite C-spaces:

PROPOSITION 5.1. *A space X is a finite C-space if and only if it satisfies the following condition: for any sequence of Banach spaces Y_n and closed nowhere dense subsets $A_n \subset Y_n$, every uniformly bounded sequence of maps $f_n\colon X \to Y_n$ is finitely removable from the set $A = \prod_{n=1}^{\infty} A_n$.*

Proof. Suppose X satisfies the condition from Proposition 5.1. To show that X is a finite C-space, take a sequence $\{\omega_n\}$ of finite functionally open covers of X. Proceeding just like in the proof of Theorem 1.2, implication (c) $\Rightarrow$ (a), we obtain ω_n-maps $f_n\colon X \to Q_n \subset \mathbb{R}^{k(n)}$, where now $Q_n = I^{k(n}$, $I = [1, 2]$. Let $K_n = [0, 3]^{n(k)}$. Since $f_n\colon X \to K_n$ is an ω_n-map, there exists positive $\epsilon_n \leqslant 1$ such that if U is an open ball in K_n with radius $< 2\epsilon_n$, then $f_n^{-1}(U)$ is contained in an element of ω_n. Next, take hyperplanes $B(j,s) = \{(x_i) \in \mathbb{R}^{k(n)} : x_j = s\}$, $1 \leqslant j \leqslant k(n)$ and $0 < s < 3$, such that their union A_n partitions K_n into open (in K_n) cubes, each having a diameter $< \epsilon_n$. Since $\{f_n\}$ is uniformly bounded and every $A_n \subset \mathbb{R}^{k(n)}$ is closed nowhere dense, $\{f_n\}$ is finitely removable from the set $A = \prod_{n=1}^{\infty} A_n$. So, there exists a finite sequence of maps $g_n\colon X \to \mathbb{R}^{k(n)}$, $n = 1, \dots, m$ such that each g_n is ϵ_n-close to f_n and the map $g = (g_n)\colon X \to \prod_{n=1}^{m} \mathbb{R}^{k(n)}$ avoids the set $A(m) = \prod_{n=1}^{m} A_n$. Observe that all g_n map X into K_n (because g_n is ϵ_n-close to f_n) and the family λ_n of all components of $K_n \backslash A_n$ is disjoint and consists of open sets in K_n with diameter $< \epsilon_n$. Consequently, $\gamma_n = g_n^{-1}(\lambda_n)$ is a disjoint and functionally open in X family which refine ω_n. It remains only to note that, since the map g avoids the set $A(m)$, $\{\gamma_n : n = 1, \dots, m\}$ covers X. Hence, $\{\gamma_n\}$ is a finite C-refinement for $\{\omega_n\}$.

To prove the inverse implication, first let show that X can be supposed to be paracompact. Indeed, let the maps $f_n\colon X \to Y_n$ and the sets $A_n \subset Y_n$ be as in Proposition 5.1. Consider the Čech–Stone extension $\beta f\colon \beta X \to \beta Y$ of $f = (f_n)$ and let $H = (\beta f)^{-1}(Y)$. Obviously, H is paracompact (as a perfect preimage of Y). Denote by $h = (h_n)\colon H \to Y$ the restriction $(\beta f)|H$. Then $h_n\colon H \to Y_n$ is uniformply bounded and, by Proposition 2.2, H is a finite C-space. Hence, if $\{h_n\}$ is finitely removable from A, so is $\{f_n\}$. Therefore, we can assume that X is paracompact.

We consider the normed space Z defined by

$$Z = \left\{ (y_n) \in Y : \sum_{n=1}^{\infty} \frac{\|y_n\|_n}{2^n} < +\infty \right\},$$

