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Dense Families of Selections and Finite-Dimensional Spaces

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Abstract. A characterization of n-dimensional spaces via continuous selections avoiding Z_n-sets is given, and a selection theorem for strongly countable-dimensional spaces is established. We apply these results to prove a generalized Ostrand's theorem, and to obtain a new alternative proof of the Hurewicz formula. It is also shown that our selection theorem yields an easy proof of a Michael's result.

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1. Introduction

One of the best characterizations of the covering dimension is given by extensions of maps in spheres of Euclidean spaces. Namely, a normal space X has a covering dimension $\dim(X) \leqslant n$ iff for every closed $A \subset X$, every continuous map $g\colon A \to \mathbb{S}^n$ can be extended to a continuous map $f\colon X \to \mathbb{S}^n$. Here, $\mathbb{S}^n$ denotes the *n-sphere*.

The parametric version of this fact is provided by the following Michael's selection theorem in [18] (see, also, [27]).

THEOREM 1.1 ([18]). *Let X be a paracompact space, with* $\dim(X) \leqslant n$*, Y be a completely metrizable space, and let $\varphi\colon X \to \mathcal{F}(Y)$ be an l.s.c. mapping such that $\{\varphi(x)\colon x \in X\}$ is equi-LC^{n-1} in Y and each $\varphi(x)$, $x \in X$, is C^{n-1}. Then, φ has a continuous selection.*

Here, $\mathcal{F}(Y) = \{S \in 2^Y : S \text{ is closed}\}$, where 2^Y is the family of all nonempty subsets of Y. A set-valued mapping $\varphi\colon X \to 2^Y$ is *lower semi-continuous*, or l.s.c., if $\varphi^{-1}(U) = \{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is open in X for every open $U \subset Y$, and a map $f\colon X \to Y$ is a *selection* for $\varphi\colon X \to 2^Y$ if $f(x) \in \varphi(x)$ for every $x \in X$.

Let $m \geqslant -1$. A family $\mathcal{S}$ of subsets of a space Y is *equi-LC^m* in Y [18] if, for every $y \in Y$ and neighbourhood U of y, there exists neighbourhood V of y such that, for every $S \in \mathcal{S}$, every continuous image of $\mathbb{S}^k$ ($k \leqslant m$) in $V \cap S$ is contractible in $U \cap S$. A space S is called C^m if every continuous image of $\mathbb{S}^k$ ($k \leqslant m$) in S is contractible in S.

In the present paper we deal with a similar problem but now concerning another property of the covering dimension. Towards this end, for spaces X and Y, we use $C(X, Y)$ to denote the set of all continuous map from X to Y. Let Y be a metrizable space, and let us recall that a closed set $F \subset Y$ (possibly empty) is called a *Z_n-set* in Y, $n < \omega$, if the set $C(\mathbb{B}^n, Y \backslash F)$ is dense in $C(\mathbb{B}^n, Y)$ with respect to the uniform topology generated by a metric on Y, see [4] and [29]. Here, $\mathbb{B}^n$ denotes the n-dimensional closed ball, and ω is the first infinite cardinal. Finally, we shall say that F is a *σZ_n-set* in Y if F is a countable union of Z_n-subsets of Y.

The σZ_n-sets are 'thin'-subsets with respect to maps from n-dimensional spaces. Namely, a normal space X has a covering dimension $\dim(X) \leqslant n$ iff for every σZ_n-set F in $\mathbb{B}^{n+1}$, the set $C(X, \mathbb{B}^{n+1} \setminus F)$ is dense in $C(X, \mathbb{B}^{n+1})$ with respect to the uniform topology, see [24]. Our main goal is now to provide a parametric version of the above characterization of $\dim(X) \leqslant n$. To state it, we need a bit more terminology about set-valued mappings. Let $\mathcal{P}(Y) = 2^Y \cup \{\emptyset\}$. To every mapping ϕ: $X \to \mathcal{P}(Y)$ we associate the following subsets of $C(X, Y)$:

$$\mathcal{S}e\ell(\phi) = \{f \in C(X, Y) : f(x) \in \phi(x) \text{ for every } x \in X\},$$

and

$$\mathcal{M}(\phi) = \{f \in C(X, Y) : f(x) \notin \phi(x) \text{ for every } x \in X\}.$$

Also, let us recall that ϕ: $X \to \mathcal{P}(Y)$ has a *closed graph* (respectively, an *F_σ-graph*) if $\mathrm{Graph}(\phi) = \{(x, y) \in X \times Y : y \in \phi(x)\}$ is closed (respectively, F_σ) in $X \times Y$.

Finally, let us recall that, for a metric space (Y, ρ), the *fine topology* on $C(X, Y)$ is the topology in which the family of all sets

$$V(f, \alpha) = \{g \in C(X, Y) : \rho(g(x), f(x)) < \alpha(x) \text{ for each } x \in X\},$$

is a local base at f (see [22]), where α runs on the positive continuous functions on X. For any space X the fine topology is finer than the uniform one (generated by ρ), and it does not depend on the metric of Y provided X is normal and countably paracompact [5] (for a paracompact X, see [16]).

The following theorem will be proven in this paper.

THEOREM 1.2. *Let X be a paracompact space, with $\dim(X) \leqslant n$, Y be a completely metrizable space, φ: $X \to \mathcal{F}(Y)$ be an l.s.c. mapping such that $\{\varphi(x) : x \in X\}$ is equi-LC^{n-1} in Y and each $\varphi(x)$, $x \in X$, is C^{n-1}, and let ψ: $X \to \mathcal{P}(Y)$ be a mapping with an F_σ-graph such that $\psi(x) \cap \varphi(x)$ is a σZ_n-set in $\varphi(x)$ for every $x \in X$. Then, the set $\mathcal{M}(\psi) \cap \mathcal{S}e\ell(\varphi)$ is a dense G_δ-subset of $\mathcal{S}e\ell(\varphi)$ with respect to the fine topology on $\mathcal{S}e\ell(\varphi)$.*

A few words about the paper. Sections 2 and 3 contain the preparation for the proof of Theorem 1.2 which will be finally accomplished in Section 4. The rest of the paper is devoted to some possible applications. For instance, in Section 5 we obtain a generalization of the classical Ostrand's theorem [23], see Theorem 5.2. In the same Section 5, we apply Theorem 1.2 to establish some properties of strongly countable-dimensional spaces in the terms of the Baire category which are similar to some results of E. Pol in [24] and [25]. We also provide an alternative proof of the Hurewicz formula that $\dim(X) \leqslant \dim f + \dim(Y)$, where $f\colon X \to Y$ is a closed continuous map and $\dim f = \sup\{\dim(f^{-1}(y)) : y \in Y\}$, see Theorem 6.3.

2. Selections Avoiding Closed Sets

Let (Y, d) be a metric space. If $S \subset Y$ and $\varepsilon > 0$, then we use $B^d_\varepsilon(S)$ to denote the *ε-neighbourhood* of S in (Y, d), i.e., $B^d_\varepsilon(S) = \{y \in Y : d(y, S) < \varepsilon\}$. Also, we use $\operatorname{diam}_d(S)$ to denote the diameter of S with respect to d, i.e., the *d-diameter* of S in Y. Now, let us recall that a family $\mathcal{S}$ of subsets of Y is *d-uniformly equi-LC^n* (for some $n \geqslant -1$) if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, for every $S \in \mathcal{S}$, every continuous image of a k-sphere ($k \leqslant n$) in S of d-diameter $< \delta(\varepsilon)$ is contractible over a subset of S of d-diameter $< \varepsilon$, [18].

In what follows, for a metrizable space Y, let $\mathcal{D}(Y)$ be the set of all metrics on Y compatible with the topology of Y. If $d, \rho \in \mathcal{D}(Y)$, then we will use $\rho \geqslant d$ to denote that $\rho(y, z) \geqslant d(y, z)$ for every $y, z \in Y$.

In this section, we provide a slight generalization of [18, Theorem 4.1] constructing controlled selections avoiding a given closed set (Theorem 2.3). To prepare for this, we need the following improvement in [18, Lemma 11.2].

LEMMA 2.1. *Let Y be a completely metrizable space, $d \in \mathcal{D}(Y)$, and let $\mathcal{S} \subset 2^Y$ be d-uniformly equi-LC^n. Also, let $\rho \in \mathcal{D}(Y)$ be a complete metric on Y such that $\rho \geqslant d$ and $\mathcal{S}' = \mathcal{S} \cup \{Y\}$ is ρ-uniformly equi-LC^n. Then, to every $\varepsilon > 0$ and $\mu > 0$ there corresponds an $\eta(\varepsilon) > 0$ and $\lambda(\varepsilon, \mu) > 0$ with the following property*: *If $k \leqslant n$ and $S \in \mathcal{S}$, then every continuous $p\colon \mathbb{S}^k \to B^\rho_{\lambda(\varepsilon,\mu)}(S)$, with* $\operatorname{diam}_d(p(\mathbb{S}^k)) < \eta(\varepsilon)$*, is homotopic to a constant map over a subset of $B^\rho_\mu(S)$ of d-diameter $< \varepsilon$.*

Proof. The proof almost repeats that of [18, Lemma 11.2]. Namely, let $\gamma(\varepsilon) \leqslant \varepsilon$ (respectively, $\kappa(\mu)$) be as in [18, Corollary 4.2] (respectively, as in [18, Lemma 11.1]) applied to $\mathcal{S}'$ as a family being ρ-uniformly equi-LC^n. Next, let $\delta(\varepsilon) \leqslant \varepsilon$ be as in the definition of d-uniformly equi-LC^n of $\mathcal{S}$. Finally, following [18, Lemma 11.2], define $\eta(\varepsilon) = \delta(\varepsilon/2)/2$ and $\lambda(\varepsilon, \mu) = \kappa(\gamma(\xi))$, where $\xi = \min\{\delta(\varepsilon/2)/4, \mu\}$. This works in our present situation. Indeed, let $k \leqslant n$, $S \in \mathcal{S}$, and let $p\colon \mathbb{S}^k \to B^\rho_{\lambda(\varepsilon,\mu)}(S)$ be continuous such that $\operatorname{diam}_d(p(\mathbb{S}^k)) < \eta(\varepsilon)$. Then, by [18, Lemma 11.1], there exists a continuous map $q\colon \mathbb{S}^k \to S$ such that $\rho(q(s),$

