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H. ARIZMENDI-PEIMBERT, A. CARRILLO-HOYO AND V. VALOV

On Q, $Q_{\mathcal{M}}$ and $Q_{\mathcal{M}\#}$-algebras

Abstract. The concepts of Q_M and $Q_{M\#}-$ algebras were defined in [4] as generalizations of $Q-$ algebras. In this paper we prove that, if X is a completely regular Hausdorff space, then the uniform topology σ is the only topology τ on $C_b(X)$ which is coarser than σ and possesses the following property: $A = (C_b(X), \tau)$ is a topological algebra and the above three concepts are equivalent for A. We also construct a B_0-algebra which is a Q_M-algebra, but it is not a Q-algebra.

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1. Introduction. Throughout this paper F is the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$ and A is a topological algebra over F with unit e and associative jointly continuous multiplication. A *locally convex* algebra is a topological algebra A which is a locally convex space, in this case its topology can be given by means of a family $\{\|.\|_\alpha : \alpha \in \Lambda\}$ of seminorms such that for each $\alpha \in \Lambda$ there exists $\beta \in \Lambda$ satisfying

$$\|x \cdot y\|_\alpha \leq \|x\|_\beta \cdot \|y\|_\beta \tag{1}$$

for all $x, y \in A$.

A complete metrizable locally convex algebra is called a B_0-algebra. For a B_0-algebra A there exists a sequence of seminorms $(\|\cdot\|_n)_{n=1}^\infty$ defining its topology and satisfying

$$\|xy\|_n \leq \|x\|_{n+1} \|y\|_{n+1} \tag{2}$$

for $n = 1, 2, \ldots$ and all $x, y \in A$.

A locally convex algebra, in particular a B_0-algebra, is said to be *locally multiplicatively convex* (shortly *m-convex*) algebra if (1) can be replaced by

$$\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha$$

for all $\alpha \in \Lambda$ and $x, y \in A$.

The set of all invertible elements in A is denoted by $G(A)$. If $G(A)$ is open, then A is called a Q-algebra. For $x \in A$ the set $\sigma_A(x) = \{\lambda \in F : \lambda e - x \notin G(A)\}$ is the *spectrum of* x in A. The *spectral radius* is defined as: $\rho_A(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$ if $\sigma_A(x) \neq \emptyset$ and $\rho_A(x) = 0$, otherwise. When no confusion can be expected we simply write $\sigma(x)$ and $\rho(x)$, respectively.

By $\mathcal{M}(A)$, (resp., $\mathcal{M}^\#(A)$) we denote the space of all non-zero continuous and multiplicative (resp., non-zero multiplicative) linear functionals on A with the weak* topology.

For any $x \in A$, $\widehat{x}$ stands for its Gelfand transform: $\widehat{x}(f) = f(x)$, $f \in \mathcal{M}^\#(A)$. Recall that $\widehat{x}(\mathcal{M}^\#(A) \subset \sigma(x)$ for all $x \in A$.

If A is a Q-algebra then $\mathcal{M}(A) = \mathcal{M}^\#(A)$ and $\sigma(x)$ is a compact set for each $x \in A$.

2. $\mathbf{Q}_{\mathcal{M}}$ - algebras and $\mathbf{Q}_{\mathcal{M}^\#}$- algebras.

For some algebras A, for example commutative complex Banach algebras, we have the following very important property:

$$\widehat{x}(\mathcal{M}(A)) = \sigma(x) \text{ for all } x \in A. \tag{m}$$

In [4] the next two conditions are setting

(m_1) $\widehat{x}\left(\mathcal{M}\left(A\right)\right)$ is dense in $\sigma\left(x\right)$ for all $x \in A$.

(m_2) $\rho\left(x\right) = \sup\{|\lambda| : \lambda \in \widehat{x}(\ \mathcal{M}(A))\}$ for all $x \in A$.

Obviously, $(m) \Rightarrow (m_1) \Rightarrow (m_2)$, if $\sigma(x) \neq \emptyset$.

Most topological algebras do not possess property (m); if A is such an algebra, then there exists $x \in A$ such that $\widehat{x}(\mathcal{M}(A))$ is a proper subset of $\sigma(x)$.

In [4] the "linear parts" $\widehat{x}\left(\mathcal{M}\left(A\right)\right)$ and $\widehat{x}\left(\mathcal{M}^{\#}\left(A\right)\right)$ of the spectrum $\sigma\left(x\right)$ are introduced. The set $\sigma_{\mathcal{M}}(x) = \widehat{x}(\mathcal{M}(A))$ $(\sigma_{\mathcal{M}^{\#}}(x) = \widehat{x}(\mathcal{M}^{\#}(A)))$ is called the *$\mathcal{M}$-spectrum* (*$\mathcal{M}^{\#}$-spectrum*) of x.