where

$$\|(y_n)\| = \sum_{n=1}^{\infty} \frac{\|y_n\|_n}{2^n}$$

is the norm in Z. Let E be the completion of Z and, for a given bounded sequence $\{\epsilon_n\}$ of positive numbers, define the lsc map $\phi\colon X \to \mathcal{F}_c(Z)$, $\phi(x) = \{(y_n) \in Y : \|y_n - f_n(x)\|_n \leqslant \epsilon_n, n \in \mathbb{N}\}$. This definition is correct because $\{f_n\}$ is uniformly bounded. The norm topology of Z coincides with the topology inherited from Y and the values of ϕ are norm-complete in Z (see [13, proof of Theorem 1.1, implication (c) $\Rightarrow$ (a)] for a similar proof). So, ϕ is a map from X into $\mathcal{F}_c(E)$. Let $D_n = A^n \times \prod_{k=n+1}^{\infty} Y_k$ and F_n be the closure in E of $D_n \cap Z$, $n \in \mathbb{N}$. Then, using a result of [5], we can show that each $F_n \cap \phi(x)$ is a Z_{n-1}-set in $\phi(x)$. Since $\{F_n\}$ is a decreasing sequence, by Theorem 1.2(c), there exists $m \in \mathbb{N}$ and a continuous map $G\colon X \to E$ such that $G(x) \in \phi(x) \backslash F_m$ for every $x \in X$. Finally, the map $g = \pi_m \circ G$ is generated by a sequence $g_n\colon X \to Y_n$, $n = 1, \dots, m$ which witnesses that $\{f_n\}$ is finitely removable from the set A.

Next corollary, which follows from the proof of Proposition 5.1, is useful for applications.

COROLLARY 5.2. *X is a finite C-space iff for any sequence $\{Q_n\}$ of finite-dimensional cubes and closed nowhere dense sets $A_n \subset Q_n$, every sequence of maps $f_n\colon X \to Q_n$ is finitely removable from the set $A = \prod_{n=1}^{\infty} A_n$.*

Following F. Ancel [1], we say that a map $f\colon X \to Y$ is approximately invertible if there exists a C^*-embedding $i\colon X \to Z$ into a space Z such that for every collection $\mathcal{W}$ of open subsets of Z which is refined by $\{i(f^{-1}(y)) : y \in Y\}$, there is a map $g\colon Y \to Z$ with $g \circ f$ is $\mathcal{W}$-close to i (i.e. the family $\{g(f(x)), i(x) : x \in X\}$ refines $\mathcal{W}$). F. Ancel [1] proved that every approximately invertible surjection between metric spaces having compact fibers preserves property C. The analogue of Ancel's result for finite C-spaces also holds.

COROLLARY 5.3. *Any approximately invertible surjection preserves finite C-space property.*

Proof. Suppose $p\colon X \to Y$ is approximately invertible and surjective, where X is a finite C-space. Let $\{Q_n\}$ be a sequence of finite-dimensional cubes, $A_n \subset Q_n$ closed nowhere dense subsets and $f_n\colon Y \to Q_n$ arbitrary sequence of maps. By Corollary 5.2, if suffices to show that $\{f_n\}$ is finitely removable from the set $A = \prod_{n=1}^{\infty} A_n$. To this end, let d_n be a metric on Q_n and $\{\epsilon_n\}$ a sequence of positive numbers. Since X is a finite C-space, according to Corollary 5.2, $\{f_n \circ p\}$ is finitely removable from A. So, there exists a number $m \in \mathbb{N}$ and a map $g = (g_n)\colon X \to Q(m) = \prod_{n=1}^{m} Q_n$ such that $g(X) \cap A(m) = \emptyset$ and $d_n(f_n(p(x)), g_n(x)) \leqslant 4^{-1}\epsilon_n$ for every $x \in X$ and $n \leqslant m$, where $A(m) = \prod_{n=1}^{m} A_n$. Now, let $i\colon X \to Z$ be a C^*-embedding into a space Z which witnesses the approximate invertibility of the map p. We identify i with the identity id_X and take an extension $\overline{g}\colon Z \to Q(m)$ of g. Let $U = \{z \in Z : \overline{g}(z) \notin A(m)\}$. For every $y \in Y$ let $W(y) = \{z \in U : d_n(\overline{g}_n(z), f_n(y)) < 3^{-1}\epsilon_n, n \leqslant m\}$ and $\mathcal{W} = \{W(y) : y \in Y\}$. Obviously, each W_y is open in Z and contains $p^{-1}(y)$. So, there is a map $q\colon Y \to Z$ such that $q \circ p$ and id_X are $\mathcal{W}$-close. This implies that $q(Y) \subset U$. Define now $h_n\colon Y \to Q_n$ by $h_n = \overline{g}_n \circ q$, $n \leqslant m$. Then $h = (h_n)\colon Y \to Q(m)$ avoids the set $A(m)$.