$p(s)) < \gamma(\xi)$ for every $s \in \mathbb{S}^k$. Hence, for every $s_1, s_2 \in \mathbb{S}^k$, we have

$$\begin{aligned} d(q(s_1), q(s_2)) &\leqslant d(q(s_1), p(s_1)) + d(p(s_1), p(s_2)) + d(p(s_2), q(s_2)) \\ &\leqslant \rho(q(s_1), p(s_1)) + d(p(s_1), p(s_2)) + \rho(p(s_2), q(s_2)) \\ &< \gamma(\xi) + \eta(\varepsilon) + \gamma(\xi) \\ &\leqslant \delta(\varepsilon/2)/4 + \delta(\varepsilon/2)/2 + \delta(\varepsilon/2)/4 = \delta(\varepsilon/2). \end{aligned}$$

So, $\operatorname{diam}_d(q(\mathbb{S}^k)) < \delta(\varepsilon/2)$ and therefore, by hypothesis, there is a homotopy h_1 of q to a constant map over a subset of S of d-diameter $< \varepsilon/2$. Let $\varphi(s,t) = Y$ for every $(s,t) \in \mathbb{S}^k \times \mathbb{I}$, where $\mathbb{I} = [0,1]$ denotes the closed unit interval. Also, let $g(s,t) = q(s)$ for every $(s,t) \in \mathbb{S}^k \times \mathbb{I}$, while $\ell : \mathbb{S}^k \times \{0,1\}$ be defined by $\ell | \mathbb{S}^k \times \{0\} = p$ and $\ell | \mathbb{S}^k \times \{1\} = q$. Since $\rho(\ell(x), g(x)) < \gamma(\xi)$ for every $x \in \mathbb{S}^k \times \{0,1\}$, by [18, Corollary 4.2], ℓ can be extended to a continuous selection h_2: $\mathbb{S}^k \times \mathbb{I} \to Y$ for φ such that $\rho(h_2(x), g(x)) < \xi \leqslant \mu$ for every $x \in \mathbb{S}^k \times \mathbb{I}$. Thus, we get a homotopy h_2 between p and q, with $h_2(\mathbb{S}^k \times \mathbb{I}) \subset B^\rho_\mu(S)$. Finally, define h to be the homotopy obtained by combining h_1 with h_2. Then, h is a homotopy of p with a constant map over a subset of $B^\rho_\mu(S)$ of d-diameter $< \varepsilon/2 + 2\xi < \varepsilon$. □

For a locally finite simplicial complex M we use $|M|$ to denote the *polytope* on M, and M^k to denote the *k-skeleton* of M. Also, for a locally finite cover $\mathcal{U}$ of X we denote by $\mathcal{N}(\mathcal{U})$ the *nerve* of $\mathcal{U}$, i.e., the simplicial complex $\mathcal{N}(\mathcal{U}) = \{\sigma \subset \mathcal{U} : \bigcap \sigma \neq \emptyset\}$.

Repeating precisely the proof of [18, Lemma 6.1] but now using Lemma 2.1 instead of [18, Lemma 11.2], we get the following generalization of [18, Lemma 6.1].

LEMMA 2.2. *Let E be a completely metrizable space, $Y \subset E$ be a G_δ-set, $d \in \mathcal{D}(E)$, and let $\mathcal{S} \subset 2^Y$ be d-uniformly equi-LC^n. Also, let $\rho \in \mathcal{D}(Y)$ be a complete metric such that $\rho \geqslant d|Y \times Y$ and $\mathcal{S}' = \mathcal{S} \cup \{Y\}$ is ρ-uniformly equi-LC^n. Then, to every $\varepsilon > 0$ there corresponds $\beta(\varepsilon) > 0$ with the following property: If X is a paracompact space, φ: $X \to \mathcal{S}$ is l.s.c., g: $X \to E$ is a continuous map, with $g(x) \in B^d_{\beta(\varepsilon)}(\varphi(x))$ for every $x \in X$, and if $\mu > 0$, then there exists a locally finite open cover $\mathcal{U}$ of X and a continuous u: $|\mathcal{N}^{n+1}(\mathcal{U})| \to Y$ such that*

$$u(|\sigma|) \subset B^\rho_\mu(\varphi(x)) \cap B^d_\varepsilon(g(x)), \quad \textit{for every } x \in \bigcap \sigma \textit{ and } \sigma \in \mathcal{N}^{n+1}(\mathcal{U}).$$

We are now ready to prove the promised generalization of [18, Theorem 4.1].

THEOREM 2.3. *Let M be a completely metrizable space, $U \subset M$ be an open set, $d \in \mathcal{D}(M)$, and let $\mathcal{S} \subset \mathcal{F}(U)$ be d-uniformly equi-LC^n. Then, for every $\varepsilon > 0$ there exists $\nu(\varepsilon) > 0$ with the following property: If X is a paracompact space, with* $\dim(X) \leqslant n + 1$, *φ: $X \to \mathcal{S}$ is l.s.c., and g: $X \to M$ is continuous, with $g(x) \in B^d_{\nu(\varepsilon)}(\varphi(x))$ for every $x \in X$, then φ has a continuous selection f such that $d(f(x), g(x)) < \varepsilon$ for every $x \in X$.*

Proof. The proof involves arguments similar to those in [18]. Namely, embed (M, d) isometrically in a Banach space (E, d), where d is the metric on E generated by the norm of E. Next, take an open set $Y \subset E$ such that $Y \cap M = U$. Then, from one hand, $\mathcal{S}$ is d-uniformly equi-LC^n as a family of subsets of Y. From another hand, $\mathcal{S}' = \mathcal{S} \cup \{Y\}$ is equi-LC^n in Y because Y is an open subset of a Banach space. Since Y is completely metrizable, there exists a complete metric $\tilde{d} \in \mathcal{D}(Y)$ such that $\tilde{d} \geqslant d|Y \times Y$. Therefore, by [18, Proposition 2.1], there also exists a metric $\rho \in \mathcal{D}(Y)$ such that $\rho \geqslant \tilde{d}$ and $\mathcal{S}'$ is ρ-uniformly equi-LC^n. In particular, ρ is a complete metric in Y and $\rho \geqslant d|Y \times Y$. Let $\gamma(\varepsilon)$ be as in [18, Theorem 4.1] applied to $\mathcal{S}'$ as a family which is ρ-uniformly equi-LC^n in Y. Also, let $\beta(\varepsilon)$ be as in Lemma 2.2 applied to $d \in \mathcal{D}(E)$, $\mathcal{S} \subset 2^Y$, and $\rho \in \mathcal{D}(Y)$. Then, take $\nu(\varepsilon) = \beta(\varepsilon/2)$, and let us check that this works. So, let $\varphi\colon X \to \mathcal{S}$ be l.s.c., and let $g\colon X \to M$ be continuous such that $g(x) \in B^d_{\beta(\varepsilon/2)}(\varphi(x))$ for every $x \in X$. Note that g as a map from X to E has the same properties because (M, d) is embedded isometrically in (E, d). Hence, by Lemma 2.2, applied with $\mu = \gamma(\varepsilon/2)$, there now exists a locally finite open cover $\mathcal{U}$ of X and a continuous $u\colon |\mathcal{N}^{n+1}(\mathcal{U})| \to Y$ such that

$$u(|\sigma|) \subset B^{\rho}_{\gamma(\varepsilon/2)}(\varphi(x)) \cap B^{d}_{\varepsilon/2}(g(x)), \quad \text{for every } x \in \bigcap \sigma \text{ and } \sigma \in \mathcal{N}^{n+1}(\mathcal{U}). \tag{1}$$

Since $\dim(X) \leqslant n + 1$, there also exists an open cover $\{V_U : U \in \mathcal{U}\}$ of X such that $V_U \subset U$ for every $U \in \mathcal{U}$, and

$$\{U \in \mathcal{U} : x \in V_U\} \in \mathcal{N}^{n+1}(\mathcal{U}), \quad \text{for every } x \in X. \tag{2}$$

Finally, take a partition of unity $\{\xi_U : U \in \mathcal{U}\}$ index-subordinated to the cover $\{V_U : U \in \mathcal{U}\}$ of X. Then, by (2), $k = u \circ \xi\colon X \to Y$ defines a continuous map, where $\xi\colon X \to |\mathcal{N}(\mathcal{U})|$ is the canonical map $\xi(x) = \sum\{\xi_U(x) \cdot U : U \in \mathcal{U}\}$, $x \in X$. According to (1) and (2), this map has also the property that

$$k(x) \in B^{\rho}_{\gamma(\varepsilon/2)}(\varphi(x)) \cap B^{d}_{\varepsilon/2}(g(x)), \quad \text{for every } x \in X.$$

Then, by [18, Theorem 4.1], there exists a continuous selection f for φ such that $\rho(f(x), k(x)) < \varepsilon/2$ for every $x \in X$. Since $\rho \geqslant d|Y \times Y$, this finally implies that, for every $x \in X$,

$$\begin{aligned} d(f(x), g(x)) &\leqslant d(f(x), k(x)) + d(k(x), g(x)) \\ &\leqslant \rho(f(x), k(x)) + d(k(x), g(x)) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

which completes the proof. □

Theorem 2.3 has some interesting consequences. One of them is related to the requirement in [18, Theorem 4.1] the family $\mathcal{S} \subset \mathcal{F}(Y)$ to be d-uniformly equi-LC^n with respect to a complete metric $d \in \mathcal{D}(Y)$. Namely, by taking $U = M$ in Theorem 2.3, we have the following immediate result.