An element $x \in A$ is said to be $\mathcal{M}$-invertible (resp., $\mathcal{M}^{\#}$-invertible) if $0 \notin \sigma_{\mathcal{M}}(x)$ (resp., $0 \notin \sigma_{\mathcal{M}^{\#}}(x)$). The set of all $\mathcal{M}$ ($\mathcal{M}^{\#}$)-invertible elements of A is denoted by $G_{\mathcal{M}}$ ($G_{\mathcal{M}^{\#}}$). Finally, we say that A is a $Q_{\mathcal{M}}$ $(Q_{\mathcal{M}^{\#}})$-algebra if $G_{\mathcal{M}}$ $(G_{\mathcal{M}^{\#}})$ is open in A.

The next result appears as part of [4, Corollary 4.3].

LEMMA 1 *If A satisfies $(m2)$, then next three properties are equivalent:*

- (a) *A is a $Q-$algebra.*
- (b) *A is a $Q_{\mathcal{M}^{\#}}$-algebra.*
- (c) *A is a $Q_{\mathcal{M}}$-algebra.*

Let X be a completely regular space. We denote by $C_b\left(X\right)$ the algebra of all continuous bounded complex functions on X. Let τ be a topology on $C_b\left(X\right)$ such that $(C_b(X), \tau)$ is a topological algebra under the pointwise operations and τ is finer than the pointwise convergence topology.

THEOREM 1 *If τ is coarser than the uniform topology σ on $C_b\left(X\right)$, then the following conditions are equivalent:*

1. *$(C_b(X), \tau)$ is a Q -algebra.*

2. *$(C_b(X),\tau)$ is a $Q_{\mathcal{M}^\#}$ -algebra.*

3. *$(C_b(X),\tau)$ is a $Q_{\mathcal{M}}$-algebra.*

4. *$\tau=\sigma$.*

Proof: Let $A=(C_b(X),\tau)$. Since τ is between the pointwise topology and σ, we have $X\subset\mathcal{M}(A)\subset\beta X=\mathcal{M}^\#(A)$, where βX is the Stone-Čech compactification of X. Moreover, $\sigma(f)=\overline{f(X)}$ for every $f\in A$. Consequently, A satisfies condition (m_1), and therefore, (m_2). Hence, by Lemma 2.1, conditions 1- 3 are equivalent. Obviously, condition 4 implies condition 1.

Finally, suppose A satisfies condition 1. Then the map $\psi: A\to(C_b(\mathcal{M}(A)),\sigma)$, $\psi(f)=\widehat{f}$, is continuous (see Corollary 3.4 from [4]). But the Banach algebra $(C_b(\mathcal{M}(A)),\sigma)$ is isomorphic to $(C_b(X),\sigma)$ because $X\subset\mathcal{M}(A)\subset\beta X$. Thus, the identity transformation from A onto $(C_b(X),\sigma)$ is a continuous bijective map. Since $\tau\subset\sigma$ we conclude that $\tau=\sigma$. Thus, condition 1 implies condition 4. □

Let $X=\mathbb{N}$ be endowed with the discrete topology. Then $C_b(X)=\ell^\infty$ and the strict topology β ([5]) on $C_b(X)$ is given by the sequence space c_0 of all null sequences, i.e., β is determined by the seminorms $\|x\|_y=\sup_{n\geq1}|x_ny_n|$ for $x=(x_n)_{n=1}^\infty$ in ℓ^∞ and $y=(y_n)_{n=1}^\infty$ in c_0. Clearly β is strictly coarser than the uniform topology σ, so $(C_b(X),\beta)$ is not a $Q-$algebra. In fact, (ℓ^∞,c_0) has the following property which is strongly opposite to that one of being a Q-algebra: the set of all non-invertible elements of this locally convex algebra is dense in (ℓ^∞,c_0). Indeed, it suffices to show that any neighborhood of the identity 1 contains a non-invertible element. To this end, we fix $\epsilon>0$ and $\mathbf{a}=(\mathbf{a_n})_{\mathbf{n=1}}^\infty$ in c_0 and choose $\eta>0$ such that $\eta<\epsilon$ and there exists $n\in\mathbb{N}$ with $\eta<|a_n|$. Then the non-invertible element $x=(x_n)_{n=1}^\infty\in\ell^\infty$ defined by $x_n=1$ if $|a_n|\geq\eta$ and $x_n=0$ otherwise, belongs to the β-neighborhood of 1 determined by $\epsilon>0$ and $\mathbf{a}$. □

3. An example.

In this section we construct a B_0-algebra which is a $Q_{\mathcal{M}}-$algebra but it is not a Q-algebra.

Let (a_{pn}), $p\geq1, n\geq0$, be an infinite matrix of positive real numbers, satisfying the following two conditions:

i) $a_{p,n} \leq a_{p+1,n}$,

ii) $a_{p,n+m} \leq a_{p+1,n} a_{p+1,m}$

for $p \geq 1$ and $n \geq 0$.