It remains to show that each h_n is ϵ_n- close to f_n, $n \leqslant m$. For any $y \in Y$ choose $x \in p^{-1}(y)$. Then x and $q(p(x)) = q(y)$ belong to $W(y')$ for some $y' \in Y$ because $q \circ p$ and id_X are $\mathcal{W}$-close. Hence, $d_n(\overline{g}_n(x), \overline{g}_n(q(y))) < 2.3^{-1}\epsilon_n$ for every $n \leqslant m$. On the other hand, since $\overline{g}_n(x) = g(x)$, we have $d_n(f_n(y), \overline{g}_n(x)) \leqslant 4^{-1}\epsilon_n$. Finally, $d_n(f_n(y), \overline{g}_n(q(y))) \leqslant \epsilon_n$. □

Hattori and Yamada [14] proved that if $f\colon X \to Y$ is a closed map from a paracompact space X onto a C-space Y and all fibers $f^{-1}(y)$ have property C, then X is a C-space. Similar result for finite C-spaces is not valid. Indeed, let X be a disjoint union of n-dimensional cubes Q_n and for every $n \in \mathbb{N}$ denote by f_n the projection of Q_n onto $I_n = [0, 1]$. Let Y be the disjoint union of all I_n and define

$f: X \to Y$ as $f|Q_n = f_n$. Then, f is a perfect map with finite-dimensional fibers and Y is a finite C-space ($\dim Y = 1$). Since every paracompact finite C-space satisfies condition (K) (see Corollary 2.6), X is not a finite C-space. But still there is an analogue of the Hattori and Yamada result.

PROPOSITION 5.4. *Let $f: X \to Y$ be a closed surjection such that X is normal, Y paracompact and* $\dim f^{-1}(y) \leqslant k$ *for every $y \in Y$. If Y is a finite C-space, so is X.*

Proof. Let $T = (\beta f)^{-1}(Y)$. Since f is closed and X normal, the closure in βX of each fiber $f^{-1}(y)$, $y \in Y$, is $(\beta f)^{-1}(y)$. Hence, $\dim(\beta f)^{-1}(y) \leqslant k$ for any $y \in Y$. Because X and T have the same Čech–Stone compactification, they are simultaneously finite C-spaces. Therefore, we can suppose that f is perfect and X is paracompact (as a perfect preimage of Y).

Since Y is a paracompact finite C-space, it satisfies condition (K) such that $K \subset Y$ has property C. Then, by the mentioned result of Hattori and Yamada, $H = f^{-1}(K)$ is a compact C-space. Hence, according to Corollary 2.6, it suffices to show that every closed $F \subset X$, which is disjoint from H, is finite-dimensional. To this end, observe that $f(F) \subset Y$ is closed and disjoint from K, so $f(F)$ is finite-dimensional. Then, by the Hurewicz formula (applied to the map $f|F: F \to f(F)$, see [25]), $\dim F \leqslant \dim f(F) + k$. □

6. Factorization Theorems

Let $f: X \to Y$ be a continuous surjection. We say that (Z, h, g) is a factorization for f if h is a continuous map from X to the space Z and g is a continuous map from Z onto Y such that $f = g \circ h$ and $w(Z) \leqslant w(Y)$. Here $w(Y)$ is the topological weight of Y. V. Chatyrko [7] proved that if $f: X \to Y$ is a map between compact spaces and X has property C, then there is a factorization (Z, h, g) such that Z is a compact C-space and $\dim_C Z \leqslant \dim_C X$, where $\dim_C$ is the Borst transfinite dimension. Another factorization theorem for C-spaces was proved by Levin, Rubin and Schapiro [17] which implies that if $f: X \to Y$ is a map between compact spaces and $\{D_n\}$ is a sequence of compact subsets of X with property C, then f admtis a factorization (Z, h, g) such that each $h(D_n)$ has property C. In this section we are presenting some factorizations for C-spaces in the special case when Y is a metric space.