COROLLARY 2.4. *Let X be a paracompact space, with* $\dim(X) \leqslant n+1$, *Y be a completely metrizable space, $d \in \mathcal{D}(Y)$, and let $\mathcal{S} \subset \mathcal{F}(Y)$ be d-uniformly equi-LC^n. Then, for every $\varepsilon > 0$ there exists $\nu(\varepsilon) > 0$ with the following property: If $\varphi\colon X \to \mathcal{S}$ is l.s.c. and $g\colon X \to Y$ is continuous, with $g(x) \in B^d_{\nu(\varepsilon)}(\varphi(x))$ for every $x \in X$, then φ has a continuous selection f such that $d(f(x), g(x)) < \varepsilon$ for every $x \in X$.*

In our next consequence, a function $\mu\colon X \to \mathbb{R}$ is *lower semi-continuous* if the set $\{x \in X : \mu(x) > r\}$ is open in X for every $r \in \mathbb{R}$.

COROLLARY 2.5. *Let M be a completely metrizable space, $U \subset M$ be an open set, $d \in \mathcal{D}(M)$, and let $\mathcal{T} \subset \mathcal{F}(U)$ be d-uniformly equi-LC^n. Then, for every paracompact space X, with* $\dim(X) \leqslant n+1$, *and every lower semi-continuous function $\mu\colon X \to (0, +\infty)$ there exists a continuous function $\alpha\colon X \to (0, +\infty)$ with the following property: If $\varphi\colon X \to \mathcal{T}$ is l.s.c. and $g\colon X \to M$ is continuous, with $g(x) \in B^d_{\alpha(x)}(\varphi(x))$ for every $x \in X$, then there exists a continuous selection f for φ such that $d(f(x), g(x)) < \mu(x)$ for every $x \in X$.*

Proof. The proof follows an idea in [18, proof that Theorem 4.1 implies Theorem 1.3]. Namely, let X and μ be as in the hypotheses. Then,

$$V_i = \{x \in X : \mu(x) > 1/i\}, \quad i \in \mathbb{N}, \tag{3}$$

defines an open increasing cover of X because μ is lower semi-continuous. Since X is paracompact (hence, normal and countably paracompact as well), there exists another open cover $\{W_i : i \in \mathbb{N}\}$ of X such that

$$\overline{W_i} \subset V_i \cap W_{i+1}, \quad \text{for every } i \in \mathbb{N}. \tag{4}$$

Next, for every $\varepsilon > 0$, let $\nu(\varepsilon) \leqslant \varepsilon$ be as in Theorem 2.3 applied to the family $\mathcal{S} = \mathcal{T} \cup \{\{y\} : y \in U\}$. Define a decreasing function $\eta\colon \mathbb{N} \to (0, +\infty)$ by letting for $i \in \mathbb{N}$ that

$$\eta(i) = \min\left\{\nu\left(\nu\left(\frac{1}{j}\right) \cdot \nu\left(\frac{1}{j+1}\right)\right) : 1 \leqslant j \leqslant i\right\}. \tag{5}$$

Now, for every $x \in X$, let $i(x) = \min\{i \in \mathbb{N} : x \in W_i\}$. Then, define a function $\beta\colon X \to (0, +\infty)$ by

$$\beta(x) = \eta(i(x)), \quad x \in X. \tag{6}$$

Note that β is lower semi-continuous. Indeed, if $x \in W_{i(x_0)}$ for some point $x_0 \in X$, then $i(x) \leqslant i(x_0)$ implies $\beta(x) = \eta(i(x)) \geqslant \eta(i(x_0)) = \beta(x_0)$. Since X is paracompact, by a result of [6] (see, also, [7, 15]), there exists a continuous function $\alpha\colon X \to (0, +\infty)$ such that $\alpha(x) \leqslant \beta(x)$ for every $x \in X$. This α is as required. Indeed, take an l.s.c. $\varphi\colon X \to \mathcal{T}$ and a continuous $g\colon X \to M$ with $d(g(x), \varphi(x)) < \alpha(x) \leqslant \beta(x)$ for every $x \in X$. Also, let $A_0 = \emptyset$, and $A_i = \overline{W_i} \setminus W_i$ for every $i \in \mathbb{N}$. Then, each A_i is a paracompact space as a closed subset of X, and

$\dim(A_i) \leqslant n + 1$. On the other hand, $x \in A_i$ implies $x \notin W_i$, so $i(x) > i$. Therefore, by (5) and (6), we now get that

$$d(g(x), \varphi(x)) < \beta(x) = \eta(i(x)) \leqslant \eta(i) \leqslant \nu(\nu(1/i) \cdot \nu(1/(i+1))).$$

Hence, by Theorem 2.3, each $\varphi|A_i$ has a continuous selection $h_i\colon A_i \to U$ with

$$d(h_i(x), g(x)) < \nu(1/i) \cdot \nu(1/(i+1)), \quad \text{for every } x \in A_i \text{ and } i \in \mathbb{N}. \tag{7}$$

Let $B_i = A_{i-1} \cup A_i$ and $X_i = \overline{W_i} \setminus W_{i-1}$ for every $i \in \mathbb{N}$, where $W_0 = \emptyset$. Next, define $k_i\colon B_i \to U$ by $k|A_j = h_j$, $j = i-1, i$. Finally, define $\varphi\colon X_i \to \mathcal{S}$ by $\varphi(x) = \{k_i(x)\}$ if $x \in B_i$ and $\varphi_i(x) = \varphi(x)$ otherwise. By [17, Example 1.3*], each φ_i is l.s.c. Then, by (7) and Theorem 2.3, each φ_i has a continuous selection $f_i\colon X_i \to U$ such that $d(f_i(x), g(x)) < 1/i$ for every $x \in X_i$. In fact, f_i is a continuous extension of k_i. Since, by (4), $X_i \cap X_{i+1} = A_i$, while $f_i|A_i = h_i = f_{i+1}|A_i$, we may now define $f\colon X \to Y$ by $f|X_i = f_i$. Clearly, f is a continuous selection for φ and, by (3) and (4), $x \in X_i \subset V_i$ implies $d(f(x), g(x)) = d(f_i(x), g(x)) < 1/i < \mu(x)$. □

3. A Construction of Set-Valued Mappings

In this section, we continue the preparation for the proof of Theorem 1.2 involving a construction of [20]. Namely, let (Y, d) be a metric space, $\psi\colon X \to \mathcal{P}(Y)$ be a mapping with a closed graph, and let $M_\psi = (X \times Y) \setminus \mathrm{Graph}(\psi)$. Then, for every $(x, y) \in M_\psi$, we consider the set $\Delta(x, y)$ of all $\delta \in (0, 1]$ for which there exists a neighbourhood U_δ of x such that $U_\delta \times B^d_\delta(y) \subset M_\psi$. Note that $\Delta(x, y) \neq \emptyset$ for every $(x, y) \in M_\psi$. Now, we define a map $u\colon X \times Y \to \mathbb{I}$ by letting for $(x, y) \in X \times Y$ that

$$u(x, y) = \begin{cases} \sup \Delta(x, y), & \text{if } (x, y) \in M_\psi, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

The following properties of u were actually demonstrated in [20, Lemma 3.1].

PROPOSITION 3.1 ([20]). *Let (Y, d) be a metric space, $\psi\colon X \to \mathcal{P}(Y)$ be a mapping with a closed graph, and let $u\colon X \times Y \to \mathbb{I}$ be defined as in* (8). *Then,*

(a) *u is a lower semi-continuous function.*
(b) $|u(x, y_1) - u(x, y_2)| \leqslant d(y_1, y_2)$ *for every $x \in X$ and $y_1, y_2 \in Y$.*

Proof. Our situation is slightly different from that in [20, Lemma 3.1], so we provide some arguments to the proof. Since $u(x, y) \geqslant 0$ for every $(x, y) \in X \times Y$, to show (a), it suffices to consider only the points of M_ψ. To this end, we repeat precisely the arguments in [12, Lemma 2.1]. Namely, take a point $(x_0, y_0) \in M_\psi$

and $a \in \mathbb{I}$, with $a < u(x_0, y_0)$. Then, there exists $\delta_0 \in (a, u(x_0, y_0)]$ and a neighbourhood U_0 of x_0 such that $U_0 \times B^d_{\delta_0}(y_0) \subset M_\psi$. Let $\rho = (\delta_0 - a)/2$ and $\delta = a + \rho$. It now follows that $\delta \in \Delta(x, y)$ for every $(x, y) \in U_0 \times B^d_\rho(y_0)$ because

$$\begin{aligned} &U_0 \times B^d_\delta(y) \subset U_0 \times B^d_{\delta+\rho}(y_0) \subset U_0 \times B^d_{\delta_0}(y_0) \subset M_\psi, \\ &\quad (x, y) \in U_0 \times B^d_\rho(y_0). \end{aligned}$$

Hence, in particular, $u(x, y) \geqslant \delta > a$ for every $(x, y) \in U_0 \times B^d_\rho(y_0)$.

The statement in (b) is the same as in [20, Lemma 3.1]. □

We are now ready to state the main result of this section which is an analogue to [20, Lemma 3.3].

THEOREM 3.2. *Let X be a space, (Y, d) be a complete metric space, φ: $X \to \mathcal{F}(Y)$ be l.s.c., with $\{\varphi(x) : x \in X\}$ d-uniformly equi-LC^{n-1}, and let ψ: $X \to \mathcal{P}(Y)$ be a mapping with a closed graph such that $\psi(x) \cap \varphi(x)$ is a σZ_n-set in $\varphi(x)$ for every $x \in X$. Define another set-valued mapping Φ: $X \to 2^{Y\times(0,1]}$ by*

$$\begin{aligned} \Phi(x) = \{&(y, t) \in Y \times (0, 1] : y \in \varphi(x) \setminus \psi(x) \text{ and} \\ &t \in (0, u(x, y)]\}, \quad x \in X. \end{aligned}$$

Then:

- (a) *Φ is l.s.c.*
- (b) *$\Phi(x)$ is a nonempty closed subset of $Y \times (0, 1]$, for every $x \in X$.*
- (c) *The family $\{\Phi(x) : x \in X\}$ is $d \times e$-uniformly equi-LC^{n-1}, where $d \times e$ is the box metric $d \times e((y_1, t_1), (y_2, t_2)) = \max\{d(y_1, y_2), e(t_1, t_2)\}$ on $Y \times (0, 1]$ generated by d and the Euclidean metric e on $(0, 1]$.*

To prepare for the proof of Theorem 3.2 we need the following simple observations about Z_n-sets in complete metric spaces, the first of which is well known.