The matrix algebra $A\left(a_{p,n}\right)$ associated with the matrix $\left(a_{p,n}\right)$ is the B_0-algebra of formal power series with coefficients in $\mathbb{C}$, defined by

$$A(a_{p,n}) = \{x = \sum_{n=0}^{\infty} x_n z^n : \sum_{n=0}^{\infty} |x_n|\, a_{p,n} < \infty \text{ for all } p \geq 1\},$$

with the topology determined by the seminorms $\|x\|_p = \sum\limits_{n=0}^{\infty} |x_n|\, a_{p,n}$, $p = 1, 2, \ldots$

By i) and ii) $A(a_{p,n})$ is a B_0 -algebra under the Cauchy multiplication and the usual linear operations.

The sequence $(z^n)_{n=0}^{\infty}$ is a basis for $A(a_{p,n})$ and it is called a cyclic basis. Since the algebra $A(a_{p,n})$ is singly generated by z, every continuous multiplicative linear functional on $A(a_{p,n})$ has the form $f_\lambda(x) = \sum\limits_{n=1}^{\infty} x_n \lambda^n$ for some fixed $\lambda \in$.

It is shown in [2] that

$$D_r\left(0\right) \subset \sigma_{\mathcal{M}}\left(z\right) \subset \overline{D_r\left(0\right)} \tag{3}$$

and

$$D_R\left(0\right) \subset \sigma\left(z\right) \subset \overline{D_R\left(0\right)}, \tag{4}$$

where $D_r\left(0\right)$ and $D_R\left(0\right)$ are the open disks in $\mathbb{C}$ around 0 with radii:

$$r = \sup_{p\geq 1}(\liminf_{n\geq 1} \sqrt[n]{\|z^n\|_p}) = \sup_{p\geq 1}(\liminf_{n\geq 1} \sqrt[n]{a_{pn}})$$

and

$$R = \sup_{p\geq 1}(\limsup_{n\geq 1} \sqrt[n]{\|z^n\|_p}) = \sup_{p\geq 1}(\limsup_{n\geq 1} \sqrt[n]{a_{pn}}).$$

Now, we need a special matrix $(b_{p,n})$ satisfying the following conditions (see [1] for the construction of such a matrix):

$$b_{p,n} \geq 1 \text{ for all } p = 1, 2, \ldots \text{and } n = 0, 1, \ldots \tag{5}$$

$$\liminf_{n\geq 1} \sqrt[n]{b_{p,n}} = 1, \; p = 1, 2, .. \tag{6}$$

$$\limsup_{n\geq 1} \sqrt[n]{b_{p,n}} = \infty, \; p = 1, 2, .. \tag{7}$$

Therefore, the matrix algebra $A(b_{p,n})$ associated to $(b_{p,n})$ is a B_0-algebra for which $r = 1$ and $R = \infty$.

It follows by (3) and (6) that every continuous multiplicative linear functional on $A(b_{p,n})$ has the form $f_\lambda(x) = \sum\limits_{n=1}^{\infty} x_n \lambda^n$, where λ is any complex number with $|\lambda| \leq 1$. Moreover, by (5) we have

$$\sum_{n=0}^{\infty} |x_n| \leq \sum_{n=0}^{\infty} |x_n| \, b_{p,n} < \infty,$$

for every $x = \sum\limits_{n=0}^{\infty} x_n z^n$ in $A(b_{p,n})$. The last inequality implies that any $x = \sum\limits_{n=0}^{\infty} x_n z^n \in A(b_{p,n})$ generates the continuous function $\phi_x \colon \overline{D_1(0)} \to \mathcal{C}$, $\phi_x(\lambda) = f_\lambda(x)$. Hence, since $\sigma_{\mathcal{M}}(x) = \{f_\lambda(x) : |\lambda| \leq 1\}$, the set $\sigma_{\mathcal{M}}(x)$ is compact for every $x \in A(b_{p,n})$.

We are in a position now to show that $A(b_{p,n})$ is a Q-algebra. Let $x = \sum\limits_{n=0}^{\infty} x_n z^n$ be an $\mathcal{M}$-invertible element of $A(b_{p,n})$. Then $0 \notin \sigma(x)$, so $\min\{|\phi_x(\lambda)| : |\lambda| \leq 1\} = c > 0$. The set $V = V(x, \|.\|_1, \frac{c}{2})$ is a neighborhood of x and for every $y \in V$ we have

$$||\phi_x(\lambda)| - |\phi_y(\lambda)|| \leq \sum_{n=0}^{\infty} |x_n - y_n| \leq \|x - y\|_1 < \frac{c}{2}.$$

So, $|\phi_y(\lambda)| > \frac{c}{2} > 0$ for any λ in $\overline{D_1(0)}$, i.e. $0 \notin \sigma(y)$. Therefore, any $y \in V$ is $\mathcal{M}$-invertible which implies that $A(a_{p,n})$ is a $Q_{\mathcal{M}}$-algebra. On the other hand, $A(b_{p,n})$ is not a Q -algebra because, by (4) and (7), $\sigma(z)$ is not bounded.

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Hugo Arizmendi-Peimbert and Angel Carrillo-Hoyo
Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Ciudad Universitaria, 04510 México, D.F.
E-mail address: `hugo@matem.unam.mx`

Vesko Valov
Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, P.O. Box 5002, North Bay, ON, P1B 8L7, Canada
E-mail address: `veskov@nipissingu.ca`

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