PROPOSITION 6.1. *Let $\{K_n\}$ be a sequence of compact C-subsets of a space X. Then every map $f: X \to Y$ into a metrizable space Y admits a factorization (Z, h, g) such that Z is metrizable and all sets $h(K_n)$ have property C.*

Proof. Recall that a map $p: Y \to S$ between metrizable spaces is strongly 0-dimensional if there is a metric on Y generating its topology such that for every $\epsilon > 0$ and every $z \in p(Y)$ there exists an open neighborhood U of z with $p^{-1}(U)$ being the union of disjoint open sets of diameter $< \epsilon$. It is well known that every metrizable space admits a strongly 0-dimensional map into Hilbert cube Q. So, we can take a strongly 0-dimensional map $p: Y \to Q$ and let $\overline{f}: \beta X \to Q$ be the Čech–Stone extension of $p \circ f$. By the Levin, Rubin and Schapiro result mentioned above, there is a factorization (Z_0, h_0, g_0) for $\overline{f}$ such that all sets $h_0(K_n)$ have property C. Let $g: Z \to Y$ and $q: Z \to h_0(X)$ be the pullback of p and $g_0|h_0(X)$ respectively, and $h: X \to Z$, $h(x) = (h_0(x), f(x))$. Then Z is a metrizable space of the same weight as Y and Z admits a compatible metric such that q is strongly 0-dimensional. Since all fibers of q are 0-dimensional, each restriction $q_n = q|h(K_n)$ is a map from $h(K_n)$ onto $h_0(K_n)$ with 0-dimensional fibers. Hence, by [14], $h(K_n)$ has property C. □

PROPOSITION 6.2. *Every map $f: X \to Y$ from a finite C-space X onto metrizable Y admits a factorization (Z, h, g) such that Z is a metrizable finite C-space.*

Proof. Let $P = (\beta f)^{-1}(Y)$ and $\overline{f}$ be the restriction $(\beta f)|P$. Then P is a paracompact finite C-space, so it satisfies condition (K) such that $K \subset P$ is a compact C-space. Let $K_Y = \overline{f}(K)$ and $K_X = (\overline{f})^{-1}(K_Y)$. Obviously, $K_X \subset P$ is compact and contains K. Next, if d is a metric on Y, let $H_n = \{y \in Y : d(y, K_Y) \geqslant n^{-1}\}$ and $F_n = (\overline{f})^{-1}(H_n)$. Since each F_n is closed in P and disjoint from K_X, it is finite-dimensional. Because $\overline{f}$ is a closed map and any closed set in Y disjoint from K_Y is contained in some H_n, we have that

(10) every closed set in P disjoint from K_X is contained in some F_n.

Observe that K_X has property C as a compact set in P. Hence, by Proposition 6.1, $\overline{f}$ admits a factorization (M, h_1, g_1) such that M is metrizable and the set $K_M = h_1(K_X)$ has property C. Now, we can apply Pasynkov's factorization theorem [20] to obtain a factorization (G, h_2, g_2) for the map h_1 such that G is metrizable and $\dim \overline{h_2(F_n)} \leqslant \dim F_n$ for every n. But each $h_2(F_n)$ is closed in G, so G is the union of the closed sets $K_G = h_2(K_X)$ and $G_n = h_2(F_n)$ with each G_n being finite-dimensional and disjoint from K_G. It follows from (10) that every closed subset of G disjoint from K_G is contained in some G_n. Finally, attaching to G the space K_M by the map $g_2|K_G$, we obtain the metrizable space Z. Obviously, K_M is homeomorhic to a subset $K_Z \subset Z$ and $Z \backslash K_Z$ is the set $G \backslash K_G$. Therefore, Z, being the union of all sets G_n, $n \in \mathbb{N}$, and K_Z, satisfies condition (K) with the compact C-set K_Z. Hence, Z is a finite C-space. It remains only to define the maps $h: X \to Z$ and $g: Z \to Y$ such that $f = g \circ h$. Let $q: G \to Z$ be the natural quotient map. Identifying K_Z with K_M we set $h(x) = q(h_2(x))$, $x \in X$, and $g(z) = g_1(g_2(z))$ provided $z \in Z \backslash K_Z$ and $g(z) = g_1(z)$ if $z \in K_Z$. □