LEMMA 3.3. *Let Y be a completely metrizable space, and let $T \subset Y$ be closed. Then, T is a σZ_n-set in Y if and only if it is a Z_n-set in Y.*

PROPOSITION 3.4. *Let (Y, d) be a complete metric space, $\mathcal{S} \subset \mathcal{F}(Y)$ be d-uniformly equi-LC^{n-1}, and let $\{Z_S \subset S : S \in \mathcal{S}\}$ be a family of closed sets in Y such that each Z_S is a Z_n-set in S, $S \in \mathcal{S}$. Then, $\{S \setminus Z_S : S \in \mathcal{S}\}$ is also d-uniformly equi-LC^{n-1}.*

Proof. In case $n = 0$ the proof is trivial. So, we suppose that $n \geqslant 1$. Let $\delta(\varepsilon) \leqslant \varepsilon$ be as in the definition of d-uniformly equi-LC^{n-1} of $\mathcal{S}$. We are going to show that $\{S \setminus Z_S : S \in \mathcal{S}\}$ is d-uniformly equi-LC^{n-1} with respect to $\delta(\varepsilon/3)$. Take $\varepsilon > 0$, $S \in \mathcal{S}$, and a continuous map p: $\mathbb{S}^m \to S \setminus Z_S$ (for some $0 \leqslant m \leqslant n - 1$) such that $\mathrm{diam}_d(p(\mathbb{S}^m)) \leqslant \delta(\varepsilon/3)$. Let

$$\mathrm{Cone}(\mathbb{S}^m) = \{(1 - t) \cdot \vartheta + t \cdot s : (s, t) \in \mathbb{S}^m \times \mathbb{I}\}$$

be the *cone* of $\mathbb{S}^m$ with a vertex ϑ. Also, take an $a \in (0, 1)$, and consider the copy

$$\mathrm{Cone}_a(\mathbb{S}^m) = \{(1-t)\cdot\vartheta + t\cdot s : (s,t) \in \mathbb{S}^m \times [0,a]\}$$

of the cone $\mathrm{Cone}(\mathbb{S}^m)$. Whenever $t > 0$, for convenience, let $\mathbb{S}^m_t = \mathbb{S}^m \times \{t\}$ be the corresponding copy of $\mathbb{S}^m$ in $\mathrm{Cone}(\mathbb{S}^m)$. Then, according to the properties of $\delta(\varepsilon/3)$, there exists a continuous $q\colon \mathrm{Cone}_a(\mathbb{S}^m) \to S$ such that

$$q|\mathbb{S}^m_a = p \quad \text{and} \quad \mathrm{diam}_d(q(\mathrm{Cone}_a(\mathbb{S}^m))) < \frac{\varepsilon}{3}. \tag{9}$$

Let $\varepsilon_1 \in (0, \varepsilon/6]$ be such that $B^d_{2\varepsilon_1}(p(\mathbb{S}^m)) \cap Z_S = \emptyset$. Next, let $\gamma(\varepsilon_1) \leqslant \varepsilon_1$ be as in [18, Corollary 4.2] applied to the family $\mathcal{S}$. Since Z_S is a Z_n-set in S, the set $C(\mathrm{Cone}_a(\mathbb{S}^m), S \setminus Z_S)$ is dense in $C(\mathrm{Cone}_a(\mathbb{S}^m), S)$ with respect to the uniform topology. Therefore, there exists a continuous $\ell\colon \mathrm{Cone}_a(\mathbb{S}^m) \to S \setminus Z_S$ such that

$$d(\ell(x), q(x)) < \gamma(\varepsilon_1), \quad x \in \mathrm{Cone}_a(\mathbb{S}^m). \tag{10}$$

For later use, let us observe that, by (9), this implies

$$\begin{aligned} \mathrm{diam}_d(\ell(\mathrm{Cone}_a(\mathbb{S}^m))) &< 2\gamma(\varepsilon_1) + \mathrm{diam}_d(q(\mathrm{Cone}_a(\mathbb{S}^m))) \\ &\leqslant \frac{2\varepsilon}{6} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}. \end{aligned} \tag{11}$$

Define now a map $k\colon \mathrm{Cone}(\mathbb{S}^m) \to S$ by letting for every $(s,t) \in \mathbb{S}^m \times \mathbb{I}$ that

$$k((1-t)\cdot\vartheta + t\cdot s) = \begin{cases} \ell((1-t)\cdot\vartheta + t\cdot s), & \text{if } t \leqslant a, \\ \ell((1-a)\cdot\vartheta + a\cdot s), & \text{otherwise.} \end{cases}$$

Also, define another map $g\colon \mathrm{Cone}_a(\mathbb{S}^m) \cup \mathbb{S}^m_1 \to S$ by $g|\mathrm{Cone}_a(\mathbb{S}^m) = \ell$, and $g|\mathbb{S}^m_1 = p$. Then, by (9) and (10), $d(g(x), k(x)) < \gamma(\varepsilon_1)$ for every $x \in \mathrm{Cone}_a(\mathbb{S}^m) \cup \mathbb{S}^m_1$. So, by [18, Corollary 4.2] (with $\varphi(x) = S$ for every $x \in \mathrm{Cone}(\mathbb{S}^m)$), there exists a continuous $h\colon \mathrm{Cone}(\mathbb{S}^m) \to S$ such that

$$\begin{aligned} &h|\mathrm{Cone}_a(\mathbb{S}^m) = \ell, \qquad h|\mathbb{S}^m_1 = p, \quad \text{and} \\ &d(h(x), k(x)) < \varepsilon_1, \quad x \in \mathrm{Cone}(\mathbb{S}^m). \end{aligned} \tag{12}$$

Clearly, h is a homotopy connecting p and a constant map. Moreover, by (11) and (12),

$$\mathrm{diam}_d(h(\mathrm{Cone}(\mathbb{S}^m))) \leqslant 2\varepsilon_1 + \mathrm{diam}_d(\ell(\mathrm{Cone}_a(\mathbb{S}^m))) < \frac{2\varepsilon}{6} + \frac{2\varepsilon}{3} = \varepsilon.$$

On the other hand, by (9) and (10),

$$\begin{aligned} h(\mathrm{Cone}(\mathbb{S}^m) \setminus \mathrm{Cone}_a(\mathbb{S}^m)) &\subset B^d_{\varepsilon_1}(k(\mathrm{Cone}(\mathbb{S}^m) \setminus \mathrm{Cone}_a(\mathbb{S}^m))) \\ &= B^d_{\varepsilon_1}(\ell(\mathbb{S}^m_a)) \\ &\subset B^d_{\varepsilon_1 + \gamma(\varepsilon_1)}(p(\mathbb{S}^m)) \\ &\subset B^d_{2\varepsilon_1}(p(\mathbb{S}^m)) \subset S \setminus Z_S. \end{aligned}$$

Since, by (12), $h(\mathrm{Cone}_a(\mathbb{S}^m)) = \ell(\mathrm{Cone}_a(\mathbb{S}^m)) \subset S \setminus Z_S$, this finally implies that $h(\mathrm{Cone}(\mathbb{S}^m)) \subset S \setminus Z_S$ which completes the proof. □

Proof of Theorem 3.2. Let X, (Y, d), φ and ψ be as in that theorem. The statements of (a) and (b) were established in [20, Lemma 3.3], so we prove only (c). According to Lemma 3.3 and Proposition 3.4, the family $\mathcal{T} = \{\varphi(x) \setminus \psi(x) : x \in X\}$ is d-uniformly equi-LC^{n-1}. Then, let $\delta(\varepsilon)$ be as in the definition of d-uniformly equi-LC^{n-1} of $\mathcal{T}$. We are going to show that $\{\Phi(x) : x \in X\}$ is $d \times e$-uniformly equi-LC^{n-1} with respect to $\delta(\varepsilon/2)$. So, take an $\varepsilon > 0$, a point $x \in X$, and a continuous map p: $\mathbb{S}^k \to \Phi(x)$ for some k, with $0 \leqslant k \leqslant n-1$, such that $\mathrm{diam}_{d\times e}(p(\mathbb{S}^k)) < \delta(\varepsilon/2)$. Also, let π_1: $Y \times (0,1] \to Y$ and π_2: $Y \times (0,1] \to (0,1]$ be the corresponding projections. Then, $p_1 = \pi_1 \circ p$: $\mathbb{S}^k \to \varphi(x) \setminus \psi(x)$ is a continuous map, with $\mathrm{diam}_d(p_1(\mathbb{S}^k)) < \delta(\varepsilon/2)$. Hence, p_1 can be extended to a continuous map q_1: $\mathbb{B}^{k+1} \to \varphi(x) \setminus \psi(x)$ such that $\mu = \mathrm{diam}_d(q_1(\mathbb{B}^{k+1})) < \varepsilon/2 < \varepsilon$. In the same way, $p_2 = \pi_2 \circ p$: $\mathbb{S}^k \to (0,1]$ is a continuous map, with $\mathrm{diam}_e(p_2(\mathbb{S}^{n-1})) < \delta(\varepsilon/2)$. According to the definition of $\Phi(x)$, we now have that $p_2(s) \leqslant u(x, p_1(s))$ for every $s \in \mathbb{S}^k$. Fix an arbitrary point $s_0 \in \mathbb{S}^k$. Then, by Proposition 3.1, we have that

$$|u(x, q_1(b)) - u(x, q_1(s_0))| \leqslant d(q_1(b), q_1(s_0)) \leqslant \mu < \frac{\varepsilon}{2}, \quad b \in \mathbb{B}^{k+1}.$$

Hence, $B^e_\mu(p_2(s_0)) \cap (0, u(x, q_1(b))] \neq \emptyset$, $b \in \mathbb{B}^{k+1}$, because $p_2(s_0) \leqslant u(x, p_1(s_0)) = u(x, q_1(s_0))$. Since, by Proposition 3.1, $u(x, q_1(b))$, $b \in \mathbb{B}^{k+1}$, is a continuous map, we may now define an l.s.c. mapping θ: $\mathbb{B}^{k+1} \to \mathcal{F}((0,1])$ by

$$\theta(b) = \overline{B^e_\mu(p_2(s_0)) \cap (0, u(x, q_1(b))]}^{(0,1]}, \quad b \in \mathbb{B}^{k+1}.$$

Then, by [17, Theorem 3.2″], θ has a continuous selection q_2: $\mathbb{B}^{k+1} \to (0,1]$ because θ is convex-valued. Note that q_2 is an extension of p_2 such that $e(q_2(b), q_2(s_0)) \leqslant \mu$ for every $b \in \mathbb{B}^{k+1}$, i.e., $\mathrm{diam}_e(q_2(\mathbb{B}^{k+1})) \leqslant 2\mu < \varepsilon$. Then, the diagonal map $q = q_1 \Delta q_2$: $\mathbb{B}^{k+1} \to \Phi(x)$ is an extension of p, with $\mathrm{diam}_{d\times e}(q(\mathbb{B}^{k+1})) < \varepsilon$. □

4. Proof of Theorem 1.2

We finalize the preparation for the proof of Theorem 1.2 with the following result which may have an independent interest.

THEOREM 4.1. *Let X be paracompact, with* $\dim(X) \leqslant n$, *Y be a completely metrizable space, φ: $X \to \mathcal{F}(Y)$ be l.s.c. such that $\{\varphi(x) : x \in X\}$ is equi-LC^{n-1}, and let ψ: $X \to \mathcal{P}(Y)$ be a mapping with a closed graph such that $\psi(x) \cap \varphi(x)$ is a σZ_n-set in $\varphi(x)$ for every $x \in X$. Also, let f be a continuous selection for φ, and let $\mu \in C(X, (0, +\infty))$. Then for every $\rho \in \mathcal{D}(Y)$, the mapping φ has a* *continuous selection g such that $g(x) \notin \psi(x)$ and $\rho(g(x), f(x)) < \mu(x)$ for every $x \in X$.*

Proof. Let $\rho \in \mathcal{D}(Y)$, and let $\tilde{d} \in \mathcal{D}(Y)$ be a complete metric, with $\tilde{d} \geqslant \rho$. Then, by [18, Proposition 2.1] (see, also, [9, Theorem 1]), there exists another complete metric $d \in \mathcal{D}(Y)$ such that $\mathcal{S} = \{\varphi(x) : x \in X\}$ is d-uniformly equi-LC^{n-1} and $d \geqslant \tilde{d} \geqslant \rho$. Now, let u: $X \times Y \to \mathbb{I}$ be defined as in (8) using the metric d, and let Φ: $X \to \mathcal{F}(Y \times (0, 1])$ be defined as in Theorem 3.2. By Proposition 3.1, the function r: $X \to \mathbb{I}$, defined by $r(x) = u(x, f(x))$, $x \in X$, is lower semi-continuous. Since X is normal, there now exists a continuous function v: $X \to \mathbb{I}$ such that $v(x) \leqslant r(x)$, $x \in X$, see [10]. Note that the diagonal map $\ell = f \Delta v$: $X \to Y \times \mathbb{I}$ is a selection for the closure of Φ, i.e., $\ell(x) \in \overline{\Phi(x)}$ for every $x \in X$. Thus, in particular, $\ell(x) \in B^{d\times e}_{\alpha(x)}(\Phi(x))$, $x \in X$, for every $\alpha \in C(X, (0, +\infty))$. On the other hand, by Theorem 3.2, the family $\{\Phi(x) : x \in X\}$ is $d\times e$-uniformly equi-LC^{n-1}. Hence, by Corollary 2.5, Φ has a continuous selection p: $X \to Y \times (0, 1]$ such that $d \times e$-distance in $Y \times \mathbb{I}$ between the points $p(x)$ and $\ell(x)$ is less than $\mu(x)$ for every $x \in X$ (recall that $Y \times (0, 1]$ is open in $Y \times \mathbb{I}$). Then, $g = \pi_1 \circ p$: $X \to Y$ is as required, where π_1: $Y \times (0, 1] \to Y$ is the projection. □

By Theorem 4.1 we have the following consequence.

COROLLARY 4.2. *Let X be paracompact, with* $\dim(X) \leqslant n$, *Y be a completely metrizable space, φ: $X \to \mathcal{F}(Y)$ be l.s.c. such that $\{\varphi(x) : x \in X\}$ is equi-LC^{n-1} and each $\varphi(x)$ is C^{n-1}, $x \in X$, and let ψ: $X \to \mathcal{P}(Y)$ be a mapping with a closed graph such that $\psi(x) \cap \varphi(x)$ is a σZ_n-set in $\varphi(x)$ for every $x \in X$. Then, the set $\mathcal{S}e\ell(\varphi)$ of all continuous selections for φ, endowed with the fine topology, is a Baire space and $\mathcal{M}(\psi) \cap \mathcal{S}e\ell(\varphi) = \{f \in \mathcal{S}e\ell(\varphi) : f(x) \notin \psi(x)$ for every $x \in X\}$ is open and dense in $\mathcal{S}e\ell(\varphi)$.*

Proof. Take a complete bounded metric $d \in \mathcal{D}(Y)$. Every subset of $C(X, Y)$ which is closed with respect to the uniform topology generated by d is a Baire space in the fine topology, see [19, Lemma 3.2]. Obviously, $\mathcal{S}e\ell(\varphi)$ is uniformly closed in $C(X, Y)$ with respect to d, so it has the Baire property. Let us show that $\mathcal{M}(\psi) \cap \mathcal{S}e\ell(\varphi)$ is open in $\mathcal{S}e\ell(\varphi)$ (note that, by [18, Theorem 1.2] and Theorem 4.1, $\mathcal{M}(\psi) \cap \mathcal{S}e\ell(\varphi) \neq \emptyset$). Take an $f \in \mathcal{M}(\psi) \cap \mathcal{S}e\ell(\varphi)$, and let u: $X \times Y \to \mathbb{I}$ be as in (8). Then, by Proposition 3.1, $\eta(x) = u(x, f(x))$, $x \in X$, defines a lower semi-continuous function η: $X \to (0, +\infty)$. Hence, by [6, 7, 15], there exists an $\alpha \in C(X, (0, +\infty))$, with $\alpha(x) < \eta(x)$ for every $x \in X$. Therefore $V(f, \alpha) \cap \mathcal{S}e\ell(\varphi) \subset \mathcal{M}(\psi) \cap \mathcal{S}e\ell(\varphi)$, so $\mathcal{M}(\psi) \cap \mathcal{S}e\ell(\varphi)$ is open in $\mathcal{S}e\ell(\varphi)$. That $\mathcal{M}(\psi) \cap \mathcal{S}e\ell(\varphi) \subset \mathcal{S}e\ell(\varphi)$ is dense, it follows by Theorem 4.1. □

Now, we complete the proof of Theorem 1.2 in the following way. Let X, Y, φ and ψ be as in that theorem. Since Graph(ψ) is an F_σ-subset of $X \times Y$, there are mappings ψ_k: $X \to \mathcal{P}(Y)$, $k \in \mathbb{N}$, such that each ψ_k, $k \in \mathbb{N}$, has a closed graph and $\psi(x) = \bigcup\{\psi_k(x) : k \in \mathbb{N}\}$. Then, by Corollary 4.2, each set $\mathcal{M}(\psi_k) \cap \mathcal{S}e\ell(\varphi)$, $k \in \mathbb{N}$, is open and dense in $\mathcal{S}e\ell(\varphi)$ with respect to the fine topology, while $\mathcal{S}e\ell(\varphi)$

is a Baire space. Therefore, $\mathcal{M}(\psi) \cap \mathcal{S}e\ell(\varphi) = \bigcap\{\mathcal{M}(\psi_k) \cap \mathcal{S}e\ell(\varphi) : k \in \mathbb{N}\}$ is a dense G_δ-subset of $\mathcal{S}e\ell(\varphi)$ which completes the proof.

Remark. It should be mentioned that Theorem 1.2 might be compared with [19, Lemma 3.2] (see, also, [19, Theorem 3.1]) which works in the special case of a convex-valued φ and a continuous ψ. In fact, one of the main goals of Theorem 1.2 is to avoid the assumption on φ to be convex-valued. Namely, in case Y is a Banach space and φ is convex-valued, Theorem 1.2 has a relative shorter proof based either on the technique developed in [20] or on that stated in [12]. Despite of that, in both cases the proof will rely on some results of [18], hence will not make the corresponding arguments simpler.

We complete this section showing that the following result of Michael [20, Theorem 5.2] can be derived from our Theorem 1.2. In what follows, for a linear topological space Y, we let $\mathcal{F}_c(Y) = \{S \in \mathcal{F}(Y) : S \text{ is convex}\}$.

THEOREM 4.3. *Let X be a paracompact space, Y be a Banach space, φ: $X \to \mathcal{F}_c(Y)$ be l.s.c., and let ψ_n: $X \to \mathcal{P}(Y)$, $n \in \mathbb{N}$, be a sequence of mappings such that each ψ_n has a closed graph and* $\dim(X) < \dim(\varphi(x)) - \dim(\operatorname{conv}(\psi_n(x) \cap \varphi(x)))$ *for every $x \in X$. Then φ has a continuous selection f such that $f(x) \notin \bigcup\{\psi_n(x) : n \in \mathbb{N}\}$ for every $x \in X$.*

Proof. Suppose that $\dim(X) = m$. To apply Theorem 1.2, it suffices to check that, in this situation, $\psi_n(x) \cap \varphi(x)$ is a Z_m-set in $\varphi(x)$ for all $x \in X$ and $n \in \mathbb{N}$. To this end, observe that, by [19, Lemma 2.1], the sets $\varphi(x)\setminus\psi_n(x)$ are C^{m-1} and, obviously, they are LC^{m-1} because each $\varphi(x)$ is convex. Moreover, if V is an open ball in $\varphi(x)$, then

$$m + \dim(\operatorname{conv}(\psi_n(x) \cap V)) < \dim(V) \quad \text{for every } n \in \mathbb{N}.$$

Hence, again by [19, Lemma 2.1], $V\setminus\psi_n(x)$ is C^{m-1}. Then, according to [29, Theorem 2.8 and Corollary 3.3], each $\psi_n(x) \cap \varphi(x)$ is a Z_m-set in $\varphi(x)$. □

5. Strongly Countable-Dimensional Spaces and Selections

In this section we provide some applications of Theorem 1.2 related to *strongly countable-dimensional spaces* (i.e., spaces which are a countable union of closed finite-dimensional subspaces). Our first result is a possible selection analogue of a result established by E. Pol [24, Corollary 4.4].