COROLLARY 6.3. *Every metrizable finite C-space has a completion which is also a finite C-space.*

Proof. Suppose X is a metrizable finite C-space and let Y be any completion of X. Extend the inclusion $X \subset Y$ to a map $p: \beta X \to \beta Y$ and denote $G = p^{-1}(Y)$ and $f = p|G: G \to Y$. Then G is a finite C-space, so, by Proposition 6.2, f admits a factorization (Z, h, g) with Z a metrizable finite C-space. Obviously, h embeds

X into Z. Finally, since g is a perfect map from Z onto Y and Y is Čech-complete, so is Z. □

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References

1. Ancel, F.: Proper hereditary shape equivalences preserve property C, *Topology Appl.* **19** (1985), 71–74.
2. Addis, D. and Gresham, J.: A class of infinite-dimensional spaces. Part I: Dimension theory and Alexandroff's problem, *Fund. Math.* **101** (1978), 195–205.
3. Alexandroff, P.: Zum allgemeinen Dimensionproblem, *Nachr. Ges. Göttingen* (1928), 25–44.
4. Alexandroff, P. and Pasynkov, B.: *Introduction to Dimension Theory,* Nauka, Moscow, 1973 (in Russian).
5. Banakh, T. and Trushchak, K.: Z_n-sets and the disjoint n-cells property in products of ANR's, Preprint.
6. Borst, P.: Some remarks concerning C-spaces, Preprint.
7. Chatyrko, V.: On factorization theorem for transfinite dimension $\dim_C$, Preprint.
8. Dieudonne, J.: Une généralisation des espaces compacts, *J. Math. Pures Appl.* **23** (1944), 65–76.
9. Dranishnikov, A., Repovš, D. and Ščepin, E.: Transversal intersection formula for compacta, *Topology Appl.* **85** (1998), 93–117.
10. Dowker, C. H.: On countably paracompact spaces, *Canad. J. Math.* **3** (1951), 219–224.
11. Engelking, R.: *Theory of Dimensions: Finite and Infinite*, Heldermann-Verlag, Lemgo, 1995.
12. Engelking, R.: *General Topology*, 2nd edn, Heldermann, Berlin, 1989.
13. Gutev, V. and Valov, V.: Continuous selections and C-spaces, *Proc. Amer. Math. Soc.*, to appear.
14. Hattori, Y. and Yamada, K.: Closed pre-images of C-spaces, *Math. Japan.* **34**(4) (1989), 555–561.
15. Haver, W.: *A Covering Property for Metric Spaces*, Lecture Notes in Math. 375, Springer-Verlag, New York, 1974.
16. Katětov, M.: On real-valued functions in topological spaces, *Fund. Math.* **38** (1951), 85–91.
17. Levin, M., Rubin, L. and Schapiro, P.: The Mardešić factorization theorem for extension theory and C-separation, Preprint.
18. Michael, E.: Continuous selecions I, *Ann. of Math.* **63** (1956), 361–382.
19. Michael, E.: Continuous selections avoiding a set, *Topology Appl.* **28** (1988), 195–213.
20. Pasynkov, B.: A factorization of maps onto metric spaces, *Dokl. Akad. Nauk SSSR* **182** (1968), 268–271 (in Russian).
21. Pasynkov, B.: A private letter, August 1999.
22. Pol, E.: The Baire-category method in some compact extension problems, *Pacific J. Math.* **122** (1986), 197–210.
23. Pol, R.: A weakly infinite-dimensional compactum which is not countable-dimensional, *Proc. Amer. Math. Soc.* **82** (1981), 634–636.

24. Ščepin, E. and Brodskii, N.: Selections of filtered multivalued mappings, *Trudy Mat. Inst. Steklova* **212** (1996), 220–249 (in Russian).
25. Skljarenko, E.: A theorem of dimension-lowering mappings, *Bull. Acad. Pol. Sci. Ser. Math.* **10** (1962), 429–432.
26. Uspenskij, V.: A selection theorem for C-spaces, *Topology Appl.* **85** (1998), 351–374.