THEOREM 5.1. *Let X_n, $n < \omega$, be closed subsets of a paracompact space X, with* $\dim(X_n) \leqslant n$, *Y be a countable product of Banach spaces, φ: $X \to \mathcal{F}_c(Y)$ be an l.s.c. mapping, and let ψ_n: $X \to \mathcal{P}(Y)$, $n < \omega$, be a sequence of mappings such that each ψ_n has an F_σ-graph and $\psi_n(x) \cap \varphi(x)$ is a σZ_n-set in $\varphi(x)$ for every $x \in X_n$ and $n < \omega$. Then, the set $\{f \in \mathcal{S}e\ell(\varphi) : f|X_n \in \mathcal{M}(\psi_n|X_n)$ for every $n < \omega\}$ is a dense G_δ-set in $\mathcal{S}e\ell(\varphi)$ equipped with the fine topology.*

Proof. By Theorem 1.2, each $\mathcal{M}(\psi_n|X_n) \cap \mathcal{S}e\ell(\varphi|X_n)$, $n < \omega$, is a dense G_δ-subset in $\mathcal{S}e\ell(\varphi|X_n)$ with respect the fine topology. Let $\pi_n\colon \mathcal{S}e\ell(\varphi) \to \mathcal{S}e\ell(\varphi|X_n)$ be the map defined by $\pi_n(f) = f|X_n$, $f \in \mathcal{S}e\ell(\varphi)$. Note that each π_n, $n < \omega$, is continuous when both $\mathcal{S}e\ell(\varphi)$ and $\mathcal{S}e\ell(\varphi|X_n)$ are equipped with the fine topology. Moreover, π_n is surjective because every partial selection for $\varphi|X_n$ can be extended to a selection for φ (see the proof of [17, Theorem 3.2″], also [27]). Let us observe that each π_n, $n < \omega$, is an open map as well. Towards this end, take a convex metric ρ on Y which is possible because Y is a metrizable locally convex topological vector space, see [14]. Now, it suffices to show that $\pi_n(V(f,\alpha) \cap \mathcal{S}e\ell(\varphi))$ is open in $\mathcal{S}e\ell(\varphi|X_n)$ for every $f \in \mathcal{S}e\ell(\varphi)$ and $\alpha \in C(X, (0, \infty))$. Let $h \in \pi_n(V(f,\alpha) \cap \mathcal{S}e\ell(\varphi))$. Then, $h = g|X_n$ for some $g \in V(f,\alpha) \cap \mathcal{S}e\ell(\varphi)$. Let $\delta(x) = (\alpha(x) - \rho(f(x), g(x)))/2$ and $\beta(x) = \rho(f(x), g(x)) + \delta(x)$, $x \in X$. We are going to show that

$$V(h,\delta) \cap \mathcal{S}e\ell(\varphi|X_n) \subset \pi_n(V(f,\alpha) \cap \mathcal{S}e\ell(\varphi)).$$

Indeed, for every $\ell \in V(h,\delta) \cap \mathcal{S}e\ell(\varphi|X_n)$ we have $\rho(\ell(x), f(x)) < \beta(x)$, $x \in X_n$. So, we may define an l.s.c. mapping $\phi\colon X \to \mathcal{F}_c(E)$ by $\phi(x) = \{\ell(x)\}$ if $x \in X_n$ and $\phi(x) = \overline{\varphi(x) \cap B^{\rho}_{\beta(x)}(f(x))}$ otherwise. By the proof of [17, Theorem 3.2″] (see, also, [27]), ϕ has a continuous selection q. Then $q \in V(f,\alpha) \cap \mathcal{S}e\ell(\varphi)$ and $\ell = \pi_n(q)$. Therefore, $V(h,\delta) \cap \mathcal{S}e\ell(\varphi|X_n)$ is a neighborhood of h in $\pi_n(V(f,\alpha) \cap \mathcal{S}e\ell(\varphi))$.

We finally accomplish the proof as follows. Since each $V_n = \mathcal{M}(\psi|X_n) \cap \mathcal{S}e\ell(\varphi|X_n)$, $n < \omega$, is a dense G_δ-set in $\mathcal{S}e\ell(\varphi|X_n)$ and π_n is continuous and open, $U_n = \pi_n^{-1}(V_n)$ is dense and G_δ in $\mathcal{S}e\ell(\varphi)$. Hence, the set $\{f \in \mathcal{S}e\ell(\varphi) : f|X_n \in \mathcal{M}(\psi|X_n),\ n < \omega\}$, being the intersection of all U_n's, is also dense and G_δ in $\mathcal{S}e\ell(\varphi)$.

Another application of our selection theorems is the following 'strongly countable-dimensional' analogue of Ostrand's theorem [23], see also [11].

THEOREM 5.2. *For a normal space X and closed subsets X_n, $n < \omega$, the following are equivalent*:

(a) $\dim(X_n) \leqslant n$, $n < \omega$.

(b) *For every sequence $\{\gamma_n : n < \omega\}$ of locally finite open covers of X there is a sequence $\{\mu_n : n < \omega\}$ of discrete open families in X such that for each $n < \omega$ we have*:

  (i) *μ_n refines γ_n,*

  (ii) *the union of any $n+1$ families of the sequence $\{\mu_k : k < \omega\}$ constitutes a cover of X_n.*

Proof. (a) ⇒ (b). We follow the proof of [30, Theorem 1.3, implication $S_2 \Rightarrow C$] and [12, Theorem 1.1, implication (c) ⇒ (a)]. Take a sequence $\{\gamma_n : n < \omega\}$ of locally finite open covers of X. We are going to prove that there exists a sequence

$\{\mu_n : n < \omega\}$ of locally finite disjoint families of open sets in X such that each μ_n, $n < \omega$, refines γ_n and the union of any $n+1$ of them is a cover of X_n. To this end, as in [12, Theorem 1.1, (c) ⇒ (a)], for every n, fix a metric space (M_n, d_n) and a continuous map $f_n\colon X \to M_n$ such that $\{f_n^{-1}(B_2^{d_n}(z)) : z \in M_n\}$ is an open cover of X refining γ_n, and $d_n(z,t) \leqslant 3$ for $z, t \in M_n$. Next, consider the disjoint union $M = \bigsqcup\{M_n : n < \omega\}$ of these spaces M_n, and let d be the metric on M defined as $d|(M_n \times M_n) = d_n$, and $d(z,t) = 3$ provided $z \in M_i$, $t \in M_j$ and $i \neq j$. Embed (M, d) isometrically into a Banach space (E, d), where d is the metric on E generated by the norm $\|\cdot\|$ of E. Let $f = \Delta\{f_n : n < \omega\}\colon X \to E^\omega$, and let $\beta f\colon \beta X \to \beta(E^\omega)$ be the corresponding Čech–Stone extension. Consider the space $H = (\beta f)^{-1}(E^\omega)$, the closure H_n of each set X_n in H, and the map $h\colon H \to E^\omega$ defined by $h = (\beta f)|H$. Then H is paracompact and $\dim(H_n) = \dim(X_n) \leqslant n$ for every n. Moreover, h is generated by a sequence of maps $h_n\colon H \to E$, $n < \omega$, such that each h_n extends f_n. For every $n < \omega$, define an l.s.c. mapping $\varphi_n\colon H \to \mathcal{F}_c(E)$ by $\varphi_n(x) = \overline{B_1^d(h_n(x))}$, $x \in H$. Next, define another mapping $\varphi\colon H \to \mathcal{F}_c(E^\omega)$ by $\varphi(x) = \prod\{\varphi_n(x) : n < \omega\}$, $x \in H$. It is easily seen that φ is l.s.c.

As in the proof in [30, Theorem 1.3] (implication $S_2 \Rightarrow C$), we now fix a closed nowhere dense set $A \subset E$ such that the family λ of all components of $E\backslash A$ consists of disjoint open cells of diameter $\leqslant 1$ and, for every $n < \omega$, the family $g^{-1}(\lambda)$ refines γ_n provided $g\colon X \to E$ is a map with $d(f_n(x), g(x)) \leqslant 1$ for all $x \in X$. Whenever $n < \omega$, let $\Omega_n = \{P \subset \omega : |P| = n+1\}$ and, for every $P \in \Omega_n$, let $F_{(i,P)} = A$ if $i \in P$ and $F_{(i,P)} = E$ otherwise. Finally, set $F_P = \prod\{F_{(i,P)} : i < \omega\}$. Then each $F_P \cap \varphi(x)$ is, in fact, the following (possibly empty) product

$$\prod\{A \cap \overline{B_1^d(f_i(x))} : i \in P\} \times \prod\{\overline{B_1^d(f_i(x))} : i \in \omega \setminus P\}.$$

On the other hand, each set $A \cap \overline{B_1(f_i(x))}$, $i < \omega$, is closed and nowhere dense in $\overline{B_1(f_i(x))}$. Therefore, by [2, Corollary 2], for every $n < \omega$ and $P \in \Omega_n$, the product $\prod\{A \cap \overline{B_1(f_i(x))} : i \in P\}$ is a Z_n-set in $\prod\{\varphi_i(x) : i \in P\}$, $x \in H$. The last yields that $F_P \cap \varphi(x)$ is a Z_n-set in $\varphi(x)$ for all $x \in H$, $n < \omega$ and $P \in \Omega_n$. We may now apply Theorem 5.1 (with X replaced by H, X_n by H_n, Y by E^ω and $\psi_n(x) = F_n = \bigcup\{F_P : P \in \Omega_n\}$, $x \in H$) to obtain a continuous selection $g\colon H \to E^\omega$ for φ such that $g(H_n) \cap F_n = \emptyset$, $n < \omega$. Let $g = \Delta\{g_n : n < \omega\}$, where each g_n is a continuous map from H into E. Then, $d(f_n(x), g_n(x)) \leqslant 1$ for all $x \in H$ and $n < \omega$. Hence, according to the properties of the set A, $\mu_n = g_n^{-1}(\lambda) \cap X$ is a disjoint open family in X refining γ_n. We can assume that each μ_n is an index refinement of γ_n, in particular, locally finite. It remains to show that for every n and $D \in \Omega_n$ the corresponding family $\{\mu_i : i \in D\}$ is a cover of X_n. This easily follows from the fact that $F_D \subset F_n$ and $g(X_n)$, being a subset of $g(H_n)$, avoids the set F_n.

(b) ⇒ (a). The implication follows directly. For a fixed n, let α be a finite open (in X) cover of X_n. To prove that $\dim(X_n) \leqslant n$, we need to find a finite open cover λ of X_n refining α and such that any $n+2$ elements of λ have an empty intersection.

To this end, let $\gamma = \alpha \cup \{X \setminus X_n\}$. Then there exists a sequence $\{\mu_k : k < \omega\}$ of disjoint, open and locally finite families in X such that each μ_k refines γ and any $n+1$ of them constitute a cover of X_n. We can suppose that every μ_k is an index refinement of γ, in particular, finite. Set $\lambda_k = \{U \in \mu_k : U \cap X_n \neq \emptyset\}$, $k < \omega$. Then $\lambda = \bigcup\{\lambda_k : 0 \leqslant k \leqslant n\}$ is as required. □

By taking $X_n = X$, $n < \omega$, in Theorem 5.2 we now get the following 'finite-dimensional' generalization of the Ostrand's theorem (recall that Ostrand originally proved his theorem for finite-dimensional spaces).

COROLLARY 5.3. *For a normal space X and $n < \omega$ the following are equivalent*:

(a) $\dim(X) \leqslant n$.
(b) *For every sequence $\{\gamma_k : k < \omega\}$ of locally finite open covers of X there is a sequence $\{\mu_k : k < \omega\}$ of discrete open families in X such that each μ_k, $k < \omega$, refines γ_k and the union of any $n+1$ elements of the sequence $\{\mu_k : k < \omega\}$ constitutes a cover of X.*

6. Closed Maps and Selections

Let M be a metrizable space. We will use the term *distance* for a possibly infinite-valued function d: $M \times M \to [0, +\infty]$ which satisfies the axioms of a metric on M. So, let $\mathcal{D}_\infty(M)$ denote the set of all distances on M which are compatible with the topology of M. Then, to every $d \in \mathcal{D}_\infty(M)$ and every positive real number $r \in \mathbb{R}$ we may associate a metric $d_r \in \mathcal{D}(M)$ on M defined by $d_r(x, y) = \min\{r, d(x, y)\}$, $x, y \in M$. Now, we shall say that $d \in \mathcal{D}_\infty(M)$ has a property $\mathcal{P}$ if d_r has $\mathcal{P}$ for every $r > 0$. For instance, *completeness* is such a property $\mathcal{P}$.

Here is a natural example of distances we will deal with. Let X be a space, and let (Y, d) be a metric space. In what follows, we will rely on the *uniform topology* on $C(X, Y)$ generated by the distance

$$\boldsymbol{d}(f, g) = \sup\{d(f(x), g(x)) : x \in X\}, \quad f, g \in C(X, Y).$$

Note that $\boldsymbol{d}_r(f, g) = \sup\{d_r(f(x), g(x)) : x \in X\}$, whenever $r > 0$. Thus, $(C(X, Y), \boldsymbol{d})$ is complete if d is a complete metric on Y (see, for instance, [8, Chapter XII]). Concerning maps ℓ: $Z \to C(X, Y)$, let us agree to denote by $\ell[z] \in C(X, Y)$ the value of ℓ in a particular point $z \in Z$.

Throughout this section, to every surjective map f: $X \to Y$ and a set-valued mapping φ: $X \to 2^E$ we associate another set-valued mapping $\Delta_{(f,\varphi)}$: $Y \to \mathcal{P}(C(X, E))$ defined by

$$\Delta_{(f,\varphi)}(y) = \{g \in C(X, E) : g|f^{-1}(y) \in \mathcal{S}e\ell(\varphi|f^{-1}(y))\}, \quad y \in Y.$$

Also, let us recall that a subset A of a space X is *P-embedded* in X if every continuous map $g\colon A \to E$ into a Banach space E is continuously extendable over X, see [1, 21, 26, 28]. Finally, by a Banach space (E, d) we mean a Banach space E and a metric d generated by the norm of E.

THEOREM 6.1. *Let X and Y be a spaces, and let $f\colon X \to Y$ be a continuous closed surjection such that each $f^{-1}(y)$, $y \in Y$, is paracompact and P-embedded in X. Also, let (E, d) be a Banach space and $\varphi\colon X \to \mathscr{F}_c(E)$ be an l.s.c. mapping. Then, $\Delta_{(f,\varphi)}$ is an l.s.c. mapping from Y with values in $\mathscr{F}_c(C(X, E))$ provided $C(X, E)$ is endowed with the uniform topology generated by the distance $\boldsymbol{d}$. Moreover, if ℓ is a continuous selection for $\Delta_{(f,\varphi)}$, then $g(x) = \ell[f(x)](x)$, $x \in X$, defines a continuous selection for φ.*

Proof. Note that, by [17, Theorem 3.2″], $\mathscr{S}e\ell(\varphi|f^{-1}(y)) \neq \emptyset$ for every $y \in Y$ because each $f^{-1}(y)$ is paracompact. Hence, $\Delta_{(f,\varphi)}(y) \neq \emptyset$ for every $y \in Y$ because each $f^{-1}(y)$ is P-embedded in X. Thus, $\Delta_{(f,\varphi)}\colon Y \to \mathscr{F}_c(C(X, E))$ because $\varphi\colon X \to \mathscr{F}_c(E)$.

To show that $\Delta_{(f,\varphi)}$ is l.s.c. take a point $y_0 \in \Delta^{-1}_{(f,\varphi)}(B^{\boldsymbol{d}}_{\varepsilon}(g_0))$, where $g_0 \in \Delta_{(f,\varphi)}(y_0)$ and $\varepsilon > 0$. It suffices to find a neighborhood of y_0 in Y which is contained in $\Delta^{-1}_{(f,\varphi)}(B^{\boldsymbol{d}}_{\varepsilon}(g_0))$. Since

$$V = \{x \in X : d(g_0(x), \varphi(x)) < \varepsilon/2\}$$

is a neighbourhood of $f^{-1}(y_0)$ and f is closed, there exists of a neighbourhood U of y_0, with $f^{-1}(U) \subset V$. Let show that $U \subset \Delta^{-1}_{(f,\varphi)}(B^{\boldsymbol{d}}_{\varepsilon}(g_0))$. Indeed, take an $y \in U$ and define an l.s.c. mapping $\Phi\colon f^{-1}(y) \to \mathscr{F}_c(E)$ by $\Phi(x) = B^{d}_{\varepsilon/2}(g_0(x)) \cap \varphi(x)$, $x \in f^{-1}(y)$. Then, by [17, Theorem 3.2″], Φ has a continuous selection h. Thus, $\varphi|f^{-1}(y)$ has a continuous selection h, with $\boldsymbol{d}(h, g_0|f^{-1}(y)) \leqslant \varepsilon/2 < \varepsilon$.

CLAIM 1. *There exists a continuous extension $g\colon X \to E$ of h such that $\boldsymbol{d}(g, g_0) < \varepsilon$.*

In order to show our claim, we first extend h to a continuous map $g_1\colon X \to E$ which is possible because $f^{-1}(y)$ is P-embedded in X. Next, let us observe that $Z = \{x \in X : d(g_0(x), g_1(x)) \geqslant 2\varepsilon/3\}$ is a zero-set of X, with $Z \cap f^{-1}(y) = \emptyset$. Since $f^{-1}(y)$ is P-embedded, by [3, Corollary 3.6.B], there now exists a continuous function $\eta\colon X \to \mathbb{I}$ such that $f^{-1}(y) \subset \eta^{-1}(0)$ and $Z \subset \eta^{-1}(1)$. Finally, we may define $g\colon X \to E$ by $g(x) = \eta(x) \cdot g_0(x) + (1 - \eta(x)) \cdot g_1(x)$, $x \in X$. This g is as required.

Now, we have that $g \in \Delta_{(f,\varphi)}(y)$ because g is an extension of h, while, by Claim 1, $y \in \Delta^{-1}_{(f,\varphi)}(B^{\boldsymbol{d}}_{\varepsilon}(g_0))$, so $U \subset \Delta^{-1}_{(f,\varphi)}(B^{\boldsymbol{d}}_{\varepsilon}(g_0))$. Therefore, $\Delta_{(f,\varphi)}$ is l.s.c.

Let prove the final part of Theorem 6.1. Suppose ℓ is a continuous selection for $\Delta_{(f,\varphi)}$, and let $g(x) = \ell[x](x)$ for every $x \in X$. Clearly g is a selection for φ. So, it only remains to show that g is continuous. Take a point $x_0 \in X$ and $\varepsilon > 0$, and let $U = \ell^{-1}(B^{\boldsymbol{d}}_{\varepsilon/2}(\ell[x_0])) \cap (\ell[x_0])^{-1}(B^{d}_{\varepsilon/2}(\ell[x_0](x_0)))$. Then, $x \in U$ implies

$$\begin{aligned} d(g(x), g(x_0)) &= d(\ell[x](x), \ell[x_0](x_0)) \\ &\leqslant d(\ell[x](x), \ell[x_0](x)) + d(\ell[x_0](x), \ell[x_0](x_0)) \\ &\leqslant \boldsymbol{d}(\ell[x], \ell[x_0]) + d(\ell[x_0](x), \ell[x_0](x_0)) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This completes the proof. □

Theorem 6.1 may have some general interest. For instance, it implies the following slight generalization of a result of Hanai [13], see also [10].

THEOREM 6.2. *Let X be a space, Y be a paracompact space, and $f\colon X \to Y$ be a continuous closed surjection such that each $f^{-1}(y)$, $y \in Y$, is paracompact and P-embedded in X. Then, X is paracompact too.*

Proof. Take an l.s.c. mapping $\varphi\colon X \to \mathcal{F}_c(E)$ for some Banach space (E, d). By [17, Theorem 3.2″], it suffices to show that φ has a continuous selection. Consider the l.s.c. mapping $\Delta_{(f,\varphi)} : Y \to \mathcal{F}_c(C(X, E))$, where $C(X, E)$ is endowed with the uniform topology generated by the distance $\boldsymbol{d}$. In fact, the same topology on $C(X, E)$ is generated by the metric $\boldsymbol{d}_1$ on $C(X, E)$ defined by $\boldsymbol{d}_1(g, h) = \min\{1, \boldsymbol{d}(g, h)\}$, see the beginning of the section. Since $(C(X, E), \boldsymbol{d}_1)$ is a Fréchet space, $\Delta_{(f,\varphi)}$ has a continuous selection [17] (see, also, [27]). Hence, Theorem 6.1 completes the proof. □

Finally, we provide an alternative proof of the dimension-lowering mapping theorem, see [11] for the proof and history of this theorem.

THEOREM 6.3. *If $f\colon X \to Y$ is a closed continuous mapping of a normal space X to a paracompact space Y and there exists an integer $k \geqslant 0$ such that $\dim(f^{-1}(y)) \leqslant k$ for every $y \in Y$, then $\dim(X) \leqslant \dim(Y) + k$.*

Proof. We may assume that X is paracompact. Indeed, we may consider the Čech–Stone extension $\beta f\colon \beta X \to \beta Y$. Then, by Theorem 6.2, $H = (\beta f)^{-1}(Y)$ is a paracompact space, while $\tilde{f} = \beta f|H$ is a perfect map such that $\tilde{f}^{-1}(y) = \overline{f^{-1}(y)}$ for every $y \in Y$. Since X is normal, we have that $\beta(f^{-1}(y)) = \tilde{f}^{-1}(y)$, so $\dim(\tilde{f}^{-1}(y)) \leqslant k$ for every $y \in Y$. Finally, $\beta H = \beta X$ implies that $\dim(H) = \dim(X)$.

Thus, let X and Y be paracompact, and let $f\colon X \to Y$ be a closed continuous surjection as in the hypothesis. Also, suppose that $\dim(Y) \leqslant m$. To show that $\dim(X) \leqslant m + k$, it suffices to show that every map $g\colon X \to \mathbb{R}^{m+k+1}$ is removable from the origin ϑ of $\mathbb{R}^{m+k+1}$. For a fixed $g_0 \in C(X, \mathbb{R}^{m+k+1})$ and $\varepsilon > 0$, we define a set-valued mapping $\varphi\colon X \to \mathcal{F}_c(\mathbb{R}^{m+k+1})$ by $\varphi(x) = \overline{B^d_\varepsilon(g_0(x))}$, $x \in X$, where d is the usual Euclidean metric on $\mathbb{R}^{m+k+1}$. Thus, by Theorem 6.1, we may get an l.s.c. mapping $\Phi = \Delta_{(f,\varphi)}\colon Y \to \mathcal{F}_c(C(X, \mathbb{R}^{m+k+1}))$.

Now we consider the mapping $\Psi\colon Y \to \mathcal{F}(C(X, \mathbb{R}^{m+k+1}))$ defined by

$$\Psi(y) = \{g \in C(X, \mathbb{R}^{m+k+1}) : \vartheta \in g(f^{-1}(y))\}, \quad y \in Y.$$

It is a routine verification that the graph of Ψ is closed. Let us show that $\Psi(y) \cap \Phi(y)$ is a Z_m-set in $\Phi(y)$ for every $y \in Y$. Take a fixed point $y \in Y$ and a continuous map $u\colon \mathbb{B}^m \to \Phi(y)$. We are going to prove that for every $\delta > 0$ there exists a continuous map $v\colon \mathbb{B}^m \to \Phi(y)$ which is δ-close to u and avoids the set $\Psi(y)$. Observe that u generates a continuous map $p\colon \mathbb{B}^m \times X \to \mathbb{R}^{m+k+1}$, $p(z, x) = u[z](x)$, such that $p(z, x) \in \varphi(x)$ for every $x \in f^{-1}(y)$ and $z \in \mathbb{B}^m$. Define $\phi\colon \mathbb{B}^m \times f^{-1}(y) \to \mathcal{F}_c(\mathbb{R}^{m+k+1})$ by $\phi(z, x) = \overline{B^d_{\delta/2}(p(z, x)) \cap \varphi(x)}$, $(z, x) \in \mathbb{B}^m \times f^{-1}(y)$. Since ϑ is a Z_{m+k}-set in each $\varphi(x)$ (as a Z_{m+k}-set in $\mathbb{R}^{m+k+1}$), it is a Z_{m+k}-set in $\phi(z, x)$ for any $(z, x) \in \mathbb{B}^m \times f^{-1}(y)$ (see [12, Lemma 2.3]). Moreover, ϕ is an l.s.c. and $\dim(\mathbb{B}^m \times f^{-1}(y)) \leqslant m + k$. Hence, by Theorem 1.2, ϕ has a continuous selection $\tilde{q}\colon \mathbb{B}^m \times f^{-1}(y) \to \mathbb{R}^{m+k+1}$ avoiding ϑ. In particular, we get that $\tilde{q}(z, x) \in \varphi(x)$ for every $x \in f^{-1}(y)$ and $z \in \mathbb{B}^m$, while $\boldsymbol{d}(\tilde{q}, p|\mathbb{B}^m \times f^{-1}(y)) \leqslant \delta/2$. So, we may extend $\tilde{q}$ to a continuous map $q\colon \mathbb{B}^m \times X \to \mathbb{R}^{m+k+1}$ such that $\boldsymbol{d}(q, p) < \delta$. Then q determines a continuous map $v\colon \mathbb{I}^m \to C(X, \mathbb{R}^{m+k+1})$, with $v[z](x) = q(z, x)$. This v is as required.

We now finish the proof as follows. By Theorem 1.2, Φ has a continuous selection $\ell\colon Y \to C(X, \mathbb{R}^{m+k+1})$ such that $\ell(y) \in \Phi(y) \backslash \Psi(y)$ for every $y \in Y$. Then the map $g\colon X \to \mathbb{R}^{m+k+1}$, $g(x) = \ell[f(x)](x)$, is ε-close to g_0 and avoids ϑ. □

References

1. Aló, R. A. and Sennott, L. I.: Extending linear space-valued functions, *Math. Ann.* **191** (1971), 79–86.
2. Banakh, T. and Trushchak, K.: Z_n-sets and the disjoint n-cells property in products of ANR's, *Mat. Stud.* **13**(1) (2000), 74–78.
3. Blair, R. L. and Hager, A. W.: Extensions of zero-sets and of real-valued functions, *Math. Z.* **136** (1974), 41–52.
4. Chigogidze, A.: *Inverse Spectra*, North-Holland, Amsterdam, 1996.
5. Di Maio, G., Holá, L'., Holý, D. and McCoy, R. A.: Topologies on the space of continuous functions, *Topology Appl.* **86** (1998), 105–122.
6. Dieudonne, J.: Une généralisation des espaces compacts, *J. Math. Pures Appl.* **23** (1944), 65–76.
7. Dowker, C. H.: On countably paracompact spaces, *Canad. J. Math.* **3** (1951), 291–224.
8. Dugundji, J.: *Topology*, Boston, 1966.
9. Dugundji, J. and Michael, E.: On local and uniformly local topological properties, *Proc. Amer. Math. Soc.* **7** (1956), 304–307.
10. Engelking, R.: *General Topology*, revised and completed edn, Heldermann Verlag, Berlin, 1989.
11. Engelking, R.: *Theory of Dimensions: Finite and Infinite*, Heldermann Verlag, Lemgo, 1995.
12. Gutev, V. and Valov, V.: Continuous selections and C-spaces, *Proc. Amer. Math. Soc.* **130** (2002), 233–242.
13. Hanai, S.: On closed mappings. II, *Proc. Japan Acad.* **32** (1956), 388–391.

14. Henderson, D. W.: Z-sets in ANR's, *Trans. Amer. Math. Soc.* **213** (1975), 205–216.
15. Katětov, M.: On real-valued functions in topological spaces, *Fund. Math.* **38** (1951), 85–91.
16. Krikorian, N.: A note concerning the fine topology on function spaces, *Compositio Math.* **21** (1969), 343–348.
17. Michael, E.: Continuous selections I, *Ann. of Math.* **63** (1956), 361–382.
18. Michael, E.: Continuous selections II, *Ann. of Math.* **64** (1956), 562–580.
19. Michael, E.: Continuous selections avoiding a set, *Topology Appl.* **28** (1988), 195–213.
20. Michael, E.: A generalization of a theorem on continuous selections, *Proc. Amer. Math. Soc.* **105** (1989), 236–243.
21. Morita, K.: On generalizations of Borsuk's homotopy extension theorem, *Fund. Math.* **88** (1975), 1–6.
22. Munkres, J.: *Topology*, Prentice Hall, Englewood Cliffs, NY, 1975.
23. Ostrand, P.: Covering dimension in general spaces, *Topology Appl.* **1** (1971), 209–221.
24. Pol, E.: Characterizing strong countable-dimensionality in terms of Baire category, *Fund. Math.* **131** (1988), 155–165.
25. Pol, E.: Residuality of the set of embeddings into Nagata's n-dimensional universal spaces, *Fund. Math.* **129** (1988), 59–67.
26. Przymusiński, T.: Collectionwise normality and extensions of continuous functions, *Fund. Math.* **98** (1978), 75–81.
27. Repovš, D. and Semenov, P. V.: *Continuous Selections of Multivalued Mappings*, Mathematics and its Applications 455, Kluwer Academic Publishers, The Netherlands, 1998.
28. Shapiro, H. L.: Extensions of pseudometrics, *Canad. J. Math.* **18** (1966), 981–998.
29. Toruńczyk, H.: Concerning locally homotopy negligible sets and characterization of l_2-manifolds, *Fund. Math.* **101** (1978), 93–110.
30. Uspenskij, V.: A selection theorem for C-spaces, *Topology Appl.* **85** (1998), 351–